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Artin-Mazur Zeta Function on Trees with Infinite Edges

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Abstract

We generalize for trees, with infinite edges and finite branching points, the Milnor-Thurston's main relationship between kneading determinant and Artin-Mazur zeta function of a piecewise monotone interval map.

1 Introduction and statement of the main result

One of the extremely useful tools for studying the periodic structure of a dynamical system was introduced by Artin and Mazur [4]. Let X be an arbitrary set and $f : X \to X$ a map. Suppose that each iterate $f^n = f \circ f \circ \cdots \circ f$ (n times) has only finitely many fixed points, then one defines the Artin-Mazur zeta function of f, $\zeta_f(t)$, to be the formal power series

$$\zeta_f(t) = \exp \sum_{n \geq 1} \frac{\#\mathrm{Fix}(f^n)}{n} t^n,$$

where $\mathrm{Fix}(f^n)$ denotes the set of all fixed points of f^n. The Artin-Mazur zeta function is a convenient way for enumerating all periodic orbits of f. Recall that an orbit $o = \{f^n(x) : n \geq 0\}$ is periodic, with period $\infty > \mathrm{per}(o) \geq 1$, if $\mathrm{per}(o)$ is the smallest positive integer verifying $f^{\mathrm{per}(o)}(x) = x$. If O denotes the set of all periodic orbits of f, then

$$\zeta_f(t)^{-1} = \prod_{o \in O} (1 - t^{\mathrm{per}(o)})$$

holds in $\mathbb{Z}[[t]]$ (ring of formal power series in the indeterminate t with integer coefficients).

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Later on, several variants of this notion were introduced by different authors (see [7]). In the general problem of computing $\zeta_f(z)$, Milnor and Thurston [8] showed that, for any expanding piecewise monotone interval map, $\zeta_f(t)$ can be computed in terms of its kneading data. Recently generalizations of this result for finite trees were obtained [2, 5]. Our goal in this paper is to present a nontrivial generalization of the same result for trees with finitely many branching points and infinite edges.

Let us begin to introduce the basic definitions. Let X be a metric space and $x \in X$. By valence of x we mean the number (not necessarily finite) $\mathrm{val}(x)$ of all arcwise connected components of $X \setminus \{x\}$. A unique arcwise connected metric space T is a tree if there exists a finite set $B(T) \subset T$ such that: 1) the closure of each arcwise connected component of $T \backslash B(T)$ is homeomorphic to the interval $[0,1]$; 2) for any $x \in T$, we have $\mathrm{val}(x) > 2$ if and only if $x \in B(T)$. If T is a tree, the elements of $B(T)$ are called the branching points of T, and the arcwise connected components of $T \backslash B(T)$ are called the edges of T. A point $x \in T$ is an end of T if $\mathrm{val}(x) = 1$. The set of all ends of T will be denoted by $E(T)$. Notice that, if T is a tree the set $B(T)$ is finite, however the set $E(T)$ and the set of all edges of T may be infinite.

Let T be a tree. A continuous map $f : T \to T$ is a piecewise monotone tree map (shortly PMT map) if there exists a finite set $C \subset T$ satisfying: 1) f is injective in each arcwise connected components of $T \backslash C$; 2) if T_i is a edge of T, then $T_i \cap C$ is a finite set; 3) the set $f(C)$ is finite. Notice that if $f : T \to T$ is a PMT map, then each of its iterates, f^n, is again a PMT.

Example 1 The disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$, with an appropriate metric, can be regarded as a tree with infinite edges. Indeed, the metric space $T = (D, d)$, where the distance $d : D \times D \to \mathbb{R}$ is defined by

$$d(z_1, z_2) = \begin{cases} |z_2 - z_1| & \text{if } \arg(z_1) = \arg(z_2), \\ |z_1| + |z_2| & \text{otherwise,} \end{cases}$$

is a tree with a single branching point at 0, infinite ends, $E(T) = \{z \in \mathbb{C} : |z| = 1\}$, and infinite edges; every edge of T has the form

$$T_\alpha = \{z \in T : 0 < |z| \leq 1 \text{ and } \arg(z) = \alpha\}, \text{ with } \alpha \in [0, 2\pi[\,.$$

It is easy to characterize the piecewise monotone maps of this tree in terms of its critical points. Let $f : T \to T$ be a continuous map. If $c \in T$, we say that c is a critical point of f if $c = 0$ or the restriction of $|f(z)|$ to $T_{\arg(c)}$ has a maximum or a minimum at c. Denoting the set of all critical points of f by C_f, we see at once that f is piecewise monotone if and only if the sets $f(C_f)$ and $C_f \cap T_\alpha$ are finite for all $\alpha \in [0, 2\pi[$.

Let T be a tree and $f : T \to T$ a PMT map. For any x, $y \in T$, denote by $\langle x, y \rangle$ the smallest arcwise connected subset of T which contains $\{x, y\}$. A fixed point x of f^n is called expanding if there exists a neighborhood $V \subset T$ of x such that: $f^n(y) \notin \langle x, y \rangle$ for all $y \in V \backslash \{x\}$. The map f is called expanding if all fixed points of f^n are expanding, for all n.

Notice that, if $f : I \to I$ is an expanding piecewise monotone interval map, then the sets $\mathrm{Fix}(f^n)$ are finite for all n. However, this need not be true for expanding PMT maps in general. In fact, if $f : T \to T$ is an expanding PMT map, the sets $\mathrm{Fix}(f^n) \cap (T \backslash E(T))$ are finite, but the sets $\mathrm{Fix}(f^n)$ may be infinite. This shows that, in particular, if every periodic orbit of f intersects $T \backslash E(T)$, then $\zeta_f(t)$ is defined.

The definition of kneading determinant, $\mathbf{D}_f(t)$, of a PMT map will be given in the next section (see also [1]), and the following result shows the importance of this determinant in the computation of $\zeta_f(t)$.

Theorem 2 *Let $f : T \to T$ be an expanding PMT map such that $o \not\subset E(T)$ and $o \not\subset B(T)$, for every periodic orbit $o \in O$ of f. Then $\zeta_f(t)\, \mathbf{D}_f(t)$ is a polynomial that can be computed in terms of the orbits of the branching points of T.*

Example 3 Let $T = (D, d)$ be the tree of Example 1. For each $b \in D$ and $\alpha \in \mathbb{R}$, let $f : T \to T$ be the expanding PMT map defined by

$$f(z) = \begin{cases} (2|z| - 1)\, e^{i(\arg(z) + \alpha 2\pi)} & \text{if } 1/2 \leq |z| \leq 1, \\ b\, |1 - |6|z| - 2|| & \text{if } 0 \leq |z| \leq 1/2. \end{cases}$$

We have $z \in C_f$ if and only if $|z| \in \{0, 1/6, 1/3, 1/2\}$, and $f(C_f) = \{0, b\}$. Following the definitions given in the next section, the kneading determinant of f, $\mathbf{D}_f(t)$, is a formal power series (in the indeterminate t) that can be computed in terms of the orbit of b. More precisely,

$$\mathbf{D}_f(t) = 1 - \omega(b)t - \sum_{n \geq 1} \left[\prod_{i=0}^{n-1} \epsilon(f^i(b)) \right] \omega(f^n(b)) t^{n+1}, \tag{1}$$

where the maps $\epsilon : D \to \mathbb{Z}$ and $\omega : D \to \mathbb{Z}$ are defined by:

$$\epsilon(z) = \begin{cases} 1 & \text{if } |z| \in \left]1/6, 1/3\right[\cup \left]1/2, 1\right], \\ -1 & \text{if } |z| \in \left]1/3, 1/2\right[\cup \left]0, 1/6\right[, \\ 0 & \text{if } |z| \in C_f; \end{cases} \qquad \omega(z) = \begin{cases} 3 & \text{if } |z| \in \left]1/3, 1\right], \\ 2 & \text{if } |z| = 1/3, \\ 1 & \text{if } |z| \in \left]0, 1/3\right[, \\ 0 & \text{if } z = 0. \end{cases} \tag{2}$$

Notice that, if $1/6 < |b| \leq 1$ and $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, then f is under the conditions of Theorem 2, and therefore $\zeta_f(t)$ can be easily computed in terms of the orbit of b.

As an example, let $b=1$ and α be any irrational number. In this case the orbit of b is infinite but $|f^n(b)|=1$, for all $n\geq 1$. Therefore $\epsilon(f^n(b))=1$ and $\omega(f^n(b))=3$, for all $n\geq 1$, and from Theorem 2 it follows that

$$\zeta_f(t)=\mathbf{D}_f(t)^{-1}=\frac{1-t}{1-4t}.$$

2 Proof of Theorem 2

The proof of Theorem 2 will be given in three main steps. In the first one we will give the general definition of kneading determinant of a PMT map. In the second one we will state a general result which establishes a main relationship between the kneading determinant and a modified zeta function, $\zeta^{MT}(z)$, called Milnor-Thurston zeta function. Finally we will prove a more general version of Theorem 2, valid for any expanding PMT map.

Let us begin by noticing that the definition of critical point given in the previous example is easily extendable to the general class of piecewise monotone tree maps. By an oriented tree we mean a tree T, where each edge T_i (which by definition is homeomorphic to an interval $I\subset\mathbb{R}$) is equipped with a linear ordering $\geq_i$ satisfying the following: there exists a homeomorphism φ_i, from T_i into $I\subset\mathbb{R}$, that satisfies $y\geq_i x$ if and only if $\varphi_i(y)\geq\varphi_i(x)$, for all $x,y\in T_i$. If T is an oriented tree, a piecewise monotone map $f:T\to T$ induces a sign map $\epsilon:T\to\{-1,0,1\}$ in a natural way: if there exist edges T_i and T_j such that $x\in T_i$ and $f(x)\in T_j$, define $\epsilon(x)=\pm1$ according to whether the restriction $f_{|T_i}$ is increasing or decreasing in a small neighborhood of x; for the remaining points of T put $\epsilon(x)=0$.

Obviously, the definition of ϵ depends upon the orientation of T, while the set $C_f=\{x\in T:\epsilon(x)=0\}$ does not. So, for any PMT map $f:T\to T$ the critical set C_f it is well defined, furthermore, as an immediate consequence of the definitions, the sets $f(C_f)$ and $C_f\cap T_i$ are finite for all edge T_i.

Next we define the kneading determinant of a PMT map, as a formal power series that can be computed in terms of the orbits of the points of $f(C_f)$. Let T be an oriented tree, and $f:T\to T$ a PMT map. For each $c\in C_f$ consider the set of sets $\mathcal{C}_c$ defined by $X\in\mathcal{C}_c$ if and only if $X=\{c\}$ or X is an arcwise connected component of $T\setminus\{c\}$. For each $X\in\mathcal{C}_c$, define

$$\nu(X)=\#(X\cap B(T)) \text{ and } \epsilon(X)=\lim_{\substack{x\to c\\ x\in X}}\epsilon(x).$$

Using these notations, we define the map $\psi_c:T\to\mathbb{Q}$ by

$$\psi_c(x)=-\epsilon(X)+[\#B(T)]^{-1}\sum_{Y\in\mathcal{C}_c}\epsilon(Y)\nu(Y),$$

for all $x \in X$ and $X \in \mathcal{C}_c$. Let d_1,..., d_k be the elements of $f(C_f)$.The $k \times k$-matrix $\mathbf{M}(t) = [\mathbf{m}_{i,j}(t)]$, with entries in $\mathbb{Q}[[t]]$ given by

$$\mathbf{m}_{i,j}(t) = \omega_i\left(d_j\right) + \sum_{n\geq 1}\left[\prod_{k=0}^{n-1}\epsilon\left(f^k(d_j)\right)\right]\omega_i\left(f^n(d_j)\right)t^n,$$

where the map $\omega_i : T \to \mathbb{Q}$ is defined by[1]

$$\omega_i = \sum_{c\in f^{-1}(d_i)\cap C_f}\psi_c,$$

is called the kneading matrix of f. Denoting the $k \times k$-identity matrix by $\mathbf{I}$, we define the kneading determinant of f to be the formal power series given by

$$\mathbf{D}_f(t) = \det(\mathbf{I} - t\mathbf{M}(t)).$$

Example 4 Let f be the map of Example 3. For any w and z lying in a same edge, T_α, of T, define $w \geq_\alpha z$ if and only if $|w| \geq |z|$. With this orientation on T, the sign map $\epsilon : T \to \{-1, 0, 1\}$ induced by f is given by (2). We have then $f(C_f) = \{d_1, d_2\}$, with $d_1 = 0$ and $d_2 = b$, furthermore $C_f \cap f^{-1}(d_1) = \{z \in \mathbb{C} : |z| \in \{1/6, 1/2\}\}$ and $C_f \cap f^{-1}(d_2) = \{z \in \mathbb{C} : |z| \in \{0, 1/3\}\}$. Using this we obtain

$$\omega_1(z) = \begin{cases} -4 & \text{if } |z| \in \,]1/2, 1[\,, \\ -3 & \text{if } |z| = 1/2, \\ -2 & \text{if } |z| \in \,]1/6, 1/2[\,, \\ -1 & \text{if } |z| = 1/6, \\ 0 & \text{if } |z| \in [0, 1/6[\,, \end{cases} \quad \text{and} \quad \omega_2(z) = \omega(z),$$

where ω is the map defined in (2). Thus, because $\epsilon(0) = \omega_1(0) = \omega(0) = 0$, the kneading matrix of f will be given by

$$\mathbf{M}(t) = \begin{bmatrix} 0 & \omega_1\left(b\right) + \sum_{n\geq 1}\left[\prod_{k=0}^{n-1}\epsilon\left(f^k(b)\right)\right]\omega_1\left(f^n(b)\right)t^n \\ 0 & \omega\left(b\right) + \sum_{n\geq 1}\left[\prod_{k=0}^{n-1}\epsilon\left(f^k(b)\right)\right]\omega\left(f^n(b)\right)t^n \end{bmatrix},$$

and therefore

$$\begin{aligned} \mathbf{D}_f(t) &= \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - t\mathbf{M}(t)\right) \\ &= 1 - \omega(b)t - \sum_{n\geq 1}\left[\prod_{i=0}^{n-1}\epsilon(f^i\,(b))\right]\omega(f^n\,(b))t^{n+1}, \end{aligned}$$

[1] The set $f^{-1}(d_i)\cap C_f$ may be infinite, nevertheless for each $x \in T$, the set $\{c \in C_f : \psi_c\,(x) \neq 0\}$ is finite, and therefore ω_i is well defined.

as in (1).

The key point in the proof of Theorem 2 is a principal relationship between $\mathbf{D}_f(t)$ and the modified zeta functions $\zeta_f^{MT}(t)$ and $\zeta_f^0(t)$. A proof of this result can be find in ([3]), and it can be stated as follows: let $f : T \to T$ be a PMT map, for each $n \geq 0$, define

$$\mathrm{Fix}_n^-(f) = \left\{ x \in \mathrm{Fix}(f^n) : \prod_{i=0}^{n-1} \epsilon\left(f^i(x)\right) = -1 \right\},$$

$$\mathrm{Fix}_n^0(f) = \left\{ x \in \mathrm{Fix}(f^n) : \prod_{i=0}^{n-1} \epsilon\left(f^i(x)\right) = 0 \right\},$$

and the formal power series[2]

$$\zeta_f^{MT}(t) = \exp \sum_{n\geq 1} \frac{2\mathrm{Fix}_n^-(f) - 1}{n} t^n \quad \text{and} \quad \zeta_f^0(t) = \exp \sum_{n\geq 1} \frac{\mathrm{Fix}_n^0(f)}{n} t^n.$$

Then the identity

$$\zeta_f^{MT}(t)\zeta_f^0(t) = \mathbf{D}_f(t)^{-1} \tag{3}$$

holds in $\mathbb{Z}[[t]]$.

Note that Identity (3) holds for any PMT map. Assume now that f is expanding. In this case there exists a simple relationship between $\zeta_f^{MT}(t)\zeta_f^0(t)$ and $\zeta_f(t)$. To see this we have to introduce some more notations. Let $f : T \to T$ be a PMT map and O_0 the finite set of all periodic orbits of f that intersect C_f, that is

$$\begin{aligned} O_0 &= \{o \in O : o \cap C_f \neq \varnothing\} \\ &= \left\{ o \in O : \epsilon(x) \ldots \epsilon\left(f^{\mathrm{per}(o)-1}(x)\right) = 0, \text{ for all } x \in o \right\}. \end{aligned}$$

Put $P = \cup_{o\in O_0} o$ and consider the set $\mathbf{P}$ defined by $(x, X) \in \mathbf{P}$ if and only if $x \in P$ and X is an arcwise connected component of $T\setminus\{x\}$. Notice that, since $f(P) = P$, f induces a map $F : \mathbf{P} \to \mathbf{P}$, defined as follows: $F(x, X) = (y, Y)$ if and only if $f(x) = y$ and there exists a neighborhood $V \subset T$ of x such that $f(V \cap X) \subset Y$. Notice that the set $\mathbf{P}$ may be infinite, nevertheless, if f is expanding, then each iterate F^n has finitely many fixed points[3]. Furthermore, if every periodic orbit of

[2] Notice that $\mathrm{Fix}(f^n)$ may be infinite, however, as f is a PMT map, $\mathrm{Fix}_n^-(f)$ and $\mathrm{Fix}_n^0(f)$ are finite for all $n \geq 0$. So, the formal power series $\zeta_f^{MT}(t)$ and $\zeta_f^0(t)$ are defined for all PMT map.

[3] Indeed, if f is expanding and $F^n(x, X) = (x, X)$, then X intersects the finite set $f^n(C_{f^n})$. Thus, since P is a finite set, it follows that $\mathrm{Fix}(F^n)$ is also finite.

f intersects $T \backslash E(T)$, then the equality[4]

$$\#\mathrm{Fix}(f^n) = 2\#\mathrm{Fix}_n^-(f) + 2\#\mathrm{Fix}_n^0(f) - \#\mathrm{Fix}(F^n) - 1$$

holds for any $n \geq 1$. Thus we may write:

$$\zeta_f(t) = \zeta_f^{MT}(t)\zeta_f^0(t)^2\zeta_F(t)^{-1},$$

and by (3) we arrive at

Theorem 5 *Let $f : T \to T$ be an expanding PMT map. If every periodic orbit of f intersects $T \backslash E(T)$, then $\zeta_f(t) = \zeta_f^0(t)\zeta_F(t)^{-1}\mathbf{D}_f(t)^{-1}$ holds in $\mathbb{Z}[[t]]$.*

Theorem 2 is a corollary of the previous result. To prove it, we introduce the following decomposition of O_0.

Definition 6 Let $f : T \to T$ be a PMT map and $o \in O_0$. The periodic orbit o lies in B_+ (resp. B_-) if and only if o intersects simultaneously $B(T)$ and $T \backslash (B(T) \cup E(T))$, and $f^{\mathrm{per}(o)}$ is preserving (resp. reversing) orientation at any $x \in o \backslash (B(T) \cup E(T))$. Define $B_0 = O_0 \backslash (B_- \cup B_+)$.

Proposition 7 *Let $f : T \to T$ be a PMT map such that $o \not\subset B(T)$, for all $o \in O_0$. Then $\zeta_f^0(t)\zeta_F(t)^{-1}$ is a polynomial, more precisely*

$$\zeta_f^0(t)\zeta_F(t)^{-1} = \prod_{o \in B_-} \left(1 + t^{\mathrm{per}(o)}\right)\left(1 - t^{2\mathrm{per}(o)}\right) \prod_{o \in B_+} \left(1 - t^{\mathrm{per}(o)}\right)$$

holds in $\mathbb{Z}[[t]]$.

Proof. Denote by $\mathbf{O}$ the set of all periodic orbits of F. By definition of $\mathbf{O}$ and O_0, we may write

$$\zeta_f^0(t) = \prod_{o \in O_0} \left(1 - t^{\mathrm{per}(o)}\right)^{-1} \quad \text{and} \quad \zeta_F(t)^{-1} = \prod_{\mathbf{o} \in \mathbb{O}} \left(1 - t^{\mathrm{per}(\mathbf{o})}\right).$$

Let $\mathbf{o} = \{(x_1, X_1), (x_2, X_2), ..., (x_p, X_p)\}$ be a periodic orbit of F. By definition of F, the set

$$\Phi(\mathbf{o}) \stackrel{def}{=} \{x_1, x_2, \ldots, x_p\}$$

[4] A similar equality on finite trees can be find in [2]. For infinite trees the proof uses the same arguments.

lies in O_0, and therefore we may consider a map

$$\begin{array}{rccc} \Phi: & \mathbf{O} & \to & O_0 \\ & \mathbf{o} & \to & \Phi(\mathbf{o}). \end{array}$$

Let $o \in O_0$ and $(x,X) \in \mathbf{P}$ with $x \in o$. Notice that, since $o \not\subset B(T)$, the orbit $\{F^n(x,X) : n \geq 0\}$ is finite, and therefore it contains a periodic orbit $\mathbf{o}$ of F, which evidently satisfies $\Phi(\mathbf{o}) = o$. This shows that the map Φ is onto, therefore $\mathbf{O} = \cup_{o\in O_0} \Phi^{-1}(o)$, and consequently

$$\zeta_f^0(t)\zeta_F(t)^{-1} = \prod_{o\in O_0}\left[\prod_{\mathbf{o}\in\Phi^{-1}(o)}\left(1-t^{\mathrm{per}(o)}\right)^{-1}\left(1-t^{\mathrm{per}(\mathbf{o})}\right)\right].$$

By Definition 6 we also have $O_0 = B_- \cup B_0 \cup B_+$ and: 1) if $o \in B_0$, then $\Phi^{-1}(o) = \{\mathbf{o}\}$ and $\mathrm{per}(\mathbf{o}) = \mathrm{per}(o)$; 2) if $o \in B_+$, then $\Phi^{-1}(o) = \{\mathbf{o}_1, \mathbf{o}_2\}$ and $\mathrm{per}(\mathbf{o}_1) = \mathrm{per}(\mathbf{o}_2) = \mathrm{per}(o)$; 3) if $o \in B_-$, then $\Phi^{-1}(o) = \{\mathbf{o}_1, \mathbf{o}_2\}$ and $\mathrm{per}(\mathbf{o}_1) = \mathrm{per}(\mathbf{o}_2) = 2\mathrm{per}(o)$. Thus

$$\begin{aligned} \zeta_f^0(t)\zeta_F(t)^{-1} &= \prod_{o\in B_-}\left(1-t^{\mathrm{per}(o)}\right)^{-1}\left(1-t^{2\mathrm{per}(o)}\right)^{2} \\ &\quad \prod_{o\in B_+}\left(1-t^{\mathrm{per}(o)}\right)^{-1}\left(1-t^{\mathrm{per}(o)}\right)^{2} \\ &= \prod_{o\in B_-}\left(1+t^{\mathrm{per}(o)}\right)\left(1-t^{2\mathrm{per}(o)}\right)\prod_{o\in B_+}\left(1-t^{\mathrm{per}(o)}\right), \end{aligned}$$

and because B_- and B_+ are finite, it follows that $\zeta_f^0(t)\zeta_F(t)^{-1}$ is a polynomial, as desired. ∎

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