5–10 July 2004, Antalya, Turkey — Dynamical Systems and Applications, Proceedings, pp. 295–309

Quintic B-Spline Galerkin Method for Numerical Solutions of the Burgers' Equation

İdris Dağ[1], Bülent Saka[2] and Ahmet Boz[1]
[1]Computer Engineering Department /[2]Mathematics Department
Osmangazi University, Eskişehir, Türkiye

Abstract

The nonlinear Burgers' equation is solved numerically by a method of Galerkin using quintic B-splines as both shape and weight functions over the finite intervals. The same method is applied to the time-split Burgers' equation. Numerical comparison of results of both algorithms and some other published numerical results is done by studying two standard problems.

Keywords: Burgers' equation, Galerkin finite element, Quintic B-spline.

1 Introduction

The nonlinear Burgers' equation has been used as test example for the numerical methods since this equation can be solved analytically for the various boundary and initial conditions. So numerical results can be compared with analytical results. Nowadays many of the numerical methods have been engaged to get the solution of the Burgers' equation with small viscosity. With these smaller constants, numerical results are likely to produce the results having non-physical oscillations unless the sizes of both space and time steps are unrealistically small. Some of the methods such as collocation method and Petrov-Galerkin finite elements have been used to obtain accurate numerical solutions for small viscosity coefficients. Spline functions, which are a class of piecewise polynomials having continuity properties of up to the degree lower than that of the spline functions play an important role of setting approximate functions. A type of splines known as B-splines are very much in use with Galerkin method to have functional approximation of the unknowns in differential equations. It provides the manageable band matrix system. This method previously has been implemented to get numerical solution of Burgers' equation. The finite element method for solutions of the Burgers' equation based on a collocation method using cubic splines as interpolation functions is set up by L. R. T. Gardner *et al.* [2]. The same method with the cubic B-splines instead of quadratic B-splines in setting up the trial functions was proposed in the paper [3]. A method of the least square was constructed to form a kind of quadratic B-splines finite element method for the Burgers' equation by the S. Kutluay *et al.* [9]. Various types of B-spline collocation finite element schemes are also proposed in the papers [4, 5].

In this paper we have written an algorithm for the numerical solution of the Burgers' equation, which is a finite element approach using Galerkin method over finite elements with quintic B-spline interpolation functions. This method will also apply to the time split Burgers' equation. The effect of both the quintic B-splines and splitting of the equation are sought in the Galerkin method.

2 Quintic B-spline Galerkin method I (QBGM1)

The Burgers' equation has the form

$$U_t + UU_x - \nu U_{xx} = 0, \tag{1}$$

where $\nu > 0$ is the coefficient of the kinematic viscosity and subscripts x and t denote differentiation. Boundary conditions are selected from

$$\begin{aligned} U(a,t) &= \alpha_1, \qquad U(b,t) = \alpha_2, \\ U_x(a,t) &= 0, \qquad U_x(b,t) = 0, \quad t \in (0,T], \\ U_{xx}(a,t) &= 0, \qquad U_{xx}(b,t) = 0. \end{aligned} \tag{2}$$

An initial condition will be prescribed in later sections.

Applying the Galerkin technique to Eq. (1) with weight functions W yields the integral equation

$$\int_a^b W\left(U_t + UU_x - \nu U_{xx}\right) dx = 0. \tag{3}$$

We consider the mesh $a = x_0 < x_1 < \cdots < x_N = b$ as a uniform partition of the solution domain $a \le x \le b$ by the knots x_m and $h = x_m - x_{m-1}, \quad m = 1, \ldots, N,$ throughout the paper.

Let $Q_m(x)$, $m = -2, \ldots, N+2$,

$$Q_m(x) = \frac{1}{h^5}\begin{cases} (x-x_{m-3})^5, & [x_{m-3}, x_{m-2}],\\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5, & [x_{m-2}, x_{m-1}],\\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5, & [x_{m-1}, x_m],\\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - 20(x-x_m)^5, & [x_m, x_{m+1}],\\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - 20(x-x_m)^5 & \\ \qquad +15(x-x_{m+1})^5, & [x_{m+1}, x_{m+2}],\\ (x-x_{m-3})^5 - 6(x-x_{m-2})^5 + 15(x-x_{m-1})^5 - 20(x-x_m)^5 & \\ \qquad +15(x-x_{m+1})^5 - 6(x-x_{m+2})^5, & [x_{m+2}, x_{m+3}]\\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

be quintic B-splines with the knots x_m, $m = -5, \ldots, N+5$ [1]. The set of quintic B-splines $Q_m(x)$ forms a basis over the region $a \leq x \leq b$. The global approximation defined using the quintic B-splines

$$U_N(x,t) = \sum_{m=-2}^{N+2} \delta_m(t) Q_m(x), \tag{5}$$

will be sought to the analytical solution U. In this approximate solution δ_m is a time-dependent parameter to be determined from the quintic Galerkin form of the Eq. (3). The nodal values of U and its derivatives of up to fourth order are given in terms of the parameters δ_m from the use of the splines (4) and the trial solution (5)

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-2} + 26\delta_{m-1} + 66\delta_m + 26\delta_{m+1} + \delta_{m+2},\\ U'_m &= U'(x_m) = \frac{5}{h}\left(\delta_{m+2} + 10\delta_{m+1} - 10\delta_{m-1} - \delta_{m-2}\right),\\ U''_m &= U''(x_m) = \frac{20}{h^2}\left(\delta_{m+2} + 2\delta_{m+1} - 6\delta_m + 2\delta_{m-1} + \delta_{m-2}\right),\\ U'''_m &= U'''(x_m) = \frac{60}{h^3}\left(\delta_{m+2} - 2\delta_{m+1} + 2\delta_{m-1} - \delta_{m-2}\right),\\ U''''_m &= U''''(x_m) = \frac{120}{h^4}\left(\delta_{m+2} - 4\delta_{m+1} + 6\delta_m - 4\delta_{m-1} + \delta_{m-2}\right). \end{aligned} \tag{6}$$

A local coordinate system can be defined using the mapping relation $\xi = x - x_m$ to transform the finite element $[x_m, x_{m+1}]$ into the interval $[0, h]$. The expressions of quintic B-spline shape functions that are independent of the element position are obtained with the relation of the global and local coordinates relation over $[0,h]$ as

$$
\begin{aligned}
Q_{m-2} &= 1-5\tfrac{\xi}{h}+10\left(\tfrac{\xi}{h}\right)^2-10\left(\tfrac{\xi}{h}\right)^3+5\left(\tfrac{\xi}{h}\right)^4-\left(\tfrac{\xi}{h}\right)^5,\\
Q_{m-1} &= 26-50\tfrac{\xi}{h}+20\left(\tfrac{\xi}{h}\right)^2+20\left(\tfrac{\xi}{h}\right)^3-20\left(\tfrac{\xi}{h}\right)^4+5\left(\tfrac{\xi}{h}\right)^5,\\
Q_{m} &= 66-60\left(\tfrac{\xi}{h}\right)^2+30\left(\tfrac{\xi}{h}\right)^4-10\left(\tfrac{\xi}{h}\right)^5,\\
Q_{m+1} &= 26+50\tfrac{\xi}{h}+20\left(\tfrac{\xi}{h}\right)^2-20\left(\tfrac{\xi}{h}\right)^3-20\left(\tfrac{\xi}{h}\right)^4+10\left(\tfrac{\xi}{h}\right)^5,\\
Q_{m+2} &= 1+5\tfrac{\xi}{h}+10\left(\tfrac{\xi}{h}\right)^2+10\left(\tfrac{\xi}{h}\right)^3+5\left(\tfrac{\xi}{h}\right)^4-5\left(\tfrac{\xi}{h}\right)^5,\\
Q_{m+3} &= \left(\tfrac{\xi}{h}\right)^5.
\end{aligned}
\tag{7}
$$

From the quintic B-splines covering six successive finite elements, typical finite elements are covered by the six quintic B-spline shape functions. So the approximation reduced over the element $[x_m, x_{m+1}]$ is

$$
U_N^e = U(\xi,t) = \sum_{i=m-2}^{m+3} \delta_i(t) Q_i(\xi), \tag{8}
$$

where δ_i, $i = m-2,\dots,m+3$, act as element parameters.

Taking the weight functions with quintic B-spline shape functions and substituting element trial function U_N^e in the integral equation (3) over the element $[0,h]$ leads to

$$
\begin{aligned}
\sum_{j=m-2}^{m+3}\Bigg\{&\left(\int_0^h Q_iQ_j\,d\xi\right)\overset{\circ}{\delta}_j+\sum_{k=m-2}^{m+3}\left[\left(\int_0^h Q_iQ_jQ_k'\,d\xi\right)\delta_k\right]\delta_j\\
&-\nu\left(\int_0^h Q_iQ_j''\,d\xi\right)\delta_j\Bigg\},
\end{aligned}
\tag{9}
$$

where j and k take only the values $m-2$, $m-1$, m, $m+1$, $m+2$, $m+3$ and $m = 0,1,\dots,N-1$, and $^\circ$ denotes derivative with respect to time, which in the matrix form is

$$
A^e\overset{\circ}{\boldsymbol{\delta}}{}^e + (\boldsymbol{\delta}^e)^T L^e \boldsymbol{\delta}^e - \nu D^e \boldsymbol{\delta}^e, \tag{10}
$$

where the element matrices A, D are 6×6, the matrix L is $6\times 6\times 6$, $\boldsymbol{\delta}^e = (\delta_{m-2},\delta_{m-1},\delta_m,\delta_{m+1},\delta_{m+2},\delta_{m+3})$ and

$$
A_{i,j} = \int_0^h Q_iQ_j\,d\xi, \quad D = \int_0^h Q_iQ_j''\,d\xi, \quad L_{i,j,k} = \int_0^h Q_iQ_jQ_k'\,d\xi. \tag{11}
$$

The matrix L is organized to be in the dimension 6×6 as matrix B

$$B_{i,j} = \sum_{k=m-2}^{m+3} L_{ijk}\delta_k, \tag{12}$$

so the matrix B^e is expressed as depending on the element parameter $\boldsymbol{\delta}^e$.

Combining all element matrices for each element we obtain a system of nonlinear ordinary differential equation:

$$\mathbf{A}\overset{\circ}{\boldsymbol{\delta}}+(\mathbf{B}-\boldsymbol{\nu}\mathbf{D})\boldsymbol{\delta}=0 \tag{13}$$

where the global element parameter is

$$\boldsymbol{\delta} = (\delta_{-2}, \delta_{-1}, \delta_0, \ldots, \delta_{N+1}, \delta_{N+2})^T. \tag{14}$$

If we use the Crank-Nicolson discretization formula for the vector of element parameter $\boldsymbol{\delta}$ and the usual finite difference equation for the time derivatives parameters $\overset{\circ}{\boldsymbol{\delta}}$ in the equation:

$$\boldsymbol{\delta} = \frac{\boldsymbol{\delta}^n + \boldsymbol{\delta}^{n+1}}{2}, \qquad \overset{\circ}{\boldsymbol{\delta}} = \frac{\boldsymbol{\delta}^{n+1} - \boldsymbol{\delta}^n}{\Delta t}, \tag{15}$$

we reach a nonlinear recurrence relation for the time parameters $\boldsymbol{\delta}$:

$$(2\mathbf{A} + \Delta t\mathbf{B} - \boldsymbol{\nu}\Delta t\mathbf{D})\,\boldsymbol{\delta}^{n+1} = (2\mathbf{A} - \Delta t\mathbf{B} + \boldsymbol{\nu}\Delta t\mathbf{D})\,\boldsymbol{\delta}^n. \tag{16}$$

This system is made up of $(N+5)$ equations in $(N+5)$ unknown parameters. We can obtain a solvable system by imposing the boundary conditions at the left end of the region $U(a,t) = U_{xx}(a,t) = 0$ and at the right end of the region $U_x(b,t) = U_{xx}(b,t) = 0$ to eliminates the parameters $\delta^n_{-2}, \delta^n_{-1}, \delta^n_{N+1}, \delta^n_{N+2}$. An 11-banded matrix system at every time step is solved with Gauss elimination method. Before moving the calculation of the next time step approximation for the time parameters, iteration should be repeated two or three times using the following corrector procedure

$$(\boldsymbol{\delta}^*)^{n+1} = \boldsymbol{\delta}^n + \frac{1}{2}(\boldsymbol{\delta}^{n+1} - \boldsymbol{\delta}^n). \tag{17}$$

To start the iteration of the recurrence relation of the system (16), the initial parameter vector δ^0 must be determined using the following initial and boundary conditions:

$$\begin{aligned} &(U_N)_x(a,0) = 0, \quad (U_N)_x(b,0) = 0, \\ &(U_N)_{xx}(a,0) = 0, \quad (U_N)_{xx}(b,0) = 0, \\ &U_N(x,0) = U(x_m,0), \quad m = 0,\ldots,N. \end{aligned} \tag{18}$$

Once the initial vector of parameters has been calculated, time evaluation of U_N can be determined from the time evolution of the vector $\boldsymbol{\delta}^n$, which is found by solving the recurrence relation (16).

3 Quintic B-spline Galerkin method II (QBGM2)

The Burgers' equation is split for the time variable as

$$\begin{aligned} U_t + 2UU_x &= 0, \\ U_t - 2\nu U_{xx} &= 0. \end{aligned} \tag{19}$$

An application of the Galerkin method to Eqs. (19) with weight functions W produces the weak form

$$\begin{aligned} &\int_a^b W\left(U_t + 2UU_x\right) dx = 0, \\ &\int_a^b W\left(U_t - 2\nu U_{xx}\right) dx = 0. \end{aligned} \tag{20}$$

Replacing the weight function W and the unknown values U by B-spline shape functions (7) and trial solution (8) respectively we obtain an element contribution to the integral equation (20) as

$$\begin{aligned} &\sum_{j=m-2}^{m+3}\left\{\left(\int_0^h Q_iQ_j\,d\xi\right)\overset{\circ}{\delta}_j + 2\sum_{k=m-2}^{m+3}\left[\left(\int_0^h Q_iQ_jQ_k'\,d\xi\right)\delta_k\right]\delta_j\right\}, \\ &\sum_{j=m-2}^{m+3}\left\{\left(\int_0^h Q_iQ_j\,d\xi\right)\overset{\circ}{\delta}_j - 2\nu\left(\int_0^h Q_iQ_j''\,d\xi\right)\delta_j\right\}, \end{aligned} \tag{21}$$

where j and k take only the values $m-2$, $m-1$, m, $m+1$, $m+2$, $m+3$ and $m = 0, 1, \ldots, N-1$, and $^\circ$ denotes derivative with respect to time. This result has the matrix form:

$$\begin{aligned} &A^e\overset{\circ}{\boldsymbol{\delta}}{}^e + 2L^{\mathbf{e}}\boldsymbol{\delta}^e, \\ &A^e\overset{\circ}{\boldsymbol{\delta}}{}^e - 2\nu D^e\boldsymbol{\delta}^e, \end{aligned} \tag{22}$$

where the element matrices A, D are 6×6, the matrix L is $6\times 6\times 6$, $\boldsymbol{\delta}^e = (\delta_{m-2}, \delta_{m-1}, \delta_m, \delta_{m+1}, \delta_{m+2}, \delta_{m+3})$ and

$$A_{i,j} = \int_0^h Q_iQ_j\,d\xi, \quad D = \int_0^h Q_iQ_j''\,d\xi, \quad L_{ijk} = \int_0^h Q_iQ_jQ_k'\,d\xi. \tag{23}$$

We use the associated 6×6 matrix B instead of L in our algorithm:

$$B_{i,j} = \sum_{k=m-2}^{m+3} L_{ijk}\delta^e. \tag{24}$$

The assembly of the element equations (22) leads to the first order matrix equation

$$\mathbf{A}\overset{\circ}{\boldsymbol{\delta}}+\mathbf{2B}\boldsymbol{\delta}=0, \tag{25}$$

$$\mathbf{A}\overset{\circ}{\boldsymbol{\delta}}-\mathbf{2}\boldsymbol{\nu}\mathbf{D}\boldsymbol{\delta}=0, \tag{26}$$

where $\boldsymbol{\delta}=(\delta_{-2},\delta_{-1},\delta_0,\ldots,\delta_{N+1},\delta_{N+2})^T$ is a global element parameter vector and $\mathbf{A},\mathbf{B},\mathbf{D}$ are derived from the corresponding element matrices A^e,B^e,D^e.

To discretize the above system, the time parameter vector $\boldsymbol{\delta}$ in the equation is interpolated using the Crank-Nicolson approximation and the time derivative parameters vector $\overset{\circ}{\boldsymbol{\delta}}$ in the equation is interpolated using the usual finite difference approximation between times n and $n+1/2$ as follows:

$$\delta_m=\frac{\delta_m^n+\delta_m^{n+1/2}}{4},\quad \overset{\circ}{\delta}_m=\frac{\delta_m^{n+1/2}-\delta_m^n}{\Delta t}\ . \tag{27}$$

We will have the following recurrence relationship:

$$\left(2\mathbf{A}+\Delta t\mathbf{B}\right)\boldsymbol{\delta}^{n+1/2}=\left(2\mathbf{A}-\Delta t\mathbf{B}\right)\boldsymbol{\delta}^n. \tag{28}$$

Similarly, Eq. (26) is discretized by applying the Crank-Nicolson method for the time parameters vector $\boldsymbol{\delta}$ and a difference approximation for the time derivatives vector $\overset{\circ}{\boldsymbol{\delta}}$ between the times $n+1/2$ and $n+1$ as follows:

$$\delta_m=\frac{\delta_m^{n+1/2}+\delta_m^{n+1}}{4},\quad \overset{\circ}{\delta}_m=\frac{\delta_m^{n+1}-\delta_m^{n+1/2}}{\Delta t}, \tag{29}$$

so that we will have the recurrence relationship

$$\left(2\mathbf{A}-\boldsymbol{\nu}\Delta t\mathbf{D}\right)\boldsymbol{\delta}^{n+1}=\left(2\mathbf{A}+\boldsymbol{\nu}\Delta t\mathbf{D}\right)\boldsymbol{\delta}^{n+1/2}. \tag{30}$$

We have systems (28), (30) having $(N+5)$ equations containing $(N+5)$ unknown parameters $\boldsymbol{\delta}^n=(\delta_{-2}^n,\delta_{-1}^n,\ldots,\delta_{N+2}^n)$. The imposition of the boundary conditions at both ends of the region $U(a,t)=U_{xx}(a,t)=0$ and $U_x(b,t)=U_{xx}(b,t)=0$ allows to eliminate the parameters $\delta_{-2}^n,\delta_{-1}^n,\delta_{N+1}^n,\delta_{N+2}^n$ from the systems (28), (30) so that the solution set becomes an 11 banded $(N+5)\times(N+5)$ matrix equation. This system is solved by the way of Gauss elimination procedure. The time evolution can be found by first implementing the recurrence relationship (28) to find the element parameters $\boldsymbol{\delta}^{n+1/2}$ from $\boldsymbol{\delta}^n$ and the second recurrence relationship to find $\boldsymbol{\delta}^{n+1}$ from $\boldsymbol{\delta}^{n+1/2}$. So the time evolution of the time parameters and the nodal values from the equations are determined by the above mentioned iteration procedure after finding initial parameters δ^0 as in the previous section.

4 Numerical examples

The two test problems are studied in order to demonstrate the robustness and numerical accuracy of the proposed methods. Accuracy is measured by using L_∞ and L_2 error norms

$$\begin{aligned} L_\infty &= |U - U_N|_\infty = \max_j |U_j - (U_N)_j^n|, \\ L_2 &= |U - U_N|^2 = h\sum_{j=0}^{N} |(U_j - (U_N)_j^n)^2| \end{aligned}$$

and $|e|_1$ error norms

$$|e|_1 = \frac{1}{N}\sum_{i=1}^{N-1} \frac{\left|U_j - (U_N)_j^n\right|}{|U_j|}.$$

L_2 and L_∞ error norms are used for numerical example 1 and comparison is made with results of the paper [2]. We used the $|e|_1$-norm for the example 2 to make the comparison with a result of the paper [9].

(a) Burgers' equation has the following form of the analytical solution.

$$U(x,t) = \frac{\frac{x}{t}}{1 + \sqrt{\frac{t}{t_0}}\exp(\frac{x^2}{4\nu t})}, \quad t \geq 1,\ 0 \leq x \leq 1, \tag{31}$$

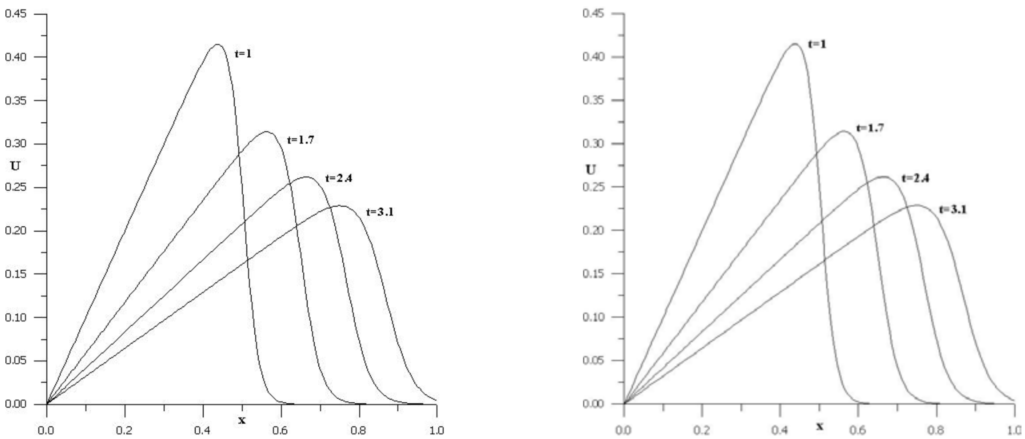
where $t_0 = \exp(\frac{1}{8\nu})$. The propagation of the shock is represented with the equation above. The initial shock which is taken when $t = 1$ in Eq. (31) will be observed as time progresses. To make comparison with earlier study [2], computation is done with parameters $\nu = 0.005$, $h = 0.005$ and $\Delta t = 0.01$ over the problem domain $[0, 1]$. Table 1 is a comparison of the exact solution with numerical values of both schemes. Comparisons are presented at time $t = 1.7, 2.4$ and 3.1 only. The accuracy in the L_2 norm obtained is measured as 2.9 $\times 10^{-5}$ at time $t = 1.7$, 2.5 $\times 10^{-5}$ at time $t = 2.4$ and 1.5 $\times 10^{-4}$ at time $t = 3.1$ for the QBGM1. When the same method is applied to the split Burgers' equation, an error norm is obtained such as 3.5 $\times 10^{-4}$ at time $t = 1.7$, 2.4 $\times 10^{-4}$ at time $t = 2.4$ and 1.9 $\times 10^{-4}$ at time $t = 3.1$ for the QBGM2, especially error becomes larger at early times. In the same table, a comparison with the collocation method using cubic splines [2] shows that the proposed methods provide a little better results for the L_2 and L_∞ error norms.

The propagation of the shock is visualized at some times in the Figs. (1), from which it is seen that the initial shock becomes steadier as the program runs. At

Table 1: Comparison of results at different times for $\nu = 0.005$ with $h = 0.005$ and $\Delta t = 0.01$

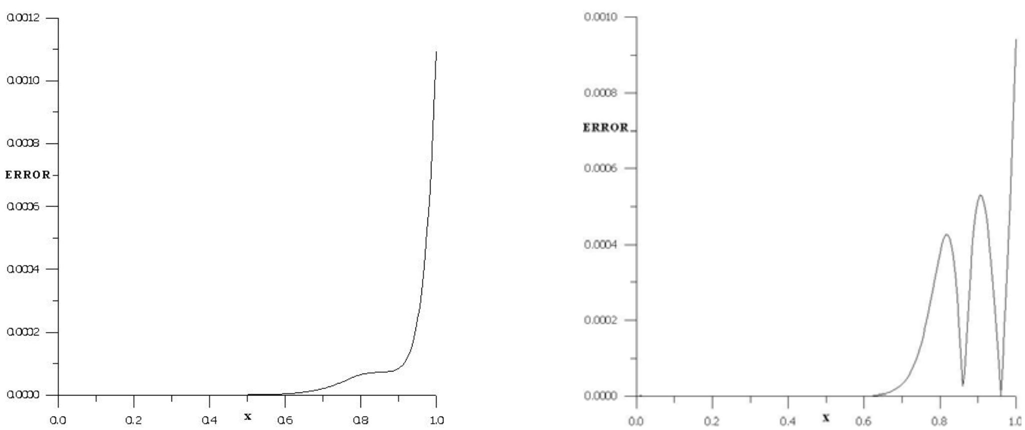
x	$t = 1.7$ QBGM1	$t = 1.7$ QBGM2	$t = 1.7$ Exact	$t = 2.4$ QBGM1	$t = 2.4$ QBGM2	$t = 2.4$ Exact
0.1	.058823	.058822	.058823	.041666	.041666	.041666
0.2	.117645	.117644	.117645	.083332	.083331	.083332
0.3	.176458	.176458	.176458	.124995	.124995	.124995
0.4	.235166	.235170	.235168	.166640	.166639	.166640
0.5	.291875	.291907	.291904	.208111	.208115	.208114
0.6	.295812	.294973	.295910	.247396	.247402	.247417
0.7	.041931	.042949	.041929	.252093	.251668	.252172
0.8	.000648	.000669	.000646	.072996	.073817	.073025
0.9	.000005	.000005	.000005	.003023	.003115	.003023
$L_2 \times 10^3$	0.02900	0.35001		.02581	0.24430	
$L_2 \times 10^3$([2])	0.857			0.423		
$L_\infty \times 10^3$	0.11314	1.21155		.07877	0.80766	
$L_\infty \times 10^3$([2])	2.576			1.242		

x	$t = 3.1$ QBGM1	$t = 3.1$ QBGM2	$t = 3.1$ Exact
0.1	.032258	.032257	.032258
0.2	.064515	.064515	.064515
0.3	.096771	.096771	.096771
0.4	.129021	.129021	.129021
0.5	.161230	.161231	.161231
0.6	.193123	.193127	.193127
0.7	.221847	.221836	.221867
0.8	.215071	.214756	.215135
0.9	.070789	.071390	.070874
$L_2 \times 10^3$	0.15713	0.19296	
$L_2 \times 10^3$([2])	0.235		
$L_\infty \times 10^3$	1.09575	0.94251	
$L_\infty \times 10^3$([2])	0.688		



a) QBGM1 b) QBGM2

Figure 1: $\nu = 0.005,\ h = 0.005,\ \Delta t = 0.01$



a) QBGM1 b) QBCM2

Figure 2: Error (|numerical − analytic solution|) at time $t = 3.1$ with $\nu = 0.005$

time $t = 3.1$, the error distribution is drawn over the domain in the Figs. 2 and there appears to be the highest error about the right-hand boundary position.

(b) Secondly we consider the Burgers' equation with initial condition

$$U(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1, \tag{32}$$

and boundary conditions

$$U(0, t) = U(1, t) = 0, \quad t \geq 0. \tag{33}$$

The exact solution was found in terms of infinite series by Cole [6]

$$U(x,t) = \frac{4\pi\nu\sum_{j=1}^{\infty} jI_j(\frac{1}{2\pi\nu})\sin(j\pi x)\exp(-j^2\pi^2\nu t)}{I_0(\frac{1}{2\pi\nu}) + 2\sum_{j=1}^{\infty} I_j(\frac{1}{2\pi\nu})\cos(j\pi x)\exp(-j^2\pi^2\nu t)} \quad (34)$$

where I_j are the modified Bessel functions. This problem gives the decay of sinusoidal disturbance.

At first, computation is carried out with parameters: viscosity constant $\nu = 1$, time step $\Delta t = 0.00001$ and various space steps at time $t = 0.1$. A comparison of the results for different space steps obtained by the present methods with exact solutions together with error norm is shown in Tables 2–3. The numerical solutions are seen to be satisfactory with the exact solutions. Errors in terms of the $|e|_1$ norm are also documented in the same tables. The numerical results for various values of the viscosity are documented in Tables 4–5. Agreement between both numerical schemes and exact values appears very satisfactory through illustration in Figs. 3a–4a. As it is known that the exact solutions for $\nu < 10^{-2}$ are not practical because of the slow convergence of the infinite series, so the numerical solutions are not compared with the exact solution in the Tables 4–5. Our numerical results for $\nu = 10^{-4}$ confirms the numerical results obtained by studies [2] and depicted in the Figs. 3b–4b. Numerical results for $\nu = 10^{-4}$ demonstrate a very sharp front near the left boundary at earlier times and as time progresses, sharpness and amplitude

Table 2: Comparison of results at $t = 0.1$ for $\nu = 1$, $\Delta t = 0.00001$ and various mesh sizes (QBGM1)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$	Exact
0.1	0.10301	0.10817	0.10915	0.10947	0.10953	0.10954
0.2	0.20623	0.20831	0.20941	0.20973	0.20978	0.20979
0.3	0.28729	0.29053	0.29156	0.29184	0.29189	0.29190
0.4	0.34471	0.34677	0.34765	0.34788	0.34792	0.34792
0.5	0.36849	0.37065	0.37136	0.37154	0.37157	0.37158
0.6	0.35690	0.35835	0.35888	0.35902	0.35904	0.35905
0.7	0.30819	0.30942	0.30979	0.30989	0.30990	0.30991
0.8	0.22680	0.22751	0.22775	0.22781	0.22782	0.22782
0.9	0.12013	0.12054	0.12065	0.12068	0.12069	0.12069
$\|e\|_1$	0.01306	0.00362	0.00094	0.00016	0.00002	
$\|e\|_1$([9])	0.012165	0.006941	0.003651	0.001858	0.000928	

Table 3: Comparison of results at $t = 0.1$ for $\nu = 1$, $\Delta t = 0.00001$ and various mesh sizes (QBGM2)

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	$h = 0.00625$	Exact
0.1	0.10260	0.10787	0.10895	0.10942	0.10952	0.10954
0.2	0.20599	0.20798	0.20921	0.20968	0.20978	0.20979
0.3	0.28699	0.29024	0.29139	0.29180	0.29188	0.29190
0.4	0.34448	0.34652	0.34750	0.34784	0.34791	0.34792
0.5	0.36828	0.37045	0.37125	0.37151	0.37157	0.37158
0.6	0.35675	0.35820	0.35880	0.35900	0.35904	0.35905
0.7	0.30807	0.30931	0.30973	0.30987	0.30990	0.30991
0.8	0.22673	0.22744	0.22771	0.22780	0.22781	0.22782
0.9	0.12009	0.12051	0.12064	0.12068	0.12068	0.12069
$\|e\|_1$	0.01392	0.00441	0.00142	0.00028	0.00005	

Table 4: Comparison of results at different times (QBGM1) $\nu = 1.0$, 0.1 and 0.01 with $h = 0.0125$ and $\Delta t = 0.0001$.

		$\nu = 1$	$\nu = 1$	$\nu = 0.1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 0.01$	$\nu = 10^{-4}$
x	t	Numer.	Exact	Numer.	Exact	Numer.	Exact	Numer.
0.25	0.4	0.01357	0.01357	0.30885	0.30889	0.34188	0.34191	0.34481
	0.6	0.00189	0.00189	0.24070	0.24074	0.26889	0.26896	0.27103
	0.8	0.00026	0.00026	0.19564	0.19568	0.22139	0.22148	0.22300
	1.0	0.00004	0.00004	0.16253	0.16256	0.18809	0.18819	0.18935
	3.0	0.00000	0.00000	0.02719	0.02720	0.07504	0.07511	0.07533
0.50	0.4	0.01923	0.01924	0.56961	0.56963	0.66071	0.66071	0.66787
	0.6	0.00267	0.00267	0.44718	0.44721	0.52941	0.52942	0.53425
	0.8	0.00037	0.00037	0.35921	0.35924	0.43913	0.43914	0.44255
	1.0	0.00005	0.00005	0.29189	0.29192	0.37440	0.37442	0.37694
	3.0	0.00000	0.00000	0.04019	0.04021	0.15013	0.15018	0.15062
0.75	0.4	0.01363	0.01363	0.62543	0.62544	0.91026	0.91026	0.92809
	0.6	0.00189	0.00189	0.48720	0.48721	0.76724	0.76724	0.77739
	0.8	0.00026	0.00026	0.37390	0.37392	0.64739	0.64740	0.65385
	1.0	0.00004	0.00004	0.28745	0.28747	0.55605	0.55605	0.56054
	3.0	0.00000	0.00000	0.02976	0.02977	0.22478	0.22481	0.22582

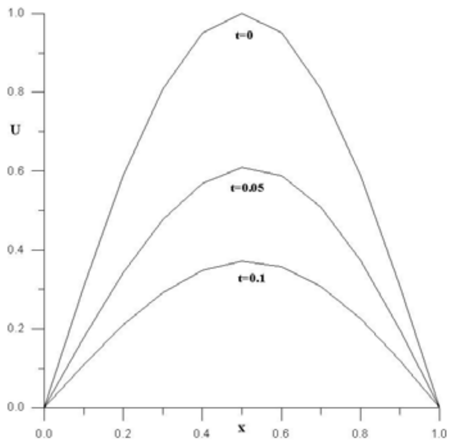


Fig. 3a: $\nu = 1,\ h = 0.1,\ \Delta t = 0.01$ (QBGM1)

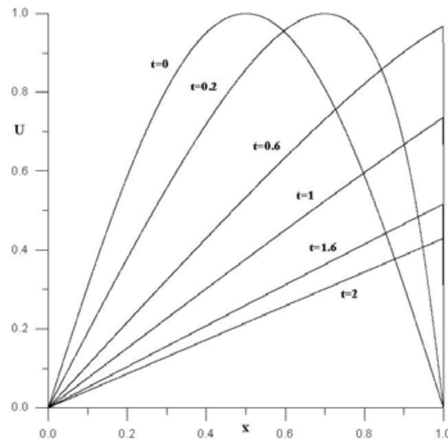


Fig. 3b: $\nu = h = \Delta t = 10^{-4}$ (QBGM1)

Table 5: Comparison of results at different times (QBGM2) $\nu = 1.0,\ 0.1$ and 0.01 with $h = 0.0125$ and $\Delta t = 0.0001$.

x	t	$\nu=1$ Numer.	$\nu=1$ Exact	$\nu=0.1$ Numer.	$\nu=0.1$ Exact	$\nu=0.01$ Numer.	$\nu=0.01$ Exact	$\nu=10^{-4}$ Numer.
0.25	0.4	0.01357	0.01357	0.30882	0.30889	0.34187	0.34191	0.34481
	0.6	0.00189	0.00189	0.24066	0.24074	0.26887	0.26896	0.27103
	0.8	0.00026	0.00026	0.19561	0.19568	0.22136	0.22148	0.22300
	1.0	0.00004	0.00004	0.16250	0.16256	0.18806	0.18819	0.18935
	3.0	0.00000	0.00000	0.02718	0.02720	0.07502	0.07511	0.07533
0.50	0.4	0.01923	0.01924	0.56960	0.56963	0.66071	0.66071	0.66787
	0.6	0.00267	0.00267	0.44716	0.44721	0.52941	0.52942	0.53425
	0.8	0.00037	0.00037	0.35919	0.35924	0.43912	0.43914	0.44255
	1.0	0.00005	0.00005	0.29186	0.29192	0.37439	0.37442	0.37694
	3.0	0.00000	0.00000	0.04018	0.04021	0.15011	0.15018	0.15062
0.75	0.4	0.01363	0.01363	0.62540	0.62544	0.91027	0.91026	0.92809
	0.6	0.00189	0.00189	0.48717	0.48721	0.76724	0.76724	0.77739
	0.8	0.00026	0.00026	0.37387	0.37392	0.64740	0.64740	0.65385
	1.0	0.00004	0.00004	0.28742	0.28747	0.55605	0.55605	0.56054
	3.0	0.00000	0.00000	0.02976	0.02977	0.22476	0.22481	0.22582

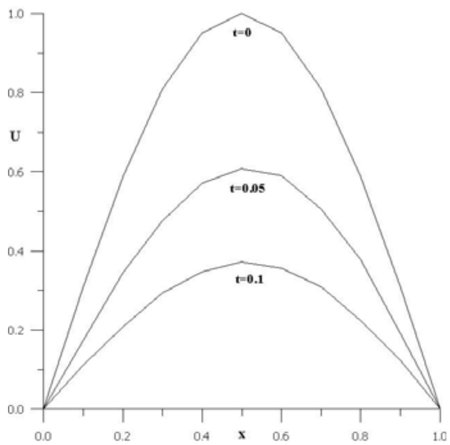


Fig. 4a: $\nu = 1$, $h = 0.1$, $\Delta t = 0.01$ (QBGM2)

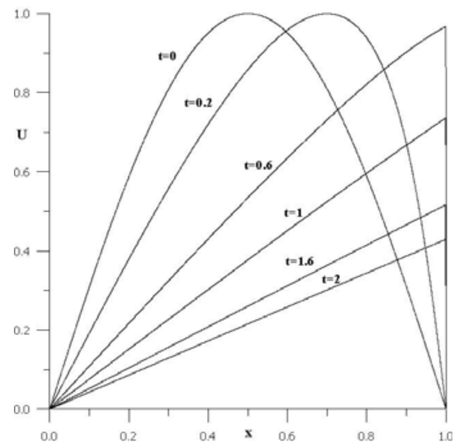


Fig. 4b: $\nu = h = \Delta t = 10^{-4}$ (QBGM2)

of the wave front start to decay. These properties of the numerical solutions from the QBGM1 and QBGM2 are in very good agreement with the finding obtained by Varoğlu and Finn [7], Kakuda and Tosaka [8].

The numerical algorithm based on Galerkin method with quintic B-splines as weight and trial functions is constructed for both Burgers' and the time-split Burgers' equation. The numerical methods appear to be capable of producing numerical solutions of high accuracy for the solution of the Burgers' equation. We have also found that there is not much effect of the time-splitting of Burgers' equation on getting the numerical solutions of the Burgers' equation for the quintic B-spline finite element method. The experimental results of both schemes are much more satisfactory in comparison with the previous results [2, 4, 9, 5]. So it can be concluded that the quintic B-spline finite element methods is both efficient and reliable for getting the numerical solutions of the partial differential equations.

References

[1] Prenter P. M., *Splines and Variational Methods*, J. Wiley, New York, 1975.

[2] Ali A. H. A., Gardner L. R. T. and Gardner G. A., *A Galerkin Approach to the Solution of Burgers' Equation*, U.C.N.W. Maths Pre-print, 90.04, 1990.

[3] Zaki S. I., *Solitary waves of the Korteweg-de-Vries-Burgers' equation*, Comput. Phys. Commun. **126** (2000), 207–218.

[4] Zaki S. I., *A quintic B-spline finite elements scheme for the KdVB equation*, Comput. Methods Appl. Engrg., **188** (2000), 121–134.

[5] Dağ I., Irk D. and Saka B., *A numerical solution of the Burgers' equation using cubic B-splines*, Appl. Math. Comput. (in press)

[6] Cole J. D., *On a quasi-linear parabolic equation occurring in aerodynamics*, Quart. Appl. Math., **9** (1951), 225–236.

[7] Varoğlu E. and Finn W. D. L., *Space-time finite elements incorporating characteristics for the Burgers' equation*, Int. J. Numer. Methods Eng., **16** (1980), 171–184.

[8] Kakuda K. and Tosaka N., *The generalised boundary element approach to Burgers' equation*, Int. J. Num. Meth. Eng., **29** (1990), 245–261.

[9] Kutluay S., Esen A. and Dağ I., *The numerical solutions of the Burgers' equation by least squares quadratic B-spline element method*, J. Comput. Appl. Math., **167** (2004), 21–33.