5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications, Proceedings, pp. 339–360

A Difference-Analytical Method for Solving Laplace's Boundary Value Problem with Singularities

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Abstract

A difference-analytical method of solving the mixed boundary value problem for Laplace's equation on polygons (which can have broken sections and be multiply connected) is described and justified. The uniform estimate for the error of the approximate solution is of order $O\left(h^2\right)$, where h is the mesh step, for the errors of the derivatives of order p, $p = 1, 2, ...$, in a finite neighbourhood of vertices, of order $O(h^2/r_j^{p-1/\lambda_j})$, where r_j is the distance from the current point to the vertex in question, $\lambda_j = 1/\alpha_j$ or $\lambda_j = 1/(2\alpha_j)$, depending on the types of boundary conditions, $\alpha_j\pi$ is the value of the angle. The last part of the paper is devoted to illustrate numerical experiments.

1 Introduction

It is well known that the use of classical finite difference or finite element methods to solve the elliptic boundary value problems with singularities becomes ineffective. A special construction is usually needed for the numerical scheme near the singularities in such a way that the order of convergence is the same as in the case of a smooth solution. Among many approaches to solve this problem, a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used, and an effective realization of the obtained system of algebraic equations is achieved (see [1], [2]–[5], and references therein).

In [2]–[4] a new combined difference-analytical method called the Block-Grid Method in solving the Laplace equation on graduated polygons is introduced. This method is a combination of two methods which takes only superiorities of each one of them: the exponentially convergent Block Method (see [6, 7] and references therein) which finely takes into account the behavior of the exact solution near the vertices of interior angles $\neq \pi/2$ of polygon (on "singular" part) and the Finite Difference Method, which has a simple structure and high accuracy on square grids of the rectangles covering the remainder, "nonsingular" part of the polygon. A sixth order gluing operator S^6 is constructed for gluing together the grids and blocks. The expression $S^6 u \equiv \sum \xi_k u_k$ contains the function values at 31 grid nodes. The pattern of the gluing operator S^6 makes restriction of the polygon as to be graduated.

In this paper, we use sufficiently simple linear interpolation as a gluing operator which contains the function values at fourth grids, and the Block-Grid Method is extended for the mixed boundary value problems on arbitrary polygons. The uniform estimate for the error of the approximate solution is of order $O(h^2)$, and for the errors of the derivatives of order p, $p = 1, 2, \dots$, in the blocks is $O(h^2/r_j^{p-\lambda_j})$, where r_j is the distance from the current point to the vertex in question, and λ_j depends on the magnitude of the interior angle and the type of boundary conditions on the sides of the angle considered. Furthermore, when the boundary conditions are either Dirichlet or mixed type on the sides of the interior angles $\neq \pi/2$, the error of the approximate solution on the block sectors decreases as $r_j^{\lambda_j} h^2$, which gives the additional accuracy of this approach near the singular points, with respect to existing finite difference or finite element modifications for the singular problems. The system of finite difference equations on the union of all rectangles may be solved by the alternating method of Schwarz with the number of iterations $O(\ln \varepsilon^{-1})$, where ε is the prescribed accuracy, by solving standard 5-point difference equations of Laplace on rectangular domain at each iteration. Finally, we illustrate the effectiveness of the method in solving the problem in an L-shaped polygon with the corner and boundary singularities, and the well known Motz problem.

2 Boundary Value Problem on Polygons

Let G be an open simply connected polygon, γ_j, $j = \overline{1, N}$, be its sides, including the endpoints, enumerated counterclockwise, $\gamma = \gamma_1 \cup \cdots \cup \gamma_N$ be the boundary of G, $\alpha_j \pi$, $0 < \alpha_j \leq 2$, be the interior angle formed by the sides γ_{j-1} and γ_j ($\gamma_0 = \gamma_N$), $A_j = \gamma_{j-1} \cap \gamma_j$ be the vertex of the $j-$th angle, r_j, θ_j be a polar system of coordinates with the pole at A_j and the angle θ_j taken counterclockwise from the side γ_j, ν_j be a parameter taking the values 0 or 1, and $\overline{\nu}_j = 1 - \nu_j$.

We consider the boundary value problem

$$\Delta u = 0 \text{ on } G, \tag{1}$$

$$\nu_j u + \overline{\nu}_j u_n^{(1)} = \nu_j \varphi_j + \overline{\nu}_j \psi_j \text{ on } \gamma_j,\ j = \overline{1, N}, \tag{2}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, $u_n^{(1)}$ is the derivative along the inner normal, φ_j and ψ_j are given functions of the arc length s taken along γ and

$$\begin{aligned} 1 &\leq \nu_1 + \nu_2 + \cdots + \nu_N \leq N, && (3)\\ \nu_j\varphi_j + \overline{\nu}_j\psi_j &\in C_{2,\lambda}(\gamma_j),\ \ 0 < \lambda < 1,\ 1 \leq j \leq N, && (4) \end{aligned}$$

at the vertices A_j $(s = s_j)$ for $\alpha_j = 1/2$ the conjugation conditions

$$\begin{aligned} \varphi_{j-1} &= \varphi_j \ \text{ for } \nu_{j-1} = \nu_j = 1;\ \ \varphi'_{j-1} = -\psi_j \text{ for } \overline{\nu}_{j-1} = \nu_j = 0;\\ \psi_{j-1} &= \varphi'_j \ \text{ for } \nu_{j-1} = \overline{\nu}_j = 0;\ \ \psi'_{j-1} = -\psi'_j \text{ for } \nu_{j-1} = \nu_j = 0, \end{aligned} \qquad (5)$$

where $\varphi'_\mu, \psi'_\upsilon$ are the derivatives along the boundary arc, are satisfied. At the vertices A_j with $\alpha_j \neq 1/2$ no compatibility conditions are required to hold for the boundary functions, in particular, the values of φ_{j-1} and φ_j at A_j might be different. In addition, we require that, when $\alpha_j \neq 1/2$, the boundary functions on γ_{j-1} and on γ_j be given as algebraic polynomials of s. We represent the given boundary functions (algebraic polynomials) on γ_{j-1} and γ_j for $\alpha_j \neq 1/2$ in the form

$$\sum_{k=0}^{\tau_{j-1}} a_{jk}^0 r_j^k \quad \text{and} \quad \sum_{k=0}^{\tau_j} b_{jk}^0 r_j^k, \qquad (6)$$

respectively, where a_{jk}^0 and b_{jk}^0 are numerical coefficients and τ_{j-1} and τ_j are the degrees of those polynomials.

Let E be the set j $(1 \leq j \leq N)$ for which $\alpha_j \neq 1/2$. In the neighborhood of A_j, $j \in E$, we construct two fixed block-sectors $T_j^i = T_j(r_{ji}) \subset G$, $i = 1, 2$, where $0 < r_{j2} < r_{j1} < \min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r,$ $0 < \theta_j < \alpha_j\pi\}$.

On the closed sector $\overline{T_j^1}$, $j \in E$, we consider a function $Q_j(r_j, \theta_j)$ with the following properties:

(a) $Q_j(r_j, \theta_j)$ is harmonic and bounded on the open sector T_j^1;

(b) continuous on $\overline{T_j^1}$ everywhere, except for the point A_j (the vertex of the sector) for $\nu_j = \nu_{j-1} = 1$ and $a_{j0}^0 \neq b_{j0}^0$, *i.e.*, for discontinuous at A_j given boundary values;

(c) continuously differentiable on $\overline{T_j^1} \setminus A_j$ and satisfies the boundary conditions (2) on $\gamma_{j-1} \cap \overline{T_j^1}$ and $\gamma_j \cap \overline{T_j^1}$, $j \in E$.

For definiteness we assume that $Q_j(r_j, \theta_j)$ with the above properties has the form (3.2)–(3.9) given in [7].

Remark 1 For the case of $\nu_{j-1} = \nu_j = 1$ we formally set the value of $Q_j(r_j, \theta_j)$ and the solution u of problem (1), (2) at the vertex A_j equal to $(a_{j0}^0 + b_{j0}^0)/2$.

We set (see [6])

$$R(m,m,r,\theta,\eta) = R(r,\theta,\eta) + (-1)^m R(r,\theta,-\eta), \tag{7}$$

$$R(1-m,m,r,\theta,\eta) = R(m,m,r,\theta,\eta) - (-1)^m R(m,m,r,\theta,\pi-\eta), \tag{8}$$

where

$$R(r,\theta,\eta) = \frac{1-r^2}{2\pi(1-2r\cos(\theta-\eta)+r^2)}, \tag{9}$$

is the kernel of the Poisson integral for a unit circle. We specify the kernel

$$R_j(r_j,\theta_j,\eta) = \lambda_j R(\nu_{j-1},\nu_{j,}\left(\frac{r_j}{r_{j2}}\right)^{\lambda_j},\lambda_j\theta_j,\lambda_j\eta),\ j \in E, \tag{10}$$

where

$$\lambda_j = \frac{1}{(2-\nu_{j-1}\nu_j - \overline{\nu}_{j-1}\overline{\nu}_j)\alpha_j} \tag{11}$$

Lemma 2 (Volkov [7]) *The solution u of the boundary value problem* (1), (2) *can be represented on $\overline{T_j^2} \setminus V_j$, $j \in E$, in the form*

$$u(r_j,\theta_j) = Q_j(r_j,\theta_j) + \int_0^{\alpha_j\pi} R_j(r_j,\theta_j,\eta)(u(r_{j2},\eta) - Q_j(r_{j2},\eta))\,d\eta, \tag{12}$$

where V_j is the curvilinear part of the boundary of T_j^2.

3 Description of the Block-Grid Method

Let us consider in addition to the sectors T_j^1 and T_j^2 (see §2) in the neighborhood of each vertex A_j, $j \in E$, of the polygon G the sectors T_j^3 and T_j^4, where $0 < r_{j4} < r_{j3} < r_{j2}$, $r_{j3} = (r_{j2}+r_{j4})/2$ and $T_k^3 \cap T_l^3 = \emptyset$, $k \neq l$, $k,l \in E$, and let $G_T = G \setminus (\cup_{j\in E} T_j^4)$.

Let $\Pi_k \subset G_T$, $k = \overline{1,M}$ $(M < \infty)$ be certain fixed open rectangles with arbitrary orientation, generally speaking, with sides a_{1k} and a_{2k}, a_{1k}/a_{2k} being rational and $G = (\cup_{k=1}^M \Pi_k) \cup (\cup_{j\in E} T_j^3)$ (see Figures 1 and 3 in Section 6). Let η_k be the boundary of the rectangle Π_k and V_j be the curvilinear part of the boundary of the sector T_j^2, and $t_j = (\cup_{k=1}^M \eta_k) \cap \overline{T_j^3}$. The following general requirement is imposed on the arrangement of the rectangles Π_k, $k = \overline{1,M}$: any point P lying on $\eta_k \cap G_T$, $1 \le k \le M$, or located on $V_j \cap G$, $j \in E$, falls inside at least one of the rectangles $\Pi_{k(p)}$, $1 \le k(p) \le M$, depending on P, and the distance from P to $G_T \bigcap \eta_{k(p)}$ is not less than some constant $\varkappa_0 > 0$ independent of P.

We call the quantity $\varkappa_0$ a *depth* of gluing of the rectangles Π_k, $k=\overline{1,M}$. We introduce the parameter $h \in (0, \varkappa_0/2]$ and define a square grid on Π_k, $k=\overline{1,M}$, with maximal possible step $h_k \le \min\{h, \min\{a_{1k}, a_{2k}\}/2\}$ such that the boundary η_k lies entirely on the grid lines. Let Π_k^h be the set of grid nodes on Π_k, and η_k^h be the set of nodes on η_k; $\overline{\Pi}_k^h = \Pi_k^h \cup \eta_k^h$, η_{k0}^h is the set of nodes on the closure of $\eta_k \cap G_T$, t_j^h is the set of nodes on t_j, and η_{k1}^h is the set of remaining nodes on η_k. We also introduce the natural number n and the quantities $n(j) = \max\{4, [\alpha_j n]\}$, $\beta_j = \alpha_j \pi / n(j)$, and $\theta_j^m = (m-1/2)\beta_j$, $j \in E$, $1 \le m \le n(j)$. On the arc V_j we choose the points (r_{j2}, θ_j^m), $1 \le m \le n(j)$, and denote the set of these points by V_j^n. Let

$$\omega^{h,n} = (\cup_{k=1}^M \eta_{k0}^h) \cup (\cup_{j\in E} V_j^n),\ \overline{G}_T^{h,n} = \omega^{h,n} \cup (\cup_{k=1}^M \overline{\Pi}_k^h).$$

Let

$$R_j^{(q)}(r_j,\theta_j) = \frac{R_j(r_j,\theta_j,\theta_j^q)}{\max\left\{1, \beta_j \sum_{p=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^p)\right\}}\ , \tag{13}$$

where $R_j(r,\theta,\eta)$ is the kernel defined by (10). It is easy to check that

$$0 \le R_j^{(q)}(r_j,\theta_j) \le R_j(r_j,\theta_j,\theta_j^q), \tag{14}$$

where $j \in E$, $0 \le q \le n(j)$. Furthermore, from the estimation (2.29) in [6] there follows the existence of the positive constants n_0 and $\sigma > 0$ such that for $n \ge n_0$,

$$\max_{\overline{T}_j^3} \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q) \le \sigma < 1, \tag{15}$$

when $\nu_{j-1} + \nu_j \ge 1$, and on the basis of (13) and (14)

$$0 \le \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}(r_j,\theta_j) \le 1, \quad j \in E, \tag{16}$$

when $\nu_{j-1} = \nu_j = 0$.

We define operators S^2, A, L and $\Re_j$, $j \in E$, on the sets $\omega^{h,n}, \Pi_k^h, \eta_{k1}^h$ and t_j^h, respectively. The operator S^2 is called a gluing operator, which is defined at each point $P \in \omega^{h,n}$ in the following way. We consider the set of all rectangles $\{\Pi_k\}$ in the intersections of which the point P lies, and we choose one of these rectangles $\Pi_{k(P)}$ part of whose boundary, situated in G^T is the furthest away from P. The value S^2u at the point P is computed according to the values of the function at the four vertices P_k, $k=1,2,3,4$, of the closure of the cell, containing the point P, of the grid constructed on $\overline{\Pi}_{k(P)}$, by multilinear interpolation in the directions of the grid lines. Thus, S^2u has the expression

$$S^2 u \equiv \sum_{\mu=1}^{4} \lambda_\mu u_\mu, \tag{17}$$

where $u = u(P)$, $u_\mu = u(P_\mu)$, and

$$\lambda_\mu \geq 0, \qquad \sum_{\mu=1}^{4} \lambda_\mu = 1. \tag{18}$$

We define on Π_k^h, $1 \leq k \leq M$, the operator A of calculating the arithmetic mean of the function at the four neighboring points of the same net.

Let L be the operator defined on the points η_{k1}^h as follows

$$\begin{aligned} L(u,\varphi,\psi) &= \varphi_m(P), \quad P \in \eta_{k1}^h \cap \gamma_m, \quad \nu_m = 1; &(19)\\ L(u,\varphi,\psi) &\equiv (u_1 + u_2 + 2u_3)/4 - h_k \psi_m(P)/2, &(20)\end{aligned}$$

when $P \in \eta_{k1}^h \cap \gamma_m \backslash (A_m \cup A_{m+1})$, $\nu_m = 0$, where P_1 and P_2 being the two points of $\overline{\Pi}_k^h$ on γ_m closest to P, and P_3 being the point on Π_k^h closest to P;

$$L(u,\varphi,\psi) \equiv (u_1 + u_2 - h_k(\psi_{j-1}(P) + \psi_j(P)))/2, \quad \nu_{j-1} = \nu_j = 0, \tag{21}$$

$P = A_j \in \eta_{k1}^h, P_1$ and P_2 being the two points in η_{k1}^h closest to P.

Let us define the operator $\Re_j$ on t_j^h, $j \in E$, as follows:

$$\Re_j(u,\varphi,\psi) \equiv Q_j(r_j,\theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}(r_j,\theta_j)(u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)), \tag{22}$$

where Q_j is the function defined in Section 2.

Consider the system of linear algebraic equation

$$\begin{aligned} u_h &= A u_h \quad \text{on} \quad \Pi_k^h, &(23)\\ u_h &= L(u_h,\varphi,\psi) \quad \text{on} \quad \eta_{k1}^h, &(24)\\ u_h(r_j,\theta_j) &= \Re(u_h,\varphi,\psi) \quad \text{on} \quad t_j^h, &(25)\\ u_h &= S^2 u_h \quad \text{on} \quad \omega^h, &(26)\end{aligned}$$

where $1 \leq k \leq M,\ j \in E$.

Definition 3 The solution of the system (23)–(26) is called a numerical solution of the problem (1), (2) on $\overline{G}_T^{h,n}$.

Definition 4 We consider the sector $T_j^* = T_j(r_j^*)$, where $r_j^* = (r_{j2} + r_{j3})/2$, $j \in E$. Let u_h be the solution of the system (23)–(26). The function

$$U_h(r_j, \theta_j) = Q_j(r_j, \theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)(u_h(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)), \tag{27}$$

defined on T_j^*, is called an approximate solution of the problem (1), (2) on the closed block $\overline{T_j^3}$, $j \in E$.

4 Analysis of the system of block-grid equations

4.1 Solvability of system (23)–(26)

Theorem 5 *There is a natural number n_0 such that, for all $n \geq n_0$ the system* (23) − (26) *has a unique solution.*

Proof. We consider a homogeneous system of finite difference equations corresponding to system (23)–(26)

$$\begin{aligned} v_h &= Av_h \text{ on } \Pi_k^h,\ v_h = L(v_h, 0, 0) \text{ on } \eta_{k1}^h \cap \gamma_m, \\ v_h(r_j, \theta_j) &= \Re(v_h, 0, 0) \text{ on } t_j^h,\ v_h = S^2 v_h \text{ on } \omega^{h,n}, \\ 1 &\leq m \leq N,\ 1 \leq k \leq M,\ j \in E. \end{aligned} \tag{28}$$

Let the system (28) have a solution $v_h \neq 0$ on $\overline{G}_T^{h,n}$, and $v_h^* = v_h(P^*) = \max_{\overline{G}_T^{h,n}} |v_h|$ $\neq 0$. If $P^* = P(r_{j^*}, \theta_{j^*}) \in t_{j^*}^h$, then from (22) and (28) we have

$$v_h(r_{j^*}, \theta_{j^*}) = \beta_{j^*} \sum_{q=1}^{n(j^*)} R_{j^*}^{(q)}(r_{j^*}, \theta_{j^*}) S^2 v_h(r_{j^*2}, \theta_{j^*}^q), \tag{29}$$

i.e., the value $v_h(P^*) = v_h^*$ is linearly represented through the values of the function v_h at the points of rectangular grids $\overline{\Pi}_{l(P^*)}^h$, $l(P^*) = 1, 2, \ldots, M'$, $M' \leq M$. Hence, when $\nu_{j^*-1} + \nu_{j^*} \geq 1$, on the basis of (14), (15), (17), (18), and (29) it follows that for $n \geq n_0$ the function v_h cannot take the value $v_h = v_h^*$ at the nodes $t_{j^*}^h$. When $\nu_{j^*-1} + \nu_{j^*} = 0$, from the gluing condition it follows that, on the right-hand side of (29) only an interior points of Π_l^h, $l = 1, 2, \ldots,$ M', $M' \leq M$, are used and by virtue of (16)–(18) at these points $v_h = v_h^*$. Similarly, if $v_h(P^*) = v_h^*$ and

$P^* = \eta^h_{k1} \cap \gamma_m,\ \nu_m = 0$, then from the constructions of the operator L by (20) and (21) the function v_h takes the value v^*_h at some interior points of some rectangular grid. Thus, we can assume that $v_h(P^*) = v^*_h \neq 0,\ P^* \in \Pi^h_{q^*}$ for some fixed $q^*, 1 \leq q^* \leq M$. But on the basis of (3), (17), (18), (28), and the principle of maximum, there exists a rectangular grid $\overline{\Pi}^h_{\mu_0}$ with the boundary $\eta^h_{\mu_0 1} \cap \gamma_m \neq \emptyset, \nu_m = 1$, at which $v_h = v^*_h = 0$. This contradicts the assumption $v^*_h \neq 0$. ∎

4.2 Error estimates on $\overline{G}^{h,n}_T$

Let

$$\varepsilon_h = u_h - u, \tag{30}$$

where u_h is a solution of system (23)–(26), and u is the trace on $\overline{G}^{h,n}_T$ of the solution of (1), (2). On the basis of (1), (2), (23)–(26) and (30) the error ε_h satisfies the system of difference equations

$$\begin{aligned}
\varepsilon_h &= A\varepsilon_h + r^1_h \text{ on } \Pi^h_k, \\
\varepsilon_h &= L(\varepsilon_h, 0, 0) + r^2_h \text{ on } \eta^h_{k1}, \\
\varepsilon_h(r_j, \theta_j) &= \beta_j \sum_{q=1}^{n(j)} R^{(q)}_j(r_j, \theta_j)\varepsilon_h(r_{j2}, \theta^q_j) + r^3_{jh}, (r_j, \theta_j) \in t^h_j, \\
\varepsilon_h &= S^2\varepsilon_h + r^4_h \text{ on } \omega^{h,n},
\end{aligned} \tag{31}$$

where $1 \leq k \leq M,\ j \in E$,

$$r^1_h = Au - u \text{ on } \cup_{k=1}^M \Pi^h_k, \tag{32}$$

$$r^2_h = L(u, \varphi, \psi) - u \text{ on } \cup_{k=1}^M \eta^h_{k1}, \tag{33}$$

$$\begin{aligned}
r^3_{jh} &= \beta_j \sum_{q=1}^{n(j)} R^{(q)}_j(r_j, \theta_j)(u(r_{j2}, \theta^q_j) - Q_j(r_{j2}, \theta^q_j)) \\
&\quad -(u(r_j, \theta_j) - Q_j(r_j, \theta_j)) \text{ on } \cup_{j\in E}\, t^h_j,
\end{aligned} \tag{34}$$

$$r^4_h = S^2u - u \text{ on } \omega^{h,n}. \tag{35}$$

In what follows and for simplicity, we will denote constants which are independent of h by c.

Lemma 6 *There exists a natural number n_0 such that, for all $n \geq \max\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\}$, where $\varkappa > 0$ is a fixed number,*

$$\max_{j\in E} \left|r^3_{jh}\right| \leq ch^2. \tag{36}$$

Proof. On the basis of (34), Lemma 2 and by virtue of $r_{j3} = (r_{j2}+r_{j4})/2 < r_{j2}$ we have

$$\begin{aligned}\left|r_{jh}^3\right| \quad &\leq \quad \Big|\beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)) \\ &\quad - \int_0^{\alpha_j \pi} R_j(r_j,\theta_j,\eta)(u(r_{j2},\eta) - Q_j(r_{j2},\eta))\, d\eta\Big| \\ &\quad + \beta_j \sum_{q=1}^{n(j)} \left|R_j^{(q)}(r_j,\theta_j) - R_j(r_j,\theta_j,\theta_j^q)\right| \left|u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)\right|.\end{aligned}$$

From this and from the Lemmas 2.5 and 2.10 given in [6] and taking the boundedness of the difference $u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)$, $1 \leq q \leq n(j)$, into account, we obtain

$$\left|r_{jh}^3\right| \leq c_j^0 \exp\left\{-d_j^0 n\right\}, j \in E, \tag{37}$$

where c_j^0 and $d_j^0 > 0$ are, independent of n, constants. Putting $c^0 = \max_{j\in E}\left\{c_j^0\right\}$, and $d = \min\{d_j\}$, from (37) we have

$$\max_{j\in E} \left|r_{jh}^3\right| \leq c^0 \exp\left\{-d^0 n\right\}. \tag{38}$$

Let n_0 be a natural number such that the inequality (15) holds. Then, for all $n \geq \max\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\}$, where $\varkappa > 0$ is a fixed number, we have the inequality (36). ■

Since the set of points $\omega^{h,n}$, according to the construction, is located from the vertices of the polygon G at a distance exceeding some positive quantity independent of h, then by virtue of (4), estimation (4.64) obtained in [8], from (35) we have

$$\max_{\omega^{h,n}} \left|r_h^4\right| \leq ch^2. \tag{39}$$

Theorem 7 *Assume that conditions* (3) – (5) *hold. Then, there exists a natural number* n_0 *such that, for all* $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$, *where* $\varkappa > 0$ *is a fixed number,*

$$\max_{\overline{G}_T^{h,n}} |u_h - u| \leq ch^2.$$

Proof. Let us take an arbitrary rectangular grid $\Pi_{k^*}^h$ and let $t_{k^*j}^h = \overline{\Pi}_{k^*}^h \cap t_j^h$. Let $t_{k^*j}^h \neq \emptyset$, and v_h be a solution of system (31) in the case when the discrepancies

r_h^1, r_h^2, r_{jh}^3, and r_h^4 in $\overline{\Pi}_{k^*}^h$ are the same as in (32)–(35), but are zero in $\overline{G}_T^{h,n} \setminus \overline{\Pi}_{k^*}^h$. By analogy to the proof of Theorem 5 one can show that

$$W = \max_{\overline{G}_T^{h,n}} |v_h| = \max_{\overline{\Pi}_{k^*}^h} |v_h| \,. \tag{40}$$

We represent the function v_h on $\overline{G}_T^{h,n}$ in the form

$$v_h = \sum_{\kappa=1}^{4} v_h^\kappa, \tag{41}$$

where the functions v_h^κ, $\kappa = 2, 3, 4$, are defined on $\overline{\Pi}_{k^*}^h$ as a solution of the system of equations

$$\begin{aligned} v_h^2 &= Av_h^2 \text{ on } \Pi_{k^*}^h, v_h^2 = L(v_h^2, 0, 0) \text{ on } \eta_{k^*1}^h, \\ v_h^2(r_j, \theta_j) &= r_{jh}^3, (r_j, \theta_j) \in t_{k^*j}^h, \quad v_h^2 = 0 \text{ on } \omega^{h,n}; \end{aligned} \tag{42}$$

$$\begin{aligned} v_h^3 &= Av_h^3 \text{ on } \Pi_{k^*}^h, v_h^3 = L(v_h^3, 0, 0) \text{ on } \eta_{k^*1}^h, \\ v_h^3(r_j, \theta_j) &= 0, (r_j, \theta_j) \in t_{k^*j}^h, \quad v_h^3 = r_h^4 \text{ on } \omega^{h,n}; \end{aligned} \tag{43}$$

$$\begin{aligned} v_h^4 &= Av_h^4 + r_h^1 \text{ on } \Pi_{k^*}^h, v_h^4 = L(v_h^4, 0, 0) + r_h^2 \text{ on } \eta_{k^*1}^h, \\ v_h^4(r_j, \theta_j) &= 0, (r_j, \theta_j) \in t_{k^*j}^h, \quad v_h^4 = 0 \text{ on } \omega^{h,n}, \end{aligned} \tag{44}$$

with

$$v_h^\kappa = 0, \ \kappa = 2, 3, 4, \text{ on } \overline{G}_T^{h,n} \backslash \overline{\Pi}_{k^*}^h. \tag{45}$$

Hence according to (41)–(45) the function v_h^1 satisfies the system of equations

$$\begin{aligned} v_h^1 &= Av_h^1 \text{ on } \Pi_k^h, \quad v_h^1 = L(v_h^1, 0, 0) \text{ on } \eta_{k1}^h, \\ v_h^1(r_j, \theta_j) &= \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}(r_j, \theta_j) \sum_{\kappa=1}^{4} v_h^\kappa(r_{j2}, \theta_j^q), (r_j, \theta_j) \in t_j^h, \\ v_h^1 &= S^2 \left(\sum_{\kappa=1}^{4} v_h^\kappa \right) \text{ on } \eta_{k0}^h, \quad 1 \le k \le M, \ j \in E, \end{aligned} \tag{46}$$

where the functions $v_h^\kappa, \kappa = 2, 3, 4$ are assumed to be known.

Taking into account (36) and (39), on the basis of (42), (43), (45) and the maximum principle we have

$$W_2 = \max_{\overline{G}_T^{h,n}} \left| v_h^2 \right| \le ch^2, \tag{47}$$

$$W_3 = \max_{\overline{G}_T^{h,n}} \left| v_h^3 \right| \le ch^2. \tag{48}$$

The function v_h^4 to be a solution of the system (44) with (45) is the error of the finite difference solution, with step $h_{k^*} \le h$, of the boundary value problem for Laplace's equation on Π_{k^*}. Then, by virtue of (45) and Theorem 3.1 in [9], we obtain

$$W_4 = \max_{\overline{G}_T^{h,n}} \left|v_h^4\right| = \max_{\overline{\Pi}_{k^*}^h} \left|v_h^4\right| \le ch^2. \tag{49}$$

We estimate the function v_h^1, which, according to Theorem 5, is the unique solution of system (46).

On the basis of (15)–(18) and the gluing condition of the rectangles Π_k, $k = 1, 2, \ldots, M$, from (46) by means of [9], there exists a real number λ^*, $0 < \lambda^* < 1$, independent of h, such that for all $n \ge \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$ we have

$$W_1 = \max_{\overline{G}_T^{h,n}} \left|v_h^1\right| \le \lambda^* W + \sum_{i=2}^{4} \max_{\overline{G}_T^{h,n}} \left|v_h^i\right|. \tag{50}$$

From (40), (41), (47)–(50) we obtain

$$W = \lambda^* W + 2\sum_{i=2}^{4} W_i \le \lambda^* W + ch^2, 0 < \lambda^* < 1,$$

i.e.,

$$W = \max_{\overline{G}_T^{h,n}} |v_h| \le ch^2. \tag{51}$$

In the case when $t_{k^*j}^h \equiv \emptyset$, the function $v_h^2 \equiv 0$ on $\overline{G}_T^{h,n}$ and the inequality (51) holds true.

Since the number of grid rectangles in $\overline{G}_T^{h,n}$ is finite, for the solution of (31) we have

$$\max_{\overline{G}_T^{h,n}} |\varepsilon_h| \le ch^2. \qquad \blacksquare$$

4.3 Convergence of the approximate solution on blocks

We consider the question of convergence of the function $U_h(r_j, \theta_j)$ defined by the formula (27). Taking into account the properties of the functions $Q_j(r_j, \theta_j), j \in E$, and the fact that the kernel $R_j(r_j, \theta_j, \eta)$ satisfies the homogenous boundary condition defined by (2) on $(\gamma_{j-1} \cup \gamma_j) \cap \overline{T}_j^2$, the function $U_h(r_j, \theta_j)$ is bounded, harmonic on T_j^* and continuous up to its boundary, except for the vertex A_j when the specified boundary values are discontinuous at A_j. In addition, on the rectilinear parts of the boundary of T_j^*, except, maybe, the vertex A_j, the function $U_h(r_j, \theta_j)$ satisfies the boundary conditions defined in (2).

Theorem 8 *There is a natural number n_0, such that for all $n \geq \max\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\}$, $\varkappa > 0$ is a fixed number, the following inequalities are valid:*

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}(U_h(r_j,\theta_j) - u(r_j,\theta_j))\right| \leq c_p h^2 \quad on \ \overline{T_j^3}, \tag{52}$$

first, for integer λ_j and any ν_{j-1} and ν_j when $p \geq \lambda_j$, second, for $\nu_{j-1} = \nu_j = 0$ and any λ_j when $p = 0$;

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}(U_h(r_j,\theta_j) - u(r_j,\theta_j))\right| \leq c_p h^2 / r^{p-\lambda_j} \quad on \ \overline{T_j^3}\ , \tag{53}$$

for any λ_j, if $\nu_{j-1}+ \nu_j \geq 1$, $0 \leq p < \lambda_j$ or $\nu_{j-1} = \nu_j = 0, 1 \leq p < \lambda_j$;

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}(U_h(r_j,\theta_j) - u(r_j,\theta_j))\right| \leq c_p h^2 / r^{p-\lambda_j} \quad on \ \overline{T_j^3}\backslash A_j, \tag{54}$$

for noninteger λ_j, and any ν_{j-1} and ν_j when $p > \lambda_j$. Everywhere $0 \leq q \leq p$, λ_j is the quantity (11), ν_{j-1} and ν_j are parameters entering into the boundary conditions (2), u is a solution of the problem (1), (2).

Proof. On the bases of (27) and Lemma 2, on the closed block $\overline{T_j^*}$, $j \in E$, we have

$$\begin{aligned} U_h(r_j,\theta_j) - u(r_j,\theta_j) &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)) \\ &\quad - \int_0^{\alpha_j\pi} R_j(r_j,\theta_j,\eta)(u(r_{j2},\eta) - Q_j(r_{j2},\eta))\, d\eta \\ &\quad + \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u_h(r_{j2},\theta_j^q) - u(r_{j2},\theta_j^q)). \end{aligned} \tag{55}$$

Since $r_j^* = (r_{j2}+r_{j3})/r_{j2}$, by analogy of the proof of Lemma 6 for $n \geq \left[\ln^{1+\varkappa} h^{-1}\right] + 1$, $\varkappa > 0$ is a fixed number, we have

$$\begin{aligned} &\left|\beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)(u(r_{j2},\theta_j^q) - Q_j(r_j,\theta_j^q)) \right. \\ &\left. \quad - \int_0^{\alpha_j\pi} R_j(r_j,\theta_j,\eta)(u(r_{j2},\eta) - Q_j(r_{j2},\eta))\, d\eta \right| \\ &\leq \ ch^2, \quad \text{on} \ \overline{T}_j^*, j \in E. \end{aligned} \tag{56}$$

On the basis of (15) and Theorem 7 for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]+1\right\}$ we obtain

$$\left|\beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q)(u_h(r_{j2}, \theta_j^q) - u(r_{j2}, \theta_j^q))\right| \leq ch^2, \quad \text{on } \overline{T_j^*}, j \in E. \tag{57}$$

From (55)–(57) for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]+1\right\}$ we have

$$|U_h(r_j, \theta_j) - u(r_j, \theta_j)| \leq ch^2 \quad \text{on } \overline{T_j^*}, j \in E. \tag{58}$$

Let

$$\varepsilon_h(r_j, \theta_j) = U_h(r_j, \theta_j) - u(r_j, \theta_j) \quad \text{on } \overline{T_j^*} \ , j \in E. \tag{59}$$

From (27), (59), and Remark 1 it follows that the function $\varepsilon_h(r_j, \theta_j)$ is continuous on $\overline{T_j^*}$, and is a solution of the boundary value problem

$$\begin{aligned} \Delta\varepsilon &= 0 \ \text{ on } \ T_j^*, \\ \nu_m \varepsilon_h + \overline{\nu}_m (\varepsilon_h)'_n &= 0 \ \text{ on } \ \gamma_m \cap \overline{T_j^*}, \ m = j-1, j, \\ \varepsilon_h(r_j^*, \theta_j) &= U_h(r_j^*, \theta_j) - u(r_j^*, \theta_j), \ 0 \leq \theta_j \leq \alpha_j \pi. \end{aligned} \tag{60}$$

Since $T_j^3 \subset \overline{T_j^*}, j \in E$, taking into account (58)-(60), from the Lemma 6.12 in [7] follows all inequalities of Theorem 8. ■

5 The use of Schwarz's alternating method to solve the system of block-grid equations

According to Definitions 3 and 4, the approximate solution of problem (1), (2) must first be found in the domain $\overline{G}_*^{h,n}$ as the solution of the system of difference equations (23)-(26), and the solution itself and its derivatives of order p, $p = 1, 2, \ldots$, at any point of $\overline{T}_j^3$, $j \in E$, except may be the vertex A_j, can then be found using formula (21). Therefore, it is sufficient to justify the possibility of finding a solution of system (23)–(26) by Schwarz's alternating method.

We denote by γ^D the union of all sides of the polygon G on which the boundary condition is of Dirichlet type. From (3) it follows that $\gamma^D \neq \emptyset$. We define the following classes Φ_τ, $\tau = 1, 2, \ldots, \tau^*$, of rectangles Π_k, $k = 1, 2, \ldots, M$ (see [3]). The class Φ_1 includes all rectangles whose intersection with γ^D contains a certain segment of positive length. The class Φ_2 contains all the rectangles which are not in the class Φ_1, whose intersection with rectangles of Φ_1 contains a segment of finite length, and so on. Let Π^h_{k0} be the set of nodes of the grid $\overline{\Pi}^h_k$ which are not less than $l_0 = \min\{\min_{1\leq k\leq M}\min\{a_{1k}, a_{2k}\}, \nu_0\}/4$ from the set η_{k0}. Let

$$\Phi^h_{\tau 0} = \bigcup_{k:\Pi_k\in\Phi_\tau} \Pi^h_{k0}, \tau = 1, 2, ..., \tau^*;\ G^h_{T0} = \bigcup_{\tau=1}^{\tau^*} \Phi^h_{\tau 0}.$$

We use Schwarz's alternating method to solve system (23)–(26) in the following form. Suppose we are given a zero approximation $u_h^{(0)}$ to the exact solution u_h of (23)–(26). Finding $u_h^{(1)}$ for all $j \in E$ with the formula (25) on t^h_j and with (26) on η_{k0}, we solve system (23)–(26) on grids $\overline{\Pi}^h_k$ constructed on rectangles belonging to the class Φ_1, and then class Φ_2 and so on. The next iteration is similar.

Consequently, we have the sequence $u_h^{(1)}$, $u_h^{(2)}, \ldots$, defined as follows

$$\begin{aligned}
u_h^{(m)}(r_j, \theta_j) &= Q_j(r_j, \theta_j) \\
&\quad + \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}(r_j, \theta_j)(u^{(m-1)}(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)) \text{ on } t^h_j, \\
u_h^{(m)} &= S^2 u_h^{(m-1)} \text{ on } \omega^{h,n}, \\
u_h^{(m)} &= A u_h^{(m)} \text{ on } \Pi^h_k,\ u_h^{(m)} = L(u_h^{(m)}, \varphi, \psi) \text{ on } \eta^h_{k1},
\end{aligned} \tag{61}$$

where $1 \leq k \leq M,\ j \in E,\ m = 1, 2, \ldots$.

Theorem 9 *For any $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$ the system (23) − (26) can be solved by Schwarz's alternating method with any accuracy $\varepsilon > 0$ in a uniform metric with the number of iterations $O(\ln \varepsilon^{-1})$, independent of h and n, where n_0 and $\varkappa$ mean the same as in Theorem 8.*

Proof. Let

$$\varepsilon^{(m)} = u_h^{(m)} - u_h \text{ on } \overline{G}^{h,n}_T, \tag{62}$$

where u_h is the exact solution of (23)–(26), and $u_h^{(m)}$ is the $m-$th iteration defined by (61), $m = 1, 2, \ldots$. Then for any m we have

$$\varepsilon_h^{(m)}(r_j, \theta_j) = \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}(r_j, \theta_j) S^2(\varepsilon_h^{(m-1)}(r_{j2}, \theta_j^q)) \text{ on } t^h_j \cap \eta_k, \tag{63}$$

$$\varepsilon_h^{(m)} = S^2(\varepsilon_h^{(m-1)}) \text{ on } \eta^h_{k0}, \tag{64}$$

$$\varepsilon_h^{(m)} = A\varepsilon_h^{(m)} \text{ on } \Pi^h_k,\ \varepsilon_h^{(m)} = L(\varepsilon_h^{(m)}, 0, 0) \text{ on } \eta^h_{k1}, \tag{65}$$

where $1 \leq k \leq M,\ j \in E$.

We denote

$$W_h^{(m)} = \max_{\overline{G}_T^{h,n}} \left| \varepsilon_h^{(m)} \right|.$$

From (15)–(18), (63), and (64) for $n \geq n_0$ we have

$$\max_{\cup_{k=1}^{M} \eta_{k,0}^h} \left| \varepsilon_h^{(m)} \right| \leq \max_{G_{T0}^h} \left| \varepsilon_h^{(m-1)} \right| \leq W_h^{(m-1)}, \tag{66}$$

$$\max_{\cup_{j \in E} t_j^h} \left| \varepsilon_h^{(m)}(r_j, \theta_j) \right| \leq \max_{G_{T0}^h} \left| \varepsilon_h^{(m-1)} \right| \leq W_h^{(m-1)},\ m = 1, 2, \ldots \tag{67}$$

By virtue of (66), (67) and the maximum principle, from (65) we obtain

$$W_h^{(0)} \geq W_h^{(1)} \geq W_h^{(2)} \geq \cdots, \tag{68}$$

$$W_h^{(m)} \leq \max_{G_{T0}^h} \left| \varepsilon_h^{(m-1)} \right|,\ m = 1, 2, \ldots \tag{69}$$

Taking into account the sequence of calculations over the classes $\Phi_\tau,\ \tau = 1, 2, \ldots, \tau^*$, and the inequalities (68) and (69), by means of [3] we obtain

$$W_h^{(m+1)} \leq \mu_{\tau^*}^m W_h^{(0)}, \quad m = 1, 2, ..., \tag{70}$$

where $\mu_{\tau^*} < 1$ is independent of m and h.

From (70) there follows the statement of Theorem 9. ∎

Remark 10 If on the sides of the right interior angles of the polygon G the boundary functions are given also as algebraic polynomials of s, then, without the conjugation conditions (5), the approximate solution in neighborhoods of the vertices of the right angles can be defined by the formula (27), and any order derivatives can be found by its simple differentiation.

Remark 11 From the error estimation formula (53) of Theorem 8 it follows that, when on the sides of the interior angles the boundary conditions are either Dirichlet or mixed type, the error of the approximate solution on the block sectors decreases as $r_j^{\lambda_j} h^2$, which gives an additional accuracy of the Block Grid method near the singular points, with respect to existing finite difference or finite element modifications for the singular problems.

Remark 12 The method and results carry out without changes to multiply connected polygons.

6 Numerical examples

To test the effectiveness of the Block-Grid method, we computed two numerical examples. In Example 13, the polygon G is L-shaped (Fig. 1), and the exact solution has both the boundary (discontinuity of the boundary functions) and the angle singularities at the vertex A_1 of the interior angle ($\alpha_1\pi = 3\pi/2$). In Example 14, the exact solution has singularities at the vertex A_1, with the interior angle $\alpha_1\pi = \pi$, caused by abrupt changes in the type of boundary conditions. In Example 13 the comparisons are made between the exact and the approximate solutions, and their derivatives. In Example 14, because of boundary conditions on the sides for $y \geq 0$ (see Fig. 3), the exact solution is unknown and comparisons are made at various points with values from the literature.

In Example 13, the four overlapping rectangles Π_k, $k = 1, \ldots, 4$, and in Example 14 three overlapping rectangles Π_k, $k = 1, 2, 3$, are taken as shown on Figs. 1 and 3 respectively. We found the solution in each rectangle on the mesh segments η_{k0}, $k = 1, \ldots, 6$, in Example 13 and $k = 1, \ldots, 4$, in Example 14. After obtaining the prescribed accuracy $\varepsilon = 0.5 \times 10^{-6}$ on the boundary of each standard rectangle, the system on each rectangle in (61) is solved by the formulas in [10], and [11], with the use of discrete fast Fourier transform. Furthermore, according to the boundary conditions on γ_0 and γ_1 for Example 13, the harmonic function $Q_1(r,\theta) = \theta$, and for Example 14, $Q_1(r,\theta) = 0$, and in both examples the radius $r_{1,2}$ of sector T_1^2 are taken 0.93. In all tables below, U_h is used as the numerical solution obtained by Block-Grid method and u is the exact solution of the given problem.

Example 13 Let G be L-shaped, and defined as follows (see Fig. 1)

$$G = \{(x,y):\ -1 < x < 1,\ -1 < y < 1\} \setminus G_1,$$

where $G_1 = \{(x,y):\ 0 \leq x \leq 1,\ -1 \leq y \leq 0\}$. Let γ be the boundary of G. Consider the following problem:

$$\Delta u = 0 \ \text{ on } \ G, \tag{71}$$

$$u = v(r,\theta) \ \text{ on } \ \gamma, \tag{72}$$

where $v(r,\theta) = \theta + r^{\frac{2}{3}} \sin\frac{2\theta}{3}$ is the exact solution of this problem.

In all tables the following notations are used: $\Pi_1^* = \overline{G} \setminus (\cup_{k=1}^{M} \overline{\Pi}_k)$, $G_{NS} = \overline{G} \setminus \Pi_1^*$ "nonsingular part" of G; $G_S = \overline{G} \cap \Pi_1^*$ "singular part" of G, $G_S^{\varrho} = G_S \cap \{r \leq \varrho\}$, $G_{NS}^{h,n} = G_{NS} \cap \overline{G}_*^{h,n}$, $\|w\|_{C(\Omega)} = \max_\Omega |w|$.

In Tables 1–3 the errors $\varepsilon_h = U_h - u$, $\varepsilon_h^{(1,0)} = r^{1/3}\Big(\dfrac{\partial U_h}{\partial x} - \dfrac{\partial u}{\partial x}\Big)$, $\varepsilon_h^{(0,1)} = r^{1/3}\Big(\dfrac{\partial U_h}{\partial y} - \dfrac{\partial u}{\partial y}\Big)$, $\varepsilon_h^{(2,0)} = r^{4/3}\Big(\dfrac{\partial^2 U_h}{\partial x^2} - \dfrac{\partial^2 u}{\partial x^2}\Big)$, $\varepsilon_h^{(1,1)} = r^{4/3}\Big(\dfrac{\partial^2 U_h}{\partial x \partial y} - \dfrac{\partial^2 u}{\partial x \partial y}\Big)$ in the

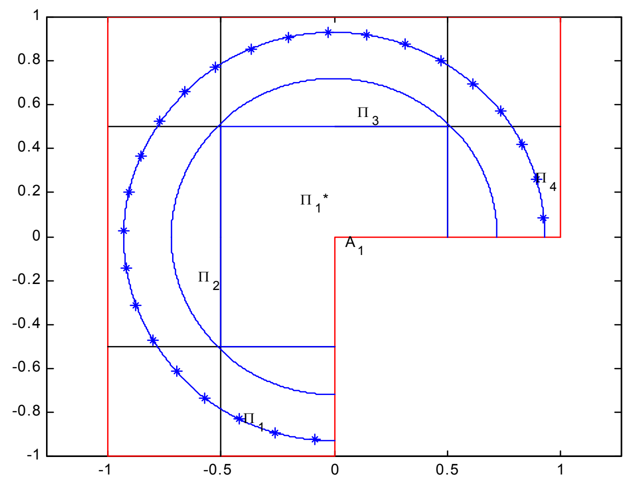

Figure 1: L-shaped domain

maximum norm between the block-grid solution U_h and the exact solution u of the problem in Example 13 are given.

(h^{-1}, n)	$\Vert\varepsilon_h\Vert_{C(G_{NS}^{h,n})}$	$\Vert\varepsilon_h\Vert_{C(G_S)}$	$\Vert\varepsilon_h\Vert_{C(G_S^{0.25})}$	$\Vert\varepsilon_h\Vert_{C(G_S^{0.125})}$	Iter.
$(8, 20)$	$1.17D-3$	$1.45D-3$	$1.32D-4$	$6.76D-5$	13
$(8, 30)$	$9.39D-4$	$1.39D-4$	$3.18D-6$	$1.63D-6$	13
$(8, 40)$	$9.17D-4$	$1.11D-4$	$8.92D-6$	$4.65D-6$	13
$(16, 60)$	$2.43D-4$	$1.98D-5$	$1.08D-6$	$5.74D-7$	13
$(32, 40)$	$6.23D-5$	$3.06D-5$	$4.81D-6$	$1.95D-6$	14
$(32, 50)$	$6.13D-5$	$6.44D-6$	$9.90D-7$	$5.02D-7$	14
$(32, 60)$	$5.93D-5$	$6.27D-6$	$1.55D-6$	$8.94D-7$	14
$(64, 50)$	$1.53D-5$	$3.34D-6$	$6.42D-7$	$2.76D-7$	15

Table 1: $\Vert\varepsilon_h\Vert$ for L-shaped problem and $r_{12} = 0.93$

(h^{-1}, n)	$\|\varepsilon_h^{(1,0)}\|_{C(G_S)}$	$\|\varepsilon_h^{(1,0)}\|_{C(G_S^{0.25})}$	$\|\varepsilon_h^{(1,0)}\|_{C(G_S^{0.125})}$
$(8, 30)$	$2.81D-3$	$4.90D-5$	$4.02D-6$
$(8, 40)$	$7.29D-4$	$2.78D-5$	$1.43D-5$
$(8, 50)$	$6.46D-4$	$4.12D-5$	$9.15D-6$
$(16, 60)$	$1.81D-4$	$7.80D-6$	$1.49D-6$
$(32, 40)$	$1.89D-4$	$1.855D-5$	$1.03D-5$
$(32, 60)$	$3.68D-5$	$2.99D-6$	$2.58D-6$
$(64, 60)$	$2.31D-5$	$7.11D-7$	$3.01D-7$
$(64, 70)$	$8.32D-6$	$1.75D-6$	$1.07D-6$

Table 2: $\|\varepsilon_h^{(1,0)}\|$ for L-shaped problem and $r_{12} = 0.93$

(h^{-1}, n)	$\|\varepsilon_h^{(2,0)}\|_{C(G_S)}$	$\|\varepsilon_h^{(2,0)}\|_{C(G_S^{0.25})}$	$\|\varepsilon_h^{(2,0)}\|_{C(G_S^{0.125})}$
$(8, 30)$	$2.28D-2$	$9.02D-5$	$7.19D-7$
$(8, 40)$	$8.18D-3$	$6.37D-5$	$4.70D-6$
$(8, 50)$	$5.83D-3$	$7.24D-5$	$1.67D-6$
$(16, 60)$	$1.46D-3$	$2.78D-5$	$4.04D-6$
$(32, 60)$	$2.70D-4$	$7.60D-6$	$3.98D-6$
$(64, 50)$	$1.93D-4$	$8.43D-6$	$2.75D-6$

Table 3: $\|\varepsilon_h^{(2,0)}\|$ for L-shaped problem and $r_{12} = 0.93$

Example 14 (Motz problem [15]). Let $G = \{(x, y) : \ -1 < x < 1, \ 0 < y < 1\}$, and γ be its boundary (Fig. 3). We consider the following problem:

$$\begin{aligned} -\Delta u &= 0 \text{ in } \ G, \\ u &= 0 \text{ on } \ y = 0, \ -1 \le x \le 0, \\ u &= 500 \ \text{ on } \ \ x = 1, \\ \frac{\partial u}{\partial n} &= 0 \ \text{ on the other boundary segments of } \gamma. \end{aligned} \tag{73}$$

The exact solution of this problem is unknown. In Table 4 the solution of the Motz problem at various points obtained by block-grid method when $(h^{-1} = 32, n = 60)$, and $(h^{-1} = 64, n = 80)$ is compared with the values from [12]. The results of [12] are presented after a correction of the 31-th coefficient (divided by 10), in the expansion of the approximate solution, discovered in [13], and are called an extremely accurate solution (see also [14]).

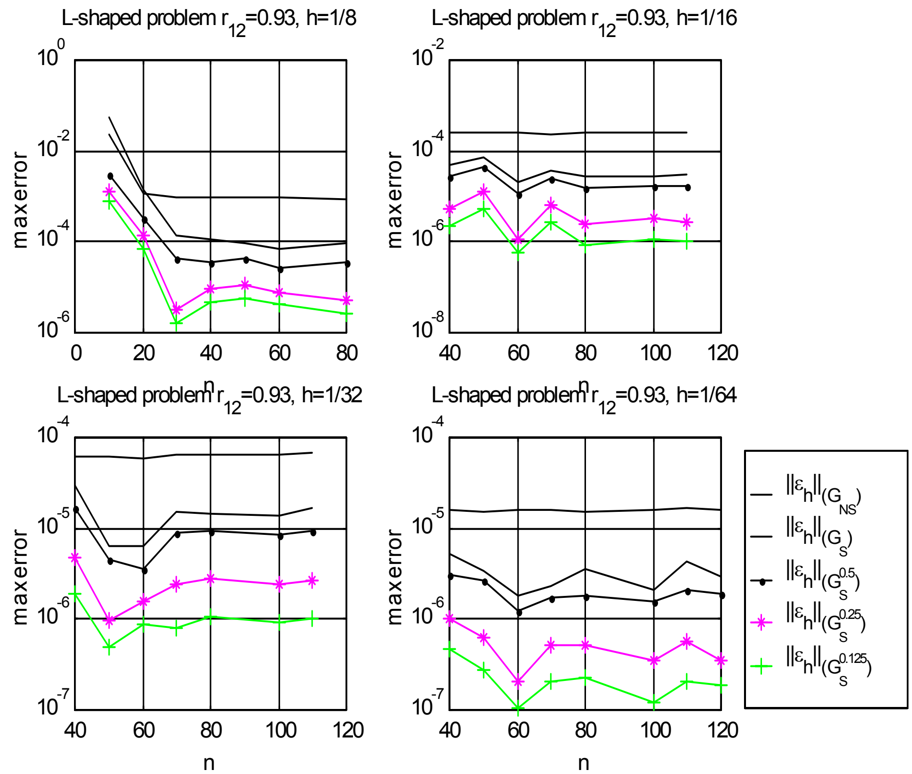

Figure 2: Decreasing error around the singular point

(x_i, y_i)	$h^{-1} = 32, n = 60$	$h^{-1} = 64, n = 80$	Li et al.[1987]
$(\frac{-2}{7}, \frac{2}{7})$	78.559747109	78.559447392	78.559230394
$(0, \frac{2}{7})$	141.559049415	141.559500712	141.559519337
$(\frac{2}{7}, \frac{2}{7})$	243.811562756	243.811877860	243.811824635
$(0, \frac{1}{7})$	103.768096701	103.768314085	103.768301476
$(\frac{-1}{28}, \frac{1}{28})$	33.591455839	33.591514936	33.591507894
$(0, \frac{1}{28})$	53.186172967	53.186268406	53.186260997
$(\frac{1}{28}, \frac{1}{28})$	83.671133081	83.671280478	83.671271165
$(\frac{1}{28}, 0)$	76.408161531	76.408294083	76.408285701
$(\frac{3}{28}, 0)$	134.446775698	134.446993973	134.446980917
$(\frac{1}{7}, 0)$	156.482209372	156.482452033	156.482436984

Table 4: Solution of the Motz problem at various points compared with values from literature

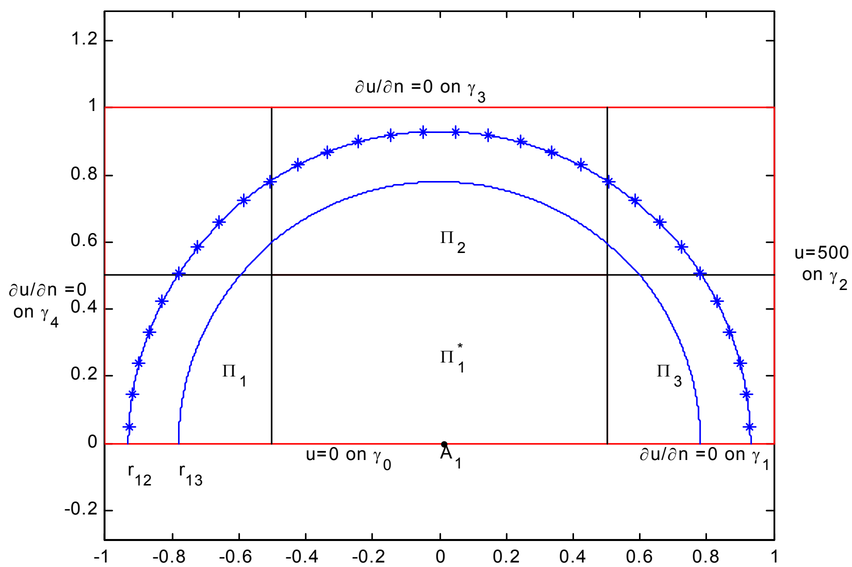

Figure 3: Motz problem

7 Concluding remarks

In the proposed method, the given polygon is decomposed into a finite number of overlapping rectangles (nonsingular parts) and sectors (singular parts). In the sectors we approximate the special integral representation of the solution which finely takes the behavior of the exact solution near the vertices of the interior angles of the polygon into account and on rectangles the 5-point scheme is used to approximate Laplace's equation on square grids which is simpler by means of sparsity. A gluing together of the subsystems is carried out, effected by a sufficiently simple linear interpolation. These properties are a prerequisite for a high rate of convergence established by Theorems 7 and 8, and an effective realization given in Sections 5 and 6. Furthermore, on the singular parts any order derivatives of the solution at any point can be approximated by simple differentiation of the function (27) (see Theorems 8 and Tables 2, 3. This issue seems to be more problematic for finite difference, finite element, and their existing modifications.

Acknowledgment

The authors thanks Professor E. A. Volkov for his attention to this work and for valuable advice.

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