5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications, Proceedings, pp. 419–424

Substitution Operators between Measurable Function Spaces

M. R. Jabbarzadeh
Department of Mathematics, University of Tabriz, Tabriz, Iran
E-mail: mjabbar@tabrizu.ac.ir

Abstract

In this paper we will consider the substitution (weighted composition operators) on measurable function spaces and Fredholmness of these type operators will be investigated.

AMS Subject Classification: Primary 47B20; Secondary 47B38.
Key words: Weighted composition operator, Conditional expectation, Multiplication operator, Fredholm operator.

1 Preliminaries and notations

In the next section we investigate a necessary and sufficient condition for a weighted composition operator $W = uC_\varphi$ to be Fredholm. Fredholm weighted composition operators have been studied by H. Takagi [7] in the $L^p(\Sigma)$ setting. By using some properties of conditional expectation operator we omit the continuity hypothesis of M_u. In other words, we do not require that $u \in L^\infty(\Sigma)$. This is stated as a hypothesis in [7].

Let (X, Σ, μ) be a σ-finite measure space. By $L(X)$, we denote the linear space of all Σ-measurable functions on X. When we consider any sub-σ-algebra $\mathcal{A}$ of Σ, we assume they are completed; *i.e.*, $\mu(A) = 0$ implies $B \in \mathcal{A}$ for any $B \subset A$. For any σ-finite algebra $\mathcal{A} \subseteq \Sigma$ and $1 \leq p \leq \infty$ we abbreviate the L^p-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ to $L^p(\mathcal{A})$, and denote its norm by $\|.\|_p$. We define the support of a measurable function f as $\sigma(f) = \{x \in X; f(x) \neq 0\}$. We understand $L^p(\mathcal{A})$ as a subspace of $L^p(\Sigma)$ and as a Banach space. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. An atom of the measure μ is an element $A \in \Sigma$ with $\mu(A) > 0$ such that for each $F \in \Sigma$, if $F \subset A$ then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. It is easy to see that every $\mathcal{A}$- measurable function $f \in L(X)$ is constant μ- almost everywhere on an atom A. So for each $f \in L(X)$

and each atom A we have $\int_A f\,d\mu = f(A)\mu(A)$. A measure with no atoms is called non-atomic.

Associated with each σ-algebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E(\cdot|\mathcal{A}) = E^{\mathcal{A}}(\cdot)$ on the set of all non-negative measurable functions f or on the set of all functions $f \in L^p(\Sigma)$, $1 \le p \le \infty$, that is uniquely determined by the conditions

(i) $E^{\mathcal{A}}(f)$ is $\mathcal{A}$- measurable, and

(ii) if A is any $\mathcal{A}$- measurable set for which $\int_A f\,d\mu$ exists, we have $\int_A f\,d\mu = \int_A E^{\mathcal{A}}(f)\,d\mu$.

The operator $E^{\mathcal{A}}$ is called conditional expectation operator. This operator is at the central idea of our work, and we list here some of its useful properties:

E1. $E^{\mathcal{A}}(f.g \circ T) = E^{\mathcal{A}}(f)(g \circ T)$.

E2. $E^{\mathcal{A}}(1) = 1$.

E3. $|E^{\mathcal{A}}(fg)|^2 \le E^{\mathcal{A}}(|f|^2)E^{\mathcal{A}}(|g|^2)$.

E4. If $f > 0$, then $E^{\mathcal{A}}(f) > 0$.

Properties E1 and E2 imply that $E^{\mathcal{A}}(\cdot)$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. So when $\mathcal{A} = \Sigma$, we have $E^{\Sigma} = I$ where I is identity operator. Suppose that φ is a mapping from X into X which is measurable, ($i.e.$, $\varphi^{-1}(\Sigma) \subseteq \Sigma$) and $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ ($\mu \circ \varphi^{-1} \ll \mu$). Let h be the Radon-Nikodym derivative, $h = \frac{d\mu \circ \varphi^{-1}}{d\mu}$. If we put $\mathcal{A} = \varphi^{-1}(\Sigma)$, it is easy to show that for each non-negative Σ-measurable function f or for each $f \in L^p(\Sigma)$ ($p \ge 1$), there exists a Σ-measurable function g such that $E^{\varphi^{-1}(\Sigma)}(f) = g \circ \varphi$. We can assume that the support of g lies in the support of h, and there exists only one g with this property. We then write $g = E^{\varphi^{-1}(\Sigma)}(f) \circ \varphi^{-1}$, though we make no assumption regarding the invertibility of φ (see [1]). For a deeper study of the properties of E see the paper [5].

Take a function u in $L(X)$ and let $\varphi : X \to X$ be a non-singular measurable transformation; $i.e.$, $\mu(\varphi^{-1}(A)) = 0$ for all $A \in \Sigma$ such that $\mu(A) = 0$. Then the pair (u, φ) induces a linear operator uC_φ from $L^p(\Sigma)$ into $L(X)$ defined by

$$uC_\varphi(f) = u.f \circ \varphi \qquad (f \in L^p(\Sigma)).$$

Here, the non-singularity of φ guarantees that uC_φ as a mapping of equivalence classes of functions on support u is well defined. If uC_φ takes $L^p(\Sigma)$ into $L^q(\Sigma)$ or uC_φ is equivalently bounded, then we say that uC_φ is a weighted composition operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ $(1 \le p, q \le \infty)$. When $u \equiv 1$, we just have the composition operator C_φ defined by $C_\varphi(f) = f \circ \varphi$. For more details see [6].

2 Fredholm weighted composition operators on L^p-spaces

Let $1 \le p < \infty$, $1 \le q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then it is a well known fact that each $g^* \in L^q(\Sigma)$ defines a bounded linear functional F_{g^*} on $L^p(\Sigma)$ by

$$F_{g^*}(f) = \int f g^* \, d\mu \qquad (f \in L^p(\Sigma)).$$

Moreover, the mapping $g^* \to F_{g^*}$ is an isometry from $L^q(\Sigma)$ onto $(L^p)^*(\Sigma)$, so the norm dual of $L^p(\Sigma)$ can be identified with $L^q(\Sigma)$. In the following theorem we compute the adjoint of uC_φ.

Proposition 1 *Let $W = uC_\varphi$ be a weighted composition operator on $L^p(\Sigma)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $W^* g^* = hE(u.g^*) \circ \varphi^{-1}$ for all $g^* \in L^q(\Sigma)$.*

Proof. Take $A \in \Sigma$ such that $0 < \mu(A) < \infty$. For $g^* \in L^q(\Sigma)$ consider a bounded linear functional F_{g^*} on $L^p(\Sigma)$ as above. Then we have

$$(W^* F_{g^*})(\chi_A) = F_{g^*}(W\chi_A) = \int (W\chi_A) g^* \, d\mu$$

$$= \int u.\chi_A \circ \varphi \; g^* \, d\mu = \int hE(u.g^*) \circ \varphi^{-1} \chi_A \, d\mu = F_{hE(u.g^*)\circ\varphi^{-1}} \chi_A.$$

Hence, $W^* F_{g^*} = F_{hE(u.g^*)\circ\varphi^{-1}}$. After identifying $(L^p)^*(\Sigma)$ with $L^q(\Sigma)$ and g^* with F_{g^*}, we can write $W^* g^* = hE(u.g^*) \circ \varphi^{-1}$ for all $g^* \in L^q(\Sigma)$. □

In the following theorem we investigate a necessary and sufficient condition for a weighted composition operator $W = uC_\varphi$ to be Fredholm. The proof of the theorem follows a similar method of proof as was used to prove Theorem 4.2 in [4] which is similar to a theorem of Takagi [7]. We use the symbols $\mathcal{N}(W)$ and $\mathcal{R}(W)$ to denote the kernel and the range of W, respectively. We recall that W is said to be a Fredholm operator if $\mathcal{R}(W)$ is closed and if $\dim \mathcal{N}(W) < \infty$ and $\operatorname{codim} \mathcal{R}(W) < \infty$.

Theorem 2 *Suppose that μ is a non-atomic measure. Let $W = uC_\varphi$ be a weighted composition operator on $L^p(\Sigma)$. Then W is a Fredholm operator if and only if $J = hE^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1} \ge \delta$ almost everywhere on X for some $\delta > 0$.*

Proof. Suppose that W is a Fredholm operator. We first claim that W is onto and takes an $f_o \in L^p(\Sigma) \setminus \mathcal{R}(W)$. Since $\mathcal{R}(W)$ is closed, we can find a functional L_{g^*} on $L^p(\Sigma)$ corresponding to $g^* \in L^q(\Sigma)$ ($\frac{1}{p} + \frac{1}{q} = 1$) which is defined as

$$L_{g^*}(f) = \int_X f g^* \, d\mu \quad \text{such that} \quad L_{g^*}(f_0) = 1 \quad \text{and} \quad L_{g^*}(\mathcal{R}(W)) = 0. \qquad (1)$$

Hence the set $E_\delta = \{x \in X : \ \mathrm{Re}\,(f_0 g^*)(x) \geq \delta\}$ must have positive measure for some $\delta > 0$. Since μ is non-atomic we can choose a sequence $\{E_n\}$ of subsets of E_δ with $0 < \mu(E_n) < \mu(E_\delta)$ and $E_n \cap E_m = \emptyset$ for $n \neq m$. Let $g_n^* = \chi_{E_n} g^*$. Then $g_n^* \in L^q(\Sigma)$ and is nonzero because

$$\mathrm{Re} \int_X f_0 g_n^* \, d\mu \geq \delta \mu(E_n) > 0 \ .$$

Evidently for any $f \in L^p(\Sigma)$, $\chi_{E_n} f$ is in $L^p(\Sigma)$, and so the right equality of (1) yields

$$\int_X f(W^* g_n^*) \, d\mu = \int_X f h E(u g_n^*) \circ \varphi^{-1} \, d\mu = \int_{E_n} f E(u g^*) \circ \varphi^{-1} \, d\mu \circ \varphi^{-1}$$
$$= \int_{\varphi^{-1}(E_n)} f \circ \varphi E(u g^*) \, d\mu = \int_{\varphi^{-1}(E_n)} u g^* f \circ \varphi \, d\mu = \int_X g^* u f \circ \varphi (\chi_{E_n} \circ \varphi) \, d\mu$$
$$= \int_X g^* u (f \chi_{E_n}) \circ \varphi \, d\mu = \int_X g^* W(f \chi_{E_n}) \, d\mu = 0.$$

This implies that $g_n^* \in \mathcal{N}(W^*)$. Thus the sequence $\{g_n^*\}$ forms a linearly independent subset of $\mathcal{N}(W^*)$. This contradicts the fact that $\dim \mathcal{N}(W^*) = \operatorname{codim} \mathcal{R}(W) < \infty$. Hence W is onto. Next we put $Z(J) = \{x : J(x) = 0\}$. Now we claim that $\mu(Z(J)) = 0$. For, if $\mu(Z(J)) > 0$, there exists a subset F of $Z(J)$ with $0 < \mu(F) < \infty$. If $\chi_F \in \mathcal{R}(W)$, then there exists $f \in L^p(\Sigma)$ such that $\chi_F = Wf$. Then

$$\mu(F) = \int_F |Wf|^p \, d\mu \int_F J|f|^p \, d\mu = 0$$

and this is a contradiction. So $\chi_F \in L^p(\Sigma) \setminus \mathcal{R}(W)$, which contradicts the fact that W is onto. Also since $\mu(Z(J)) = 0$ and $\mu \circ \varphi^{-1} \ll \mu$ we have $\mu(Z(J \circ \varphi)) = 0$. For each $n = 1, 2, \ldots$ let

$$H_n = \left\{ x \in X : \frac{\|J \circ \varphi\|_\infty}{(n+1)^2} < J \circ \varphi(x) \leq \frac{\|J \circ \varphi\|_\infty}{n^2} \right\},$$

and $H = \{n : \mu(H_n) > 0\}$. Then the H_n's are pairwise disjoint and $X = \cup_{n=1}^\infty H_n$. Define

$$f(x) = \begin{cases} \left(\frac{J \circ \varphi(x)}{\mu(H_n)} \right)^{\frac{1}{p}} & \text{if } \ x \in H_n, n \in H, \\ 0 & \text{elsewhere.} \end{cases}$$

Then

$$\int_X |f|^p \, d\mu = \sum_{n \in H} \int_{H_n} \frac{J \circ \varphi(x)}{\mu(H_n)} \, d\mu \leq \sum_{n \in H} \frac{\|J \circ \varphi\|_\infty}{n^2} \leq \|J \circ \varphi\|_\infty \sum_{n=1}^\infty \frac{1}{n^2} < \infty,$$

so $f \in L^p(\Sigma)$. If $g \in L^p(\Sigma)$ is such that $Wg = f$, then

$$\int_X E^{\varphi^{-1}(\Sigma)}(|u|^p)|g|^p \circ \varphi \, d\mu = \int_X E^{\varphi^{-1}(\Sigma)}(|f|^p) \, d\mu.$$

It follows that

$$\int_X hE^{\varphi^{-1}(\Sigma)}(|u|^p) \circ \varphi^{-1}|g|^p \, d\mu = \int_X hE^{\varphi^{-1}(\Sigma)}(|f|^p) \circ \varphi^{-1} \, d\mu.$$

Thus $|g|^p = \frac{hE^{\varphi^{-1}(\Sigma)}(|f|^p)\circ\varphi^{-1}}{J}$ on off $Z(J)$. Since $\mu(Z(J)) = 0$, it follows that

$$\int_X |g|^p \, d\mu = \int_X \frac{E^{\varphi^{-1}(\Sigma)}(|f|^p) \circ \varphi^{-1}}{J} \, d\mu \circ \varphi^{-1} = \int_X \frac{E^{\varphi^{-1}(\Sigma)}(|f|^p)}{J \circ \varphi} \, d\mu$$

$$= \int_X \frac{|f|^p}{J \circ \varphi} \, d\mu = \sum_{n \in H} \int_{H_n} \frac{d\mu}{\mu(H_n)} = \sum_{n \in H} 1.$$

This implies that H must be a finite set. Thus there is an n_0 such that $n \geq n_0$ implies $\mu(H_n) = 0$ and so

$$\mu\left(\left\{x \in X : J \circ \varphi(x) \leq \frac{\|J \circ \varphi\|_\infty}{n_0^2}\right\}\right) = \mu\left(\bigcup_{n=n_0}^{\infty} H_n \cup Z(J \circ \varphi)\right) = 0.$$

Therefore we obtain $J \circ \varphi \geq \frac{\|J\circ\varphi\|_\infty}{n_0^2}$ almost everywhere on X. Since $\mathcal{N}(W) = L^p(Z(J))$, $\mu(Z(J)) = 0$, so dim $\mathcal{N}(W) = \{0\}$ and then φ is essentially surjective. Hence $J \geq \frac{\|J\|_\infty}{n_0^2}$ $(= \delta)$ almost everywhere on X.

Conversely, suppose that $J \geq \delta$ almost everywhere for some $\delta > 0$. Since $h > 0$ and for each $f \in L^p(\Sigma)$, $\|Wf\|_p = (\int_X J|f|^p \, d\mu)^{1/P} \geq \delta^{1/p}\|f\|_p$, it follows that W and C_φ are injective and $\mathcal{R}(W)$ is closed. Also since $W = M_u C_\varphi$, we deduce that M_u is injective and so $\mu(Z(u)) = 0$. Now let $g^* \in \mathcal{N}(W^*)$. Then $W^* g^* = hE^{\varphi^{-1}(\Sigma)}(ug^*) \circ \varphi^{-1} = 0$ and so $E^{\varphi^{-1}(\Sigma)}(ug^*) \circ \varphi^{-1} = 0$. It follows that $g^* = 0$. Thus codim $\mathcal{R}(W) = \dim \mathcal{N}(W^*) = 0$. Therefore the theorem is proved. □

Corollary 3 *Suppose M_u and C_φ are both bounded linear operators on $L^p(\Sigma)$ and μ is a non-atomic measure. Then*

(i) M_u is Fredholm if and only if $|u| \geq \delta$ on X for some $\delta > 0$.

(ii) C_φ is Fredholm if and only if $h \geq \delta$ on X for some $\delta > 0$.

Remark 4 One of the interesting features of a weighted composition operator is that the multiplication operator alone may not define a bounded operator between two $L^p(\Sigma)$ spaces. As an example, let X be $(0,1)$, Σ be the Borel sets, and μ be the Lebesgue measure. Let φ be the map $\varphi(x) = \sqrt[3]{x}$ and $u(x) = 1/\sqrt{x}$ on $(0,1)$. Then M_u dos not define a bounded operator from $L^1(\Sigma)$ into $L^1(\Sigma)$. However a simple computation shows that $J(x) = 3\sqrt{x} \in L^\infty(\Sigma)$ and so $Wf(x) = 1/\sqrt{x}f(\sqrt[3]{x})$ is bounded operator on $L^1(\Sigma)$.

References

[1] Campbell J. and Jamison J., *On some classes of weighted composition operators*, Glasgow Math. J., **32** (1990), 87–94.

[2] Hoover T., Lambert A. and Quinn J., *The Markov process determined by a weighted composition operator*, Studia Math. (Poland), **LXXII** (1982), 225–235.

[3] Jabbarzadeh M. R. and Pourreza E., *A note on weighted composition operators on L^p-spaces*, Bull. Iranian Math. Soc., **29** (2002), 1–8.

[4] Komal B. S. and Gupta S., *Multiplication operators between Orlicz spaces*, Integr. Eqns. Oper. Theory, **41** (2001), 324–330.

[5] Lambert A., *Localising sets for sigma-algebras and related point transformations*, Proc. Royal Soc. of Edinburgh Ser. A, **118** (1991), 111–118.

[6] Singh R. K. and Manhas J. S., *Composition Operators on Function Spaces*, North-Holland, 1993.

[7] Takagi H., *Fredholm weighted composition operators*, Integr. Eqns. Oper. Theory, **16** (1993), 267–276.

[8] Takagi H. and Yokouchi K., *Multiplication and composition operators between two L^p-spaces*, Contemporary Math., **232** (1999), 321–338.

[9] Taylor A. E. and Lay D. C., *Introduction to Functional Analysis*, 2nd ed., Wiley, 1980.

[10] Zaanen A. C., *Integration*, 2nd ed., North-Holland, Amsterdam, 1967.