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Asymptotic Properties of Differential Systems with Unbounded Delays

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Abstract

The paper presents the asymptotic estimates of all solutions of the vector differential equation

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\tau_j(t)), \quad t \in I = [t_0, \infty)$$

with continuous real matrices A, B_j and unbounded lags. Assuming that the equation

$$\dot{y}(t) = A(t)y(t)$$

is uniformly asymptotic stable, we derive the asymptotic bounds valid for all solutions y of the considered delay differential system. These estimates are formulated by means of a solution of an auxiliary scalar functional equation and inequality.

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1 Introduction and preliminaries

In this paper, we discuss the asymptotic properties of the vector delay differential equation

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\tau_j(t)), \quad t \in I = [t_0, \infty), \tag{1.1}$$

where A, B_j are $n \times n$ matrices with real and continuous entries and τ_j are real, continuous and increasing functions on I such that $\tau_j(t) < t$ and $\tau_j(t) \to \infty$ as

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$t \to \infty$ $(j = 1, \ldots, m)$. Some additional requirements on A, B_j and τ_j will be imposed later. Particularly, the main result we formulate for equations (1.1) with unbounded lags, *i.e.*, such that $t - \tau_j(t) \to \infty$ as $t \to \infty$ $(j = 1, \ldots, m)$.

Various special cases of equation (1.1) with unbounded lags have been studied because of numerous interesting applications as well as the specific qualitative properties. The special attention was paid to equations with proportional time delays, particularly to the equation

$$\dot{y}(t) = Ay(t) + By(\lambda t), \qquad 0 < \lambda < 1, \qquad t \geq 0, \tag{1.2}$$

where A, B are constant real or complex matrices. Equation (1.2) is usually referred to as the pantograph equation because of its relevance to the study of the motion of the pantograph head of an electric locomotive (see [12]). The theoretical study of the pantograph equation can be found in papers [7, 9, 13] and many others.

The qualitative investigation of (1.2) has been extended also to some more general cases. Iserles [5] studied the qualitative properties of the vector neutral equation

$$\dot{y}(t) = Ay(t) + By(\lambda t) + C\dot{y}(\lambda t), \qquad 0 < \lambda < 1, \qquad t \geq 0.$$

Using the idea of expansion of solutions into Dirichlet series form many results known from the "pure" delayed case have been extended to this neutral case. This approach turned out to be very effective especially in the study of autonomous linear differential equations involving proportionally delayed argument (for other related results see, *e.g.*, [6] or [10]). However, considering these equations with variable coefficients and/or with a general form of a delayed argument, it is often very difficult to apply this technique (see [4, 11] or [2, 3]).

In this paper, we utilize proof methods based on some results of the transformation theory of delay differential equations and some results of the theory of functional equations to describe the asymptotic properties of all solutions of (1.1). The main results are formulated in Section 3, where we derive the asymptotic estimates of all solutions of (1.1) in terms of a solution of some scalar functional equation and inequality. These results generalize several earlier asymptotic estimates due to Lim [9], Pandolfi [13], Makay and Terjéki [11] and some others. We also present several illustrating examples.

Let $t_{-1} := \min\{\tau_j(t_0),\, j = 1, 2 \ldots, m\}$ and $I^* := [t_{-1}, \infty)$. By a solution of (1.1) we understand a real valued function $y \in C(I^*) \cap C^1(I)$ such that y satisfies (1.1) on I. Under the above mentioned assumptions on A, B_j, τ_j there exists a unique solution y of (1.1) coinciding with a given initial function on the initial interval $[t_{-1}, t_0]$. We note that all of the results of this paper remain valid if we admit $\tau_j(t_0) = t_0$ for some (or all) $j = 1, \ldots, m$.

We define the norm of a vector $y = (y_1, \dots, y_n)$ as $|y| = \max\{|y_i|,\ i = 1, \dots, n\}$ and the norm of a matrix A as $\|A\| = \sup\{|Ay|,\ y \in \mathbb{R}^n,\ |y| = 1\}$.

2 Some auxiliary results

We consider the linear functional inequality

$$-a\omega(t) + \sum_{j=1}^{m} b_j(t)\omega(\tau_j(t)) \le 0, \qquad t \in I, \tag{2.1}$$

where $a > 0$ is a real constant, b_j, τ_j are continuously differentiable functions on I such that $b_j(t) \ge 0$, $\sum_{j=1}^{m} b_j(t) > 0$, τ_j are increasing on I, $\tau_j(t) < t$ for every $t \in I$ and $\tau_j(t) \to \infty$ as $t \to \infty$ $(j = 1, \dots, m)$.

Set $t_{-1} := \min\{\tau_j(t_0),\ j = 1, \dots, m\}$, $t_k := \sup\{s,\ \tau_j(s) < t_{k-1}$ for all $j = 1, \dots, m\}$, $k = 1, 2, \dots$ and let $I_k = [t_{k-1}, t_k]$, $k = 0, 1, \dots$. Then $I = \cup_{k=1}^{\infty} I_k$ and $\tau_j(I_{k+1}) \subset \cup_{i=0}^{k} I_i$ for any $j = 1, \dots, m$.

Now let ω_0 be a positive and continuously differentiable function on I_0 and let

$$\omega_0(t_0) = \sum_{j=1}^{m} \frac{b_j(t_0)}{a}\omega_0(\tau_j(t_0)),$$

$$\dot{\omega}_0(t_0) = \sum_{j=1}^{m} \left(\frac{\dot{b}_j(t_0)}{a}\omega_0(\tau_j(t_0)) + \frac{b_j(t_0)}{a}\dot{\omega}_0(\tau_j(t_0))\dot{\tau}_j(t_0) \right).$$

If we put

$$\omega(t) = \sum_{j=1}^{m} \frac{b_j(t)}{a}\omega_0(\tau_j(t)) \qquad \text{for all } t \in I_1,$$

and inductively

$$\omega(t) = \sum_{j=1}^{m} \frac{b_j(t)}{a}\omega(\tau_j(t)) \qquad \text{for all } t \in I_k,$$

$k = 2, 3, \dots$, it is easy to verify that ω defines the positive and continuously differentiable solution of (2.1).

Thus we have

Proposition 2.1 *Consider the inequality* (2.1) *and let the above introduced assumptions on a, b_j and τ_j be fulfilled. Then there exists a positive and continuously differentiable solution ω of* (2.1).

Remark 2.2 Notice that we derived the function ω fulfilling relation (2.1) in the form of an equality.

To obtain a more applicable form of a solution of (2.1), we consider the following additional assumptions on τ_j: $\dot\tau_j$ are positive and nonincreasing on I, $\tau_i\circ\tau_j=\tau_j\circ\tau_i$ on I for any pair $i,j=1,\dots,m$ and let $\tau_j(t_0)=t_0$. Then, by [1], there exists a common solution φ to the system of the simultaneous functional equations

$$\varphi(\tau_j(t))=\lambda_j\varphi(t), \qquad \lambda_j=\dot\tau_j(t_0)<1, \qquad t\in I, \tag{2.2}$$

$j=1,\dots,m$, which is positive on (t_0,∞) and has a continuous, positive and bounded derivative on I.

The verification of the following assertion is now easy.

Proposition 2.3 *Consider the inequality* (2.1), *where $a>0$ is a constant, b_j are nonnegative functions fulfilling $b_j(t)\le\beta_j$ for every $t\in I$, τ_j are continuously differentiable functions on I such that $\tau_j(t)<t$ for every $t>t_0$, $\tau_j(t_0)=t_0$, $\tau_j(t)\to\infty$ as $t\to\infty$, $\dot\tau_j$ are positive and nonincreasing on I $(j=1,\dots,m)$ and let $\tau_i\circ\tau_j=\tau_j\circ\tau_i$ on I for any pair $i,j=1,\dots,m$. Then the function $\omega(t)=(\varphi(t))^\alpha$, where φ is a solution of* (2.2) *with the above described properties and α is a (unique) real root of*

$$\sum_{j=1}^{m}\beta_j\lambda_j^\alpha=a\,,$$

defines a solution of (2.1) *which is positive and continuously differentiable on* (t_0,∞).

We recall that the functional equation of the form

$$\varphi(\tau(t))=\lambda\varphi(t), \qquad \lambda=\dot\tau(t_0)<1, \qquad t\in I, \tag{2.3}$$

is usually referred to as the Schröder equation. It is known (see, *e.g.*, [8]) that if τ has a continuous and positive derivative on I, $\tau(t)<t$ for every $t\in I$ and $\tau(t)\to\infty$ as $t\to\infty$, then there exists a positive and unbounded solution φ of (2.3) with a continuous and positive derivative on I. Moreover, this solution is unique up to a multiplicative constant and has a bounded derivative on I provided $\tau(t_0)=t_0$ and $\dot\tau$ is nonincreasing on I.

We note that the assumptions on τ ensuring the existence of a solution of (2.3) with given properties can be formulated in a weaker form. However, our formulation is more suitable with the respect to its utilizing in the next section.

3 Asymptotic estimates of solutions

In this section, we present the asymptotic estimate of all solutions of equation (1.1). Throughout this section we assume that the equation

$$\dot{y}(t) = A(t)y(t) \tag{3.1}$$

is uniformly asymptotic stable, *i.e.*, there exist real constants a, $L > 0$ such that

$$\|Y(u,v)\| \le L\exp\{a(v-u)\} \qquad \text{for all } u \ge v \ge t_0\,, \tag{3.2}$$

where $Y(u,v)$ is the evolution matrix of (3.1).

Theorem 3.1 *Consider equation* (1.1)*, where $A \in C(I)$, $\|B_j\|$, $\tau_j \in C^1(I)$, $\tau_j(t) < t$ for every $t \in I$, $\tau_j(t) \to \infty$ as $t \to \infty$ and τ_j are increasing on I $(j = 1,\dots,m)$. Further, assume that there exists $\tau \in C^1(I)$ such that $\tau(t) \ge \max\{\tau_j(t), j = 1,\dots,m\}$, $\tau(t) < t$ for every $t \in I$, $\dot{\tau}$ is positive on I and $\dot{\tau}(t_0) < 1$. Let φ be a positive solution of* (2.3) *with a continuous and positive derivative on I and let a, L be given by* (3.2)*. If $\sum_{j=1}^m \|B_j(t_0)\| \ge a$ and $\sum_{j=1}^m \|B_j\|$ is nondecreasing on I, then the estimate*

$$y(t) = O\left(\omega(t)(\varphi(t))^{\log L/\log\lambda^{-1}}\right) \qquad \text{as } t \to \infty, \tag{3.3}$$

where $\omega \in C^1(I)$ is a positive solution of (2.1) *with $b_j(t) = \|B_j(t)\|$, holds for any solution y of* (1.1)*.*

Proof. First note that the solution ω of (2.1) can be chosen as nondecreasing on I. Set

$$s = \log\varphi(t), \qquad z(s) = \frac{y(t)}{\omega(t)}$$

in (1.1) to obtain

$$\begin{aligned} z'(s) = {} & \left(A(h(s))h'(s) - \frac{\omega'(h(s))h'(s)}{\omega(h(s))}I\right)z(s) \\ & + \sum_{j=1}^m B_j(h(s))h'(s)\frac{\omega(\tau_j(h(s)))}{\omega(h(s))}z(\xi_j(s)), \end{aligned} \tag{3.4}$$

where $h(s) = \varphi^{-1}(\exp\{s\})$, $\xi_j(s) = \log\varphi(\tau_j(h(s))) \le s + \log\lambda$, " $'$ " stands for the derivative with the respect to s and I is the identity matrix.

If y is a solution of (1.1) defined on an interval I, then z is a solution of (3.4) defined on $[q_0,\infty)$, where $q_0 = \log\varphi(t_0)$. Put $q_{-1} := \log\varphi(t_{-1})$, where $t_{-1} :=$

$\min\{\tau_j(t_0),\ j=1,\dots,m\}$ and let us define $q_k := q_0 + k\log\lambda^{-1}$, $k=1,2,\dots$, $J_k := [q_{k-1},q_k]$ and $M_k := \sup\{|z(s)|,\ s\in\cup_{i=0}^{k}J_i\}$, $k=0,1,\dots$. Then $[q_0,\infty)=\cup_{i=1}^{\infty}J_i$ and $\xi_j(J_{k+1})\subset\cup_{i=0}^{k}J_i$ for any $j=1,\dots,m$.

Now we consider the system

$$z'(s) = (A(h(s))h'(s) - \frac{\omega'(h(s))h'(s)}{\omega(h(s))}I)z(s) \tag{3.5}$$

and let H be the evolution matrix of (3.5). Then

$$\|H(s,p)\| = \frac{\omega(h(p))}{\omega(h(s))}\|Y(h(s),h(p))\| \le L\frac{\omega(h(p))}{\omega(h(s))}\exp\{-a(h(s)-h(p))\}\,. \tag{3.6}$$

Considering arbitrary $s\in J_{k+1}$ and integrating (3.4), we have

$$z(s) = H(s,q_k)z(q_k) + \int_{q_k}^{s}\sum_{j=1}^{m}H(s,u)B_j(h(u))h'(u)\frac{\omega(\tau_j(h(u)))}{\omega(h(u))}z(\xi_j(u))\,du\,.$$

Using (3.6), we obtain the estimate

$$\begin{aligned}|z(s)| \le{}& LM_k\Big(\frac{\omega(h(q_k))}{\omega(h(s))}\exp\{-a(h(s)-h(q_k))\}\\ &+\int_{q_k}^{s}\sum_{j=1}^{m}\|B_j(h(u))\|h'(u)\frac{\omega(\tau_j(h(u)))}{\omega(h(s))}\exp\{-a(h(s)-h(u))\}\,du\Big)\,,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{|z(s)|}{L} \le{}& M_k\Big(\frac{\omega(h(q_k))}{\omega(h(s))}\exp\{-a(h(s)-h(q_k))\}\\ &+\int_{q_k}^{s}\frac{\omega(h(u))}{\omega(h(s))}\exp\{-a(h(s)-h(u))\}ah'(u)\,du\Big)\,.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{|z(s)|}{L} \le{}& M_k\Big(\frac{\omega(h(q_k))}{\omega(h(s))}\exp\{-a(h(s)-h(q_k))\}\\ &+\frac{\exp\{-ah(s)\}}{\omega(h(s))}\int_{q_k}^{s}\omega(h(u))\frac{d}{du}[\exp\{ah(u)\}]\,du\Big)\,.\end{aligned} \tag{3.7}$$

Integrating by parts and using the fact that ω is nondecreasing, we can estimate the last integral as

$$\int_{q_k}^{s}\omega(h(u))\frac{d}{du}[\exp\{ah(u)\}]\,du \le \omega(h(u))\exp\{ah(u)\}\Big|_{q_k}^{s}\,.$$

Substituting this back into (3.7), we have $|z(s)| \le LM_k$ for any $s \in J_{k+1}$, *i.e.*,

$$z(s) = O(L^{s/\log \lambda^{-1}}) \qquad \text{as } s \to \infty\,.$$

Then

$$y(t) = z(s)\omega(t) = z(\log\varphi(t))\omega(t) = O(\omega(t)(\varphi(t))^{\log L/\log \lambda^{-1}}) \qquad \text{as } t \to \infty. \ \square$$

Example 3.2 To illustrate the previous result we consider equation (1.1) under the assumptions of Theorem 3.1, where particularly we assume that $\max\{\tau_j(t), j = 1,\dots,m\} \le \lambda t$ for a suitable $0 < \lambda < 1$ and every $t \in I$. Further, let $\|B_j(t)\| = b_j t^\gamma$, $b_j > 0$ $(j = 1,\dots,m)$, $\gamma \ge 0$ are reals and let a, L be given by (3.2). Then it is easy to check that the function

$$\omega(t) = t^{\frac{\log(b/a)+\frac{\gamma}{2}\log t}{\log\lambda^{-1}}+\frac{\gamma}{2}},$$

where $b = \sum_{j=1}^m b_j$, defines the required solution of (2.1) for all t large enough and $\varphi(t) = t$ is a solution of the corresponding equation (2.3). Consequently, substituting this ω and φ into (3.3), we obtain the effective asymptotic formula valid for all solutions of (1.1).

In the sequel we consider the case when either the requirement $\sum_{j=1}^m \|B_j\|$ nondecreasing or $\sum_{j=1}^m \|B_j(t_0)\| \ge a$ is not valid. First we show that in such a case the previous asymptotic estimate is not true.

Example 3.3 We consider the scalar equation with constant coefficients and a constant delay of the form

$$\dot{y}(t) = \frac{1-2\exp\{2\}}{2\exp\{2\}-2}y(t) + \frac{1}{2\exp\{3\}-2\exp\{1\}}y(t-1)\,, \qquad t \in I\,. \tag{3.8}$$

Then, by (3.3), we have the estimate $y(t) = O(\exp\{\beta t\})$ as $t \to \infty$, where $\beta = -1-\log(2\exp\{2\}-1) < -1$. However, equation (3.8) admits the solution $y(t) = \exp\{-t\}$, which contradicts the asymptotic property (3.3). Notice that the assumption $\sum_{j=1}^m \|B_j(t_0)\| \ge a$ was not fulfilled in the case of (3.8).

In the next theorem we show that estimate (3.3) holds (also if $\sum_{j=1}^m \|B_j(t_0)\| < a$) for a wide class of equations (1.1) with unbounded lags.

Theorem 3.4 *Consider equation* (1.1)*, where* A, $B_j \in C(I)$, $\tau_j \in C^1(I)$, $\tau_j(t) < t$ *for every* $t > t_0$, $\tau_j(t_0) = t_0$, $\tau_j(t) \to \infty$ *as* $t \to \infty$, $\dot{\tau}_j$ *are positive and nonincreasing on* I, $\dot{\tau}_j(t_0) < 1$ $(j = 1,\dots,m)$ *and* $\tau_i \circ \tau_j = \tau_j \circ \tau_i$ *on* I *for any pair* $i, j = 1,\dots,m$.

Further, let φ be a solution of (2.2) *with a continuous and positive derivative and let a, L be given by* (3.2). *If $\|B_j(t)\| \le Q_j a$ for suitable reals Q_j $(j = 1, \ldots, m)$ and every $t \in I$, then the estimate*

$$y(t) = O\left((\varphi(t))^{\alpha + \log L / \log \lambda^{-1}}\right) \qquad \text{as } t \to \infty,$$

where α is a (unique) real root of $\sum_{j=1}^m Q_j \lambda_j^\alpha = 1$, holds for any solution y of (1.1).

Proof. It is easy to verify that the function $\omega(t) = (\varphi(t))^\alpha$ defines a solution of the inequality (2.1), where $b_j(t) = \|B_j(t)\|$ (see also Proposition 2.3). Indeed,

$$\sum_{j=1}^m b_j(t)\omega(\tau_j(t)) \le \sum_{j=1}^m Q_j a(\varphi(\tau_j(t)))^\alpha = \sum_{j=1}^m Q_j a \lambda_j^\alpha (\varphi(t))^\alpha = a\omega(t)\,.$$

Similarly as in the proof of Theorem 3.1 we can put

$$s = \log \varphi(t), \qquad z(s) = \frac{y(t)}{\omega(t)}$$

in (1.1) and obtain (3.4). Following this proof, we derive using the same line of argument inequality (3.7). To estimate the integral

$$\int_{q_k}^s \omega(h(u)) \frac{d}{du}[\exp\{ah(u)\}]\, du = \int_{q_k}^s \exp\{\alpha u\} \frac{d}{du}[\exp\{ah(u)\}]\, du$$

occurring in (3.7), we must now proceed in a different way (ω can be decreasing provided $\sum_{j=1}^m Q_j > 1$). Before we carry out this estimate, we note that differentiating (2.2) we can check that $\dot\varphi(t)$ is bounded as $t \to \infty$. Then it holds

$$\frac{1}{h'(s)} = \frac{\dot\varphi(h(s))}{\varphi(h(s))} = O(\exp\{-s\}) \qquad \text{as } s \to \infty\,.$$

Using this property, we have the following estimates:

$$\begin{aligned}
&\int_{q_k}^s \exp\{\alpha u\} \frac{d}{du}[\exp\{ah(u)\}]\, du \\
&\le \exp\{\alpha u + ah(u)\}\Big|_{q_k}^s + |\alpha| \int_{q_k}^s \exp\{\alpha u + ah(u)\}\, du \\
&\le \exp\{\alpha u + ah(u)\}\Big|_{q_k}^s + \int_{q_k}^s \frac{|\alpha|}{\alpha + ah'(u)} \frac{d}{du}[\exp\{\alpha u + ah(u)\}]\, du \\
&\le (1 + O(\exp\{-q_k\})) \exp\{\alpha u + ah(u)\}\Big|_{q_k}^s \quad \text{as } k \to \infty.
\end{aligned}$$

Substituting this back into (3.7), we get

$$\frac{|z(s)|}{L} \le M_k(1 + O(\exp\{-q_k\})) \qquad \text{as } k \to \infty$$

for any $s \in J_{k+1}$, *i.e.*,

$$z(s) = O(L^{s/\log \lambda^{-1}}) \qquad \text{as } s \to \infty\,.$$

From here we get

$$y(t) = z(s)\omega(t) = z(\log \varphi(t)(\varphi(t))^{\alpha} = O((\varphi(t))^{\alpha+\log L/\log \lambda^{-1}}) \qquad \text{as } t \to \infty. \ \square$$

If we put $m = 1$ and $\tau(t) = \lambda t$ in Theorem 3.4, then $\varphi(t) = t$ and we have

Corollary 3.5 *Let y be a solution of the equation*

$$\dot{y}(t) = A(t)y(t) + B(t)y(\lambda t), \qquad 0 < \lambda < 1, \qquad t \ge 0,$$

where A, $B \in C(I)$ and let a, L be given by (3.2). *If $\|B(t)\| \le Qa$ for a suitable real $Q > 0$ and every $t \ge 0$, then*

$$y(t) = O\left(t^{\alpha+\log L/\log \lambda^{-1}}\right) \qquad \text{as } t \to \infty, \tag{3.9}$$

where $\alpha = \log Q/(\log \lambda^{-1})$.

We note that estimate (3.9) was derived in [13, Theorem 2]. Hence, Theorem 3.4 generalizes this asymptotic result.

If we put $n = 1$, $A(t) \equiv -a$ and $\tau_j(t) = t^{\gamma_j}$ $(j = 1, \dots, m)$ in Theorem 3.4, then $L = 1$, $\varphi(t) = \log t$ and we have

Corollary 3.6 *Consider the scalar equation*

$$\dot{y}(t) = -ay(t) + \sum_{j=1}^{m} b_j(t)y(t^{\gamma_j}), \qquad 0 < \gamma_j < 1, \qquad t \ge 1, \tag{3.10}$$

where $a > 0$ is a real constant, $b_j \in C(I)$ and $|b_j(t)| \le Q_j a$ for suitable reals Q_j and every $t \ge 1$ $(j = 1, \dots, m)$. Then for any solution y of (3.10) *it holds*

$$y(t) = O\left((\log t)^{\alpha}\right) \qquad \text{as } t \to \infty,$$

where α is a (unique) real root of $\sum_{j=1}^{m} Q_j \gamma_j^{\alpha} = 1$.

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