# Properties of Solutions of the Matrix Equation $X+A^{*} X^{-2^{k}} A=Q^{*}$ 

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#### Abstract

The positive definite solutions of the nonlinear matrix equations $X+A^{*} X^{-n} A=Q$ are investigated. We extend and improve the results to the equation $X+A^{*} X^{-2^{k}} A=I$ proved in [4]. We consider an iterative method which defines different matrix sequences depending on an initial point. The new results are illustrated by numerical examples.


AMS Subject Classifications: 15A24, 15A45, 65F35.
Keywords: Nonlinear matrix equation, Positive definite solution.

## 1 Introduction

Consider the nonlinear matrix equation

$$
\begin{equation*}
X+A^{*} X^{-n} A=Q \tag{1}
\end{equation*}
$$

where $Q$ is an $m \times m$ Hermitian positive definite matrix, $A$ is an $m \times m$ complex matrix and $n$ is a positive integer.

Many authors have considered more general nonlinear matrix equations $[1,5]$. We investigate a monotone iterative method for computing a positive definite solution of this equation under some restrictions on the matrices $A, Q$. In the case $n=2^{k}$ where $k$ is a positive integer we extend the results proved in $[2,3,4]$. In $[2,3]$ the equation $X+A^{*} X^{-2} A=I$ is considered and one iterative method with three different initial points is investigated. It is proved that the iterative method with every initial point leads to a positive definite solution which is denoted $X_{\tilde{\alpha}}, X_{\gamma}, X_{\tilde{\beta}}$. In [4] the equation $X+A^{*} X^{-2^{k}} A=I$ is considered. The authors have proved that there exists a positive definite solution.

[^0]In this paper we consider the equation (1) assuming $A$ is nonsingular and the corresponding matrix sequence

$$
X_{s+1}=\sqrt[n]{A\left(Q-X_{s}\right)^{-1} A^{*}}, \quad s=0,1,2, \ldots
$$

where $X_{0}$ is suitably chosen. For a different initial point $X_{0}$ we obtain three different matrix sequences $\left\{X_{s}\right\}$. Two of these sequences are monotonic and convergent to a positive definite solution of (1). Under additional restrictions on the matrices $A$ and $Q$ the third sequence converges to a positive definite solution of the equation $X+A^{*} X^{-2^{k}} A=Q$. The rate of convergence of these methods depends on two parameters. Numerical examples are discussed and some results of the experiments are given.

We start with some notations which we use throughout this paper. We shall use $\|A\|$ to denote the spectral norm of the matrix $A$, and $\sigma_{1}(Q), \sigma_{m}(Q)$ denote the biggest and the smallest singular value of $Q$, and $\lambda_{i}\left(A A^{*}\right)$ denotes the corresponding eigenvalue of $A A^{*}$. Let the matrices $R$ and $S$ be Hermitian. The notation $R>$ $S(R \geq S)$ means that $R-S$ is positive definite (semidefinite). The assumption $R \geq S>0$ implies $R^{-1} \leq S^{-1}$ and $\sqrt[l]{R} \geq \sqrt[l]{S}$, where $l$ is a positive integer.

The equation $X+A^{*} X^{-2} A=I$ and the matrix sequence

$$
\begin{equation*}
X_{0}=\gamma I, \quad X_{s+1}=\sqrt{A\left(I-X_{s}\right)^{-1} A^{*}}, \quad s=0,1,2, \ldots, \tag{2}
\end{equation*}
$$

are considered in [3]. The following theorem is proved.

Theorem 1 (Theorem $3[3]$, 2001) Let $\tilde{\alpha}$ and $\tilde{\beta}$ be solutions of the scalar equations $\tilde{\alpha}^{2}(1-\tilde{\alpha})=\min \lambda_{i}\left(A A^{*}\right)$ and $\tilde{\beta}^{2}(1-\tilde{\beta})=\max \lambda_{i}\left(A A^{*}\right)$, respectively. Assume $0<\tilde{\alpha} \leq \tilde{\beta} \leq \frac{2}{3}$. Consider $\left\{X_{s}\right\}$ defined by (2). Then
(i) If $\gamma \in[0, \tilde{\alpha}]$, then $\left\{X_{s}\right\}$ is monotonically increasing and converges to a positive definite solution $X_{\tilde{\alpha}}$;
(ii) If $\gamma \in\left[\tilde{\beta}, \frac{2}{3}\right]$, then $\left\{X_{s}\right\}$ is monotonically decreasing and converges to a positive definite solution $X_{\tilde{\beta}}$;
(iii) If $\gamma \in(\tilde{\alpha}, \tilde{\beta})$ and $\frac{\tilde{\beta}^{2}}{2 \tilde{\alpha}(1-\tilde{\beta})}<1$, then $\left\{X_{s}\right\}$ converges to a positive definite solution $X_{\gamma}$.

Later, the equation $X+A^{*} X^{-2^{k}} A=I$ where $k$ is integer was considered by the authors El-Sayed and El-Alem [4]. They have proved the following theorems.

Theorem 2 (Theorem 4 [4], 2002) If there exist numbers $\alpha, \beta$ satisfying $0<\alpha<$ $\beta<1$ and the inequalities

$$
\alpha^{2^{k}}(1-\alpha) I<A A^{*}<\beta^{2^{k}}(1-\beta) I
$$

hold, then the equation $X+A^{*} X^{-2^{k}} A=I$ has a positive definite solution.
Theorem 3 (Theorem 5 [4], 2002) If there exist numbers $\alpha, \beta$ satisfying $0<\alpha<$ $\beta<1$ and the following conditions
(i) $\alpha^{2^{k}}(1-\alpha) I<A A^{*}<\beta^{2^{k}}(1-\beta) I$,
(ii) $q(\alpha, \beta)=\frac{\beta^{2}}{2^{k} \alpha^{k}(1-\beta)}<1$
hold, then

$$
\left\|X_{s}-X\right\| \leq[q(\alpha, \beta)]^{s}\left\|X_{0}-X\right\| \leq[q(\alpha, \beta)]^{s}(\beta-\alpha),
$$

where $X$ is a positive definite solution of $X+A^{*} X^{-2^{k}} A=I$ and $X_{s}$ is defined by

$$
\begin{equation*}
X_{0}=\beta I, \quad X_{s+1}=\sqrt[2^{k}]{A\left(I-X_{s}\right)^{-1} A^{*}}, \quad s=0,1,2, \ldots . \tag{3}
\end{equation*}
$$

## 2 Properties and convergence of matrix sequences

Consider the iterative method

$$
\begin{equation*}
X_{0}=\gamma Q, \quad X_{s+1}=\sqrt[n]{A\left(Q-X_{s}\right)^{-1} A^{*}}, \quad s=0,1,2, \ldots \tag{4}
\end{equation*}
$$

Theorem 4 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be solutions of the scalar equations

$$
\alpha^{n}(1-\alpha)=\sigma_{m}^{2}\left(Q^{-n / 2} A Q^{-1 / 2}\right) \quad \text { and } \quad \beta^{n}(1-\beta)=\sigma_{1}^{2}\left(Q^{-n / 2} A Q^{-1 / 2}\right),
$$

respectively. Assume $0<\tilde{\alpha} \leq \tilde{\beta} \leq \frac{n}{n+1}$. Consider $\left\{X_{s}\right\}$ defined by (4). Then
(i) If $X_{0}=\gamma Q$ and $\gamma \in[0, \tilde{\alpha}]$, then $\left\{X_{s}\right\}$ is monotonically increasing and converges to a positive definite solution $X_{\tilde{\alpha}}$ with $X_{\tilde{\alpha}} \leq \tilde{\beta} Q$;
(ii) If $X_{0}=\gamma Q$ and $\gamma \in\left[\tilde{\beta}, \frac{n}{n+1}\right]$, then $\left\{X_{s}\right\}$ is monotonically decreasing and converges to a positive definite solution $X_{\tilde{\beta}}$ with $\frac{n}{n+1} Q \geq X_{\tilde{\beta}} \geq \tilde{\alpha} Q$;
(iii) If $\gamma \in(\tilde{\alpha}, \tilde{\beta}), n=2^{k}$, where $k$ is an arbitrary integer and

$$
r(\tilde{\alpha}, \tilde{\beta})=\left(\frac{\left\|Q^{-1}\right\|}{2 \tilde{\alpha}}\right)^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\tilde{\beta}}\right)^{2}<1
$$

then $\left\{X_{s}\right\}$ converges to a positive definite solution $X_{\gamma}$.

Proof. Since the function $\varphi(x)=x^{n}(1-x)$ is monotonically increasing, where $x \in\left[0, \frac{n}{n+1}\right]$, we have $0<\alpha \leq \tilde{\alpha} \leq \tilde{\beta} \leq \beta \leq \frac{n}{n+1}$ and the inequalities

$$
\begin{gathered}
\alpha^{n}(1-\alpha) I \leq Q^{-n / 2} A Q^{-1} A^{*} Q^{-n / 2} \leq \beta^{n}(1-\beta) I, \\
\alpha^{n}(1-\alpha) Q^{n} \leq A Q^{-1} A^{*} \leq \beta^{n}(1-\beta) Q^{n}
\end{gathered}
$$

are satisfied.
(i) We have $X_{0}=\gamma Q \leq \tilde{\beta} Q$ and $\gamma \in[0, \tilde{\alpha}]$. Thus

$$
\begin{gathered}
X_{1}=\sqrt[n]{A(Q-\gamma Q)^{-1} A^{*}}=\sqrt[n]{\frac{A Q^{-1} A^{*}}{1-\gamma}}, \\
X_{1} \leq \sqrt[n]{\frac{1}{1-\gamma} \tilde{\beta}^{n}(1-\tilde{\beta}) Q^{n}} \leq \tilde{\beta} Q \\
X_{1}=\sqrt[n]{\frac{A Q^{-1} A^{*}}{1-\gamma}} \geq \sqrt[n]{\frac{\tilde{\alpha}^{n}(1-\tilde{\alpha})}{1-\gamma} Q^{n}} \geq \gamma Q=X_{0} .
\end{gathered}
$$

We have $X_{0} \leq X_{1} \leq \tilde{\beta} Q$. We assume

$$
X_{s-1} \leq X_{s} \leq \tilde{\beta} Q .
$$

Hence

$$
\begin{gathered}
\left(Q-X_{s-1}\right)^{-1} \leq\left(Q-X_{s}\right)^{-1} \leq(Q-\tilde{\beta} Q)^{-1}=\frac{1}{1-\tilde{\beta}} Q^{-1}, \\
\sqrt[n]{A\left(Q-X_{s-1}\right)^{-1} A^{*}} \leq \sqrt[n]{A\left(Q-X_{s}\right)^{-1} A^{*}} \leq \sqrt[n]{\frac{1}{1-\tilde{\beta}} A Q^{-1} A^{*}}, \\
X_{s} \leq X_{s+1} \leq \sqrt[n]{\frac{1}{1-\tilde{\beta}} \tilde{\beta}^{n}(1-\tilde{\beta}) Q^{n}}=\tilde{\beta} Q .
\end{gathered}
$$

The sequence $\left\{X_{s}\right\}$ is monotonically increasing and converges to a positive definite solution $X_{\tilde{\alpha}}$ with $X_{\tilde{\alpha}} \leq \tilde{\beta} Q$.
(ii) Let $\gamma \in\left[\tilde{\beta}, \frac{n}{n+1}\right]$. Hence $\frac{n}{n+1} Q \geq X_{0}=\gamma Q \geq \tilde{\alpha} Q$. We have

$$
\begin{aligned}
& X_{1}=\sqrt[n]{A(Q-\gamma Q)^{-1} A^{*}}=\sqrt[n]{\frac{A Q^{-1} A^{*}}{1-\gamma}} \\
& X_{1} \leq \sqrt[n]{\frac{1}{1-\gamma} \tilde{\beta}^{n}(1-\tilde{\beta}) Q^{n}} \leq \gamma Q=X_{0}
\end{aligned}
$$

$$
X_{1}=\sqrt[n]{\frac{A Q^{-1} A^{*}}{1-\gamma}} \geq \sqrt[n]{\frac{1}{1-\gamma} \tilde{\alpha}^{n}(1-\tilde{\alpha}) Q^{n}} \geq \tilde{\alpha} Q
$$

Thus $X_{0} \geq X_{1} \geq \tilde{\alpha} Q$. We assume

$$
\begin{gathered}
X_{s-1} \geq X_{s} \geq \tilde{\alpha} Q \\
\left(Q-X_{s-1}\right)^{-1} \geq\left(Q-X_{s}\right)^{-1} \geq(Q-\tilde{\alpha} Q)^{-1}=\frac{1}{1-\tilde{\alpha}} Q^{-1} \\
\sqrt[n]{A\left(Q-X_{s-1}\right)^{-1} A^{*}} \geq \sqrt[n]{A\left(Q-X_{s}\right)^{-1} A^{*}} \geq \sqrt[n]{\frac{A Q^{-1} A^{*}}{1-\tilde{\alpha}}} \\
X_{s} \geq X_{s+1} \geq \sqrt[n]{\frac{\tilde{\alpha}^{n}(1-\tilde{\alpha}) Q^{n}}{1-\tilde{\alpha}}}=\tilde{\alpha} Q
\end{gathered}
$$

The sequence $\left\{X_{s}\right\}$ is monotonically decreasing and converges to a positive definite solution $X_{\tilde{\beta}}$ with the property $\frac{n}{n+1} Q \geq X_{\tilde{\beta}} \geq \tilde{\alpha} Q$.
(iii) Assume $\gamma \in(\tilde{\alpha}, \tilde{\beta})$. Hence $\tilde{\alpha} Q<X_{0}=\gamma Q<\tilde{\beta} Q$. We will prove that $\left\{X_{s}\right\}$ is a Cauchy sequence. Following the proof of cases (i) and (ii) we obtain $\tilde{\alpha} Q<X_{s}<\tilde{\beta} Q$.

Consider the difference $X_{s+p}-X_{s}$ for which

$$
X_{s+p}-X_{s}=\sqrt[2^{k}]{A\left(Q-X_{s+p-1}\right)^{-1} A^{*}}-\sqrt[2^{k}]{A\left(Q-X_{s-1}\right)^{-1} A^{*}}
$$

We put $R=A\left(Q-X_{s+p-1}\right)^{-1} A^{*}$ and $S=A\left(Q-X_{s-1}\right)^{-1} A^{*}$ and use the identity

$$
\sqrt[2^{k}]{R}(\sqrt[2^{k}]{R}-\sqrt[2^{k}]{S})+(\sqrt[2^{k}]{R}-\sqrt[2^{k}]{S}) \sqrt[2^{k}]{S}=\sqrt[2^{k-1}]{R}-\sqrt[2^{k-1}]{S}
$$

We have that $Y=\sqrt[2^{k}]{R}-\sqrt[2^{k}]{S}$ is a solution of the matrix equation

$$
\sqrt[2^{k}]{R} Y+Y \sqrt[2^{k}]{S}=\sqrt[2^{k-1}]{R}-\sqrt[2^{k-1}]{S}=C
$$

Thus the solution $Y$ can be expressed

$$
Y=\int_{0}^{\infty} e^{-\sqrt[2^{k}]{R} t} C e^{-\sqrt[2^{k}]{S} t} d t
$$

For the spectral norm of $X_{s+p}-X_{s}$ we have

$$
\left\|X_{s+p}-X_{s}\right\|=\|\sqrt[2^{k}]{R}-\sqrt[2^{k}]{S}\|=\|Y\| \leq \int_{0}^{\infty}\|C\|\left\|e^{-\sqrt[2^{k}]{R} t}\right\|\left\|e^{-\sqrt[2^{k}]{S} t}\right\| d t
$$

We know that $\tilde{\alpha} Q \leq X_{j} \leq \tilde{\beta} Q$ for $j=0,1, \ldots$ Hence

$$
\frac{1}{1-\tilde{\alpha}} Q^{-1} \leq\left(Q-X_{j}\right)^{-1} \leq \frac{1}{1-\tilde{\beta}} Q^{-1}
$$

and

$$
\begin{gathered}
\left\|\left(Q-X_{s+p-1}\right)^{-1}\right\| \leq \frac{1}{1-\tilde{\beta}}\left\|Q^{-1}\right\|, \quad\left\|\left(Q-X_{s-1}\right)^{-1}\right\| \leq \frac{1}{1-\tilde{\beta}}\left\|Q^{-1}\right\| \\
R=A\left(Q-X_{s+p-1}\right)^{-1} A^{*} \geq \frac{1}{1-\tilde{\alpha}} A Q^{-1} A^{*} \geq \tilde{\alpha}^{2^{k}} Q^{2^{k}} \geq \tilde{\alpha}^{2^{k}}\left(\lambda_{\min }(Q)\right)^{2^{k}} I \\
S=A\left(Q-X_{s+p-1}\right)^{-1} A^{*} \geq \tilde{\alpha}^{2^{k}}\left(\lambda_{\min }(Q)\right)^{2^{k}} I
\end{gathered}
$$

Thus

$$
\begin{aligned}
\left\|X_{s+p}-X_{s}\right\| & \leq\|C\| \int_{0}^{\infty}\left\|e^{-\tilde{\alpha} \lambda_{\min }(Q) I t}\right\|\left\|e^{-\tilde{\alpha} \lambda_{\min }(Q) I t}\right\| d t \\
& =\|C\| \frac{\left\|Q^{-1}\right\|}{2 \tilde{\alpha}}=\|C\| \delta
\end{aligned}
$$

So

$$
\begin{aligned}
\left\|X_{s+p}-X_{s}\right\|=\|\sqrt[2^{k}]{R}-\sqrt[2^{k}]{S}\| & \leq\|C\| \delta=\delta\|\sqrt[2^{k-1}]{R}-\sqrt[2^{k-1}]{S}\| \\
& \leq \delta^{2}\|\sqrt[2^{k-2}]{R}-\sqrt[2^{k-2}]{S}\| \leq \cdots \\
& \leq \delta^{k}\|R-S\|
\end{aligned}
$$

Consider

$$
\begin{aligned}
\|R-S\| & =\left\|A\left[\left(Q-X_{s+p-1}\right)^{-1}-\left(Q-X_{s-1}\right)^{-1}\right] A^{*}\right\| \\
& =\left\|A\left(Q-X_{s-1}\right)^{-1}\left(X_{s+p-1}-X_{s-1}\right)\left(Q-X_{s+p-1}\right)^{-1} A^{*}\right\| \\
& \leq\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\tilde{\beta}}\right)^{2}\left\|X_{s+p-1}-X_{s-1}\right\|
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\left\|X_{s+p}-X_{s}\right\| & \leq \delta^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\tilde{\beta}}\right)^{2}\left\|X_{s+p-1}-X_{s-1}\right\| \\
& \leq \cdots \\
& \leq\left[\left(\frac{\left\|Q^{-1}\right\|}{2 \tilde{\alpha}}\right)^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\tilde{\beta}}\right)^{2}\right]^{s}\left\|X_{p}-X_{0}\right\|
\end{aligned}
$$

Since

$$
r(\tilde{\alpha}, \tilde{\beta})=\left(\frac{\left\|Q^{-1}\right\|}{2 \tilde{\alpha}}\right)^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\tilde{\beta}}\right)^{2}<1,
$$

we have

$$
\begin{aligned}
\left\|X_{p}-X_{0}\right\| & \leq\left\|X_{p}-X_{p-1}\right\|+\left\|X_{p-1}-X_{p-2}\right\|+\cdots+\left\|X_{1}-X_{0}\right\| \\
& \leq\left([r(\tilde{\alpha}, \tilde{\beta})]^{p-1}+\cdots+1\right)\left\|X_{1}-X_{0}\right\| \\
& <\frac{1}{1-r(\tilde{\alpha}, \tilde{\beta})}\left\|X_{1}-X_{0}\right\| .
\end{aligned}
$$

Hence the sequence $\left\{X_{s}\right\}$ forms a Cauchy sequence considered in the Banach space $\mathcal{C}^{n \times n}$ (where $X_{s}$ are $n \times n$ positive definite matrices). Hence this sequence has a positive definite limit which is a positive definite solution of (1).

Theorem 5 Let $\alpha_{1}$ and $\beta_{1}$ be real for which the inequalities
(i) $\alpha_{1}^{n}\left(1-\alpha_{1}\right) Q^{n} \leq A Q^{-1} A^{*} \leq \beta_{1}^{n}\left(1-\beta_{1}\right) Q^{n}$,
(ii) $r\left(\alpha_{1}, \beta_{1}\right)=\left(\frac{\left\|Q^{-1}\right\|}{2 \alpha_{1}}\right)^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\beta_{1}}\right)^{2}<1$,
are satisfied.
For each two $\gamma_{1}$, $\gamma_{2}$ with $0<\alpha_{1} \leq \gamma_{1} \leq \gamma_{2} \leq \beta_{1} \leq \frac{2^{k}}{2^{k}+1}$ the recurrence equation (4) defines two matrix sequences $\left\{X_{s}^{\prime}\right\}$ and $\left\{X_{s}^{\prime \prime}\right\}$ with initial points $X_{0}^{\prime}=\gamma_{1} Q$ and $X_{0}^{\prime \prime}=\gamma_{2} Q$. These sequences converge to the same limit $X_{\gamma}$ which is a positive definite solution of (1).

Proof. We have $\alpha_{1} Q \leq X_{s}^{\prime} \leq \beta_{1} Q$ and $\alpha_{1} Q \leq X_{s}^{\prime \prime} \leq \beta_{1} Q$. We put $R=A\left(Q-X_{s-1}^{\prime}\right)^{-1} A^{*}$ and $S=A\left(Q-X_{s-1}^{\prime \prime}\right)^{-1} A^{*}$ and for $\left\|X_{s}^{\prime}-X_{s}^{\prime \prime}\right\|$ we obtain

$$
\begin{aligned}
\left\|X_{s}^{\prime}-X_{s}^{\prime \prime}\right\| & =\|\sqrt[2^{k}]{R}-\sqrt[2^{k}]{S}\| \\
& =\left\|\int_{0}^{\infty} e^{-\sqrt[2^{k}]{R} t}(\sqrt[2^{k-1}]{R}-\sqrt[2^{k-1}]{S}) e^{-\sqrt[2^{k}]{S} t} d t\right\| \\
& \leq\|\sqrt[2^{k-1}]{R}-\sqrt[2^{k-1}]{S}\| \int_{0}^{\infty} e^{-2 \alpha_{1} \lambda_{\min }(Q) t} d t \\
& \leq \frac{1}{2 \alpha_{1} \lambda_{\min }(Q)}\|\sqrt[2^{k-1}]{R}-\sqrt[2^{k-1}]{S}\|
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left(\frac{\left\|Q^{-1}\right\|}{2 \alpha_{1}}\right)^{k}\|R-S\| \\
& \leq\left(\frac{\left\|Q^{-1}\right\|}{2 \alpha_{1}}\right)^{k}\left\|A\left[\left(Q-X_{s-1}^{\prime}\right)^{-1}-\left(Q-X_{s-1}^{\prime \prime}\right)^{-1}\right] A^{*}\right\| \\
& =\left(\frac{\left\|Q^{-1}\right\|}{2 \alpha_{1}}\right)^{k}\left\|A\left(Q-X_{s-1}^{\prime \prime}\right)^{-1}\left(X_{s-1}^{\prime}-X_{s-1}^{\prime \prime}\right)\left(Q-X_{s-1}^{\prime}\right)^{-1} A^{*}\right\| \\
& \leq\left(\frac{\left\|Q^{-1}\right\|}{2 \alpha_{1}}\right)^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\beta_{1}}\right)^{2}\left\|X_{s-1}^{\prime}-X_{s-1}^{\prime \prime}\right\|
\end{aligned}
$$

But

$$
\left(\frac{\left\|Q^{-1}\right\|}{2 \alpha_{1}}\right)^{k}\left(\frac{\|A\|\left\|Q^{-1}\right\|}{1-\beta_{1}}\right)^{2}<1
$$

Consequently, the sequences $\left\{X_{k}^{\prime}\right\}$ and $\left\{X_{k}^{\prime \prime}\right\}$ have a common limit.

## 3 Numerical Experiments

We have made numerical experiments to compute a positive definite solution of the equation $X+A^{*} X^{-2^{k}} A=Q$. The solution was computed for different matrices $A, Q$ and different values of $m$. Computations were done on a PENTIUM IV, 2.1 GHz computer. All programs were written in MATLAB. We denote

$$
\varepsilon(Z)=\left\|Z+A^{*} Z^{-2^{k}} A-Q\right\|_{\infty}
$$

We have tested our iteration processes for solving the equation $X+A^{*} X^{-4} A=$ $Q(k=2)$ on the following $m \times m$ matrices. We use the stopping criterion $\varepsilon(Z)<$ tol $(t o l=1.0 e-15)$ and let $s_{X_{0}}$ be the smallest number $s$ for which $\varepsilon\left(X_{s}\right)<t o l$ for the method (3) and (4) with an initial point $X_{0}$.

Example 1 (Example 1 [4]) Consider the equation $X+A^{*} X^{-4} A=Q(k=2)$ with $Q=I$ and

$$
A=\operatorname{diag}\left[\frac{1}{1+8 m}, \frac{2}{2+8 m}, \cdots, \frac{m}{m+8 m}\right] .
$$

The results are given in Table 1. We apply Theorem 3 for computing a positive definite solution of $X+A^{*} X^{-4} A=I$ for different $m$. We compute $q(\tilde{\alpha}, \beta)$ for $\tilde{\alpha}$ such that $\tilde{\alpha}^{2 k}(1-\tilde{\alpha})=\sigma_{m}^{2}(A)$ and $\beta=0.477$. For computing a positive definite solution we use the recurrence equation (3) with $X_{0}=\beta I$ and $X_{0}=\tilde{\beta} I$, where
$\tilde{\beta}^{2 k}(1-\tilde{\beta})=\sigma_{1}^{2}(A)<\frac{2^{k}}{2^{k}+1}$. Further, we apply Theorem 4 for finding a positive definite solution of the same equation with $X_{0}=\tilde{\alpha} I$. According to Theorem 5 we compute $r\left(\alpha_{1}, \beta_{1}\right)$.

For the case $m=5$ we obtain: the matrix sequence (3) with $X_{0}=\beta I$ converges to $X_{\beta}$; the matrix sequence (3) with $X_{0}=\tilde{\beta} I$ converges to $X_{\tilde{\beta}}$ and the monotone matrix sequence (4) with $X_{0}=\tilde{\alpha} I$ converges to $X_{\tilde{\alpha}}$. Calculating $r\left(\alpha_{1}, \beta_{1}\right)$ we conclude that $X_{\beta} \equiv X_{\tilde{\beta}} \equiv X_{\tilde{\alpha}}$ because $r\left(\alpha_{1}, \beta_{1}\right)=0.5695<1\left(\alpha_{1}=\tilde{\alpha}, \beta_{1}=\right.$ $\beta$ ). Thus, all matrix sequences converge to the same positive definite solution of $X+A^{*} X^{-4} A=I$.

For the case $m=15$ we obtain: the matrix sequence (3) with $X_{0}=\beta I$ does not converge because $q(\tilde{\alpha}, \beta)=2.8527>1$; the matrix sequence (3) with $X_{0}=\tilde{\beta} I$ converges to $X_{\tilde{\beta}}$ and the monotone matrix sequence (4) with $X_{0}=\tilde{\alpha} I$ converges to $X_{\tilde{\alpha}}$. Since $r\left(\alpha_{1}, \beta_{1}\right)=0.0137<1\left(\alpha_{1}=\tilde{\alpha}, \beta_{1}=\tilde{\beta}\right)$, we obtain that $X_{\tilde{\beta}} \equiv X_{\tilde{\alpha}}$.

Table 1. Iterative method (3) with different initial points, $t o l=1.0 e-15$.

| $m$ | $q(\alpha, \beta)$ | initial point | number of <br> iterations | error |
| :---: | :---: | :---: | :---: | :---: |
| 5 | $q(\tilde{\alpha}, \beta)=0.9285$ | $\beta=0.477$ | $s_{X_{\beta}}=18$ | $\varepsilon\left(X_{\beta}\right)=9.9920 e-16$ |
|  | $q(\tilde{\alpha}, \tilde{\beta})=0.2947$ | $\tilde{\beta}=0.3745$ | $s_{X_{\tilde{\beta}}}=17$ | $\varepsilon\left(X_{\tilde{\beta}}\right)=3.3307 e-16$ |
|  |  | $\tilde{\alpha}=0.1633$ | $s_{X_{\tilde{\alpha}}}=19$ | $\varepsilon\left(X_{\tilde{\alpha}}\right)=3.3307 e-16$ |
| 15 | $q(\tilde{\alpha}, \beta)=2.8527$ | $\beta=0.4770$ | $*$ | $*$ |
|  | $q(\tilde{\alpha}, \tilde{\beta})=0.9055$ | $\tilde{\beta}=0.3745$ | $s_{X_{\tilde{\beta}}}=17$ | $\varepsilon\left(X_{\tilde{\beta}}\right)=5.5511 e-16$ |
|  |  | $\tilde{\alpha}=0.0932$ | $s_{X_{\tilde{\alpha}}}=19$ | $\varepsilon\left(X_{\tilde{\alpha}}\right)=5.5511 e-16$ |

Example 2 Consider the equation $X+A^{*} X^{-4} A=Q$ with $Q=I, m=25$ and the elements $a_{i j}=\frac{b_{i j}}{6}$ of the matrix $A$ are computed by

$$
b_{i j}= \begin{cases}\frac{j-i-2^{k}}{900} & \text { if } i<j, \\ \frac{i+j-2^{k}}{900} & \text { if } i>j, \\ \frac{i+j+2^{k}}{m 2^{k}+400} & \text { if } i=j\end{cases}
$$

The results are given in Table 2. For this example we obtain: the matrix sequence (3) with $X_{0}=\beta I$ does not converge because $q(\tilde{\alpha}, \beta)=6.09 e+03>1$;
the matrix sequence (3) with $X_{0}=\tilde{\beta} I$ converges to $X_{\tilde{\beta}}$ and the monotone matrix sequence (4) with $X_{0}=\tilde{\alpha} I$ converges to $X_{\tilde{\alpha}}$. Since $r\left(\alpha_{1}, \beta_{1}\right)=5.1239 e-16<$ $1\left(\alpha_{1}=\tilde{\alpha}, \beta_{1}=\tilde{\beta}\right)$, we obtain that $X_{\tilde{\beta}} \equiv X_{\tilde{\alpha}}$.

Table 2. Iterative method (3) with different initial points, $t o l=1.0 e-14$.

| $m$ | $q(\alpha, \beta)$ <br> Theorem 3 | initial point | number of <br> iterations | error |
| :---: | :---: | :---: | :---: | :---: |
| 25 | $q(\tilde{\alpha}, \beta)=6.09 e+03$ | $\beta=0.9927$ | $*$ | $*$ |
|  | $q(\tilde{\alpha}, \tilde{\beta})=0.7034$ | $\tilde{\beta}=0.3196$ | $s_{X_{\tilde{\beta}}}=13$ | $\varepsilon\left(X_{\tilde{\beta}}\right)=2.9126 e-14$ |
|  |  | $\tilde{\alpha}=0.0738$ | $s_{X_{\tilde{\alpha}}}=13$ | $\varepsilon\left(X_{\tilde{\alpha}}\right)=9.2511 e-14$ |

Example 3 Consider the equation $X+A^{*} X^{-4} A=Q$ and the elements $a_{i j}$ of the matrix $A$ are computed by

$$
a_{i j}= \begin{cases}\frac{i+j-m}{420} & \text { if } i<j, \\ \frac{j-i}{420} & \text { if } i>j, \\ \frac{i+j-m / 2}{m 2^{k} 2^{k}} & \text { if } i=j .\end{cases}
$$

We define the $m \times m$ matrix $Q$ as follows

$$
Q=U^{*} \operatorname{diag}\left[1.2+(-1)^{1} \frac{1}{20 m}, 1.2+(-1)^{2} \frac{2}{20 m}, \cdots, 1.2+(-1)^{m} \frac{m}{20 m}\right] U
$$

where $U=I-2 v^{\prime} v$ and $v=\left(\frac{1}{\sqrt{m}}, \cdots, \frac{1}{\sqrt{m}}\right)$.
We use the iterative method (4) for computing a positive definite solution with different initial points. The results are given in Table 3. The convergence rate is $r(\tilde{\alpha}, \tilde{\beta})=0.8388$ (Theorem 4, (iii)). According to Theorem 5 we have $q\left(\alpha_{1}, \beta_{1}\right)=$ $0.8388<1\left(\alpha_{1}=\tilde{\alpha}, \beta_{1}=\tilde{\beta}\right)$ and thus $X_{\tilde{\alpha}} \equiv X_{\tilde{\beta}} \equiv X_{\gamma}$.

Table 3. Iterative method (4) with different initial points, $t o l=1.0 e-14$.

| $m$ | initial point | number of <br> iterations | error |
| :---: | :---: | :---: | :---: |
| 12 | $X_{0}=\tilde{\alpha} Q, \tilde{\alpha}=0.0999$ | $s_{X_{\tilde{\alpha}}}=38$ | $\varepsilon\left(X_{\tilde{\alpha}}\right)=9.7647 e-15$ |
|  | $X_{0}=\tilde{\beta} Q, \tilde{\beta}=0.3525$ | $s_{X_{\tilde{\beta}}}=69$ | $\varepsilon\left(X_{\tilde{\beta}}\right)=9.5893 e-15$ |
|  | $X_{0}=\gamma Q, \gamma=\frac{\tilde{\alpha}+\tilde{\beta}}{2}$ | $s_{X_{\gamma}}=35$ | $\varepsilon\left(X_{\gamma}\right)=9.2079 e-15$ |

## 4 Conclusion

We consider the nonlinear matrix equation $X+A^{*} X^{-n} A=Q$. We improve the results proved by El-Sayed and El-Alem [4]. First, we consider this equation with a right-hand Hermitian positive definite matrix $Q$ while they have considered the case $Q=I$. Second, they have proved that the iterative method (3) where $0<\beta<1$ (Theorems 2,3) converges to a positive definite solution. Here we propose to choose $X_{0}=\tilde{\beta} Q$ (Theorem 4) where $\tilde{\beta}$ is a solution of a special scalar equation and $\tilde{\beta}<\frac{n}{n+1}$. Examples 1 and 2 show that the choice $X_{0}=\tilde{\beta} Q$ instead of $X_{0}=\beta I$ leads to a smaller rate of convergence.

## References

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