King Fahd University of Petroleum and Minerals

Department of Mathematics

Math 102 Sem II (062)

Second Major Exam Sun 6 / 5 / 2007 Time $1\frac{1}{2}$ hours

Name: Key I.D.#: ____________ Serial #: ______

Section 7 16 19

Answer all the questions

Show all of your work

Question #	1	2	3	4	5	6	7	8	9	10	11	12	13	Total
Grade														/50

1. $\int_1^3 \frac{x}{(x-2)^2}\,dx = \int_1^2 \frac{x}{(x-2)^2}\,dx + \int_2^3 \frac{x}{(x-2)^2}\,dx$ (4 *points*)

$\int_1^2 \frac{x}{(x-2)^2}\,dx = \lim_{t\to 2^-} \int_1^t \frac{x}{(x-2)^2}\,dx$

let $u = x-2$, $x = u+2$, $dx = du$

$x = 1 \Rightarrow u = -1$,

$x = 0 \Rightarrow u = 0$

$= \lim_{t\to 0^+} \int_{-1}^{t} \frac{u+2}{u^2}\,du = \lim_{t\to 0^+}\left[\int_{-1}^{t} \frac{1}{u}\,du + \int_{-1}^{t} \frac{2}{u^2}\,du\right]$

$= \lim_{t\to 0^+}\left[\ln|t| - \ln|-1| - \frac{2}{t} + \frac{2}{-1}\right]$ divergent

So $\int_1^3 \frac{x}{(x-2)^2}\,dx$ is divergent

2. $\int_1^3 \frac{\tan^{-1}\sqrt{X}}{\sqrt{X}}\,dx$ let $w=\sqrt{x} \Rightarrow x=w^2$, $dx=2w\,dw$ (4 *points*)

$x=1 \Rightarrow w=1$

$x=3 \Rightarrow w=\sqrt{3}$

$$=\int_1^{\sqrt{3}} \frac{\tan^{-1}w}{\cancel{w}} \cdot 2\cancel{w}\,dw = \int_{+1}^{\sqrt{3}} 2\tan^{-1}w\,dw$$

Use By parts! let $u=\tan^{-1}w$, $dv=2\,dw$

$du=\frac{1}{1+w^2}\,dw$, $v=2w$

$$=\int_{+1}^{\sqrt{3}} 2\tan^{-1}w\,dw = 2w\tan^{-1}w - \int \frac{2w\,dw}{1+w^2}\Big]_{+1}^{\sqrt{3}}$$

$$=\left[2w\tan^{-1}w - \ln|1+w^2|\right]_{+1}^{\sqrt{3}}$$

$$=2\sqrt{3}\tan^{-1}\sqrt{3} - \ln|4| - 2\tan^{-1}1 + \ln|2|$$

$$=2\sqrt{3}\cdot\frac{\pi}{3} - \ln 2 - \cancel{2}\cdot\frac{\pi}{\cancel{4}\,2}$$

3. $\int \frac{1}{\sqrt{X+X^{\frac{3}{2}}}}\,dx$, $x^{\frac{1}{2}}=t$, $x=t^2$, $dx=2t\,dt$ (4 *points*)

$$=\int \frac{2t\,dt}{\sqrt{t^2+t^3}} = \int \frac{2\cancel{t}\,dt}{\cancel{t}\sqrt{1+t}} = 2\cdot\frac{(1+t)^{1/2}}{1/2} + C$$

$$=4\sqrt{1+t}+C = 4\sqrt{1+\sqrt{x}}+C$$

4. Only one of the following is FALSE: (3 *points*)

a. The average value of the function $f(x) = x^2$ on $[-2,2]$ is equal to $\frac{4}{3}$

$$\frac{1}{4}\frac{x^3}{3}\Big]_{-2}^{2} = \frac{1}{4}\left[\frac{8}{3}+\frac{8}{3}\right] = \frac{16}{12} = \frac{4}{3}$$

b. $\frac{x(x^2+4)}{x^2-4}$ can be put in the form $\frac{A}{x-2}+\frac{B}{x+2}$

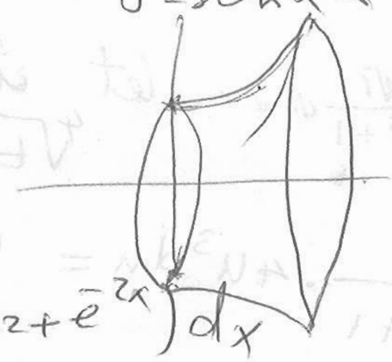
c. $\int_1^\infty \frac{1}{x^{\sqrt{2}}}\,dx$ is convergent

d. The following identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ is correct

e. The integral $\int \sqrt{1-x^2}\,dx$ can be solved by assuming $x = \sin\theta$

5. Find the surface area generated if the curve $y = \cosh x$, $0 \le x \le 1$ is revolving about the $x-$ *axis*. (4 *points*)

$y = \cosh x,\quad y' = \sinh x$



$$A = 2\pi\int_0^1 \cosh x\sqrt{1+\sinh^2 x}\,dx$$

$$= 2\pi\int_0^1 \cosh^2 x\,dx$$

$$= 2\pi\int_0^1 \left(\frac{e^x+e^{-x}}{2}\right)^2 dx = \frac{2\pi}{4}\int_0^1 \left(e^{2x}+2+e^{-2x}\right)dx$$

$$= \frac{\pi}{2}\left[\frac{e^{2x}}{2}+2x-\frac{e^{-2x}}{2}\right]_0^1 = \frac{\pi}{2}\left[\frac{e^2}{2}+2-\frac{e^{-2}}{2}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}\right]$$

6. $\int \frac{2x^2 - x + 2}{x(x^2+1)}\,dx =$ Use partial Fraction (4 points)

$$\frac{2x^2-x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+x(Bx+C)}{x(x^2+1)}$$

$$2x^2 - x + 2 = A(x^2+1) + x(Bx+C)$$

$x=0$: $\boxed{2 = A}$

Coeff. of x^2 : $2 = A + B \Rightarrow B = 0$

Coeff. of x : $\boxed{-1 = C}$

$$\int \frac{2x^2-x+2}{x(x^2+1)}\,dx = \int \frac{2}{x}\,dx - \int \frac{1}{x^2+1}\,dx$$

$$= 2\ln|x| - \tan^{-1}x + C$$

7. $\int \frac{\sqrt[4]{t}}{\sqrt{t}+1}\,dt =$ let $u = t^{4}$, $dt = 4u^3\,du$ (4 points)

$\sqrt[4]{t} = u$, $\sqrt{t} = u^2$

$$= \int \frac{u}{u^2+1}\cdot 4u^3\,du = \int \frac{4u^4}{u^2+1}\,du$$ Use long Division

$$u^2+1 \overline{\smash{)}\,4u^4} \quad 4u^2 - 4 \quad \mp 4u^4 \mp 4u^2 \quad -4u^2 \quad \pm 4u^2 \pm 4 \quad 4$$

$$= \int \left[4u^2 - 4 + \frac{4}{u^2+1}\right] du$$

$$= 4\frac{u^3}{3} - 4u + 4\tan^{-1}u + C$$

$$= \frac{4}{3}t^{3/4} - 4t^{1/4} + 4\tan^{-1}t^{1/4} + C$$

8. $\int \frac{\cos x}{\sin^2 x+\sin x-2}\,dx$ $\quad u=\sin x,\ du=\cos x\,dx$ (4 *points*)

$$=\int \frac{du}{u^2+u-2} = \int \frac{\cancel{du}}{(u+2)(u-1)}\,du$$

$$\frac{1}{(u+2)(u-1)} = \frac{A}{u+2}+\frac{B}{u-1} = \frac{A(u-1)+B(u+2)}{(u+2)(u-1)}$$

$$1=A(u-1)+B(u+2)$$

$$u=1;\ 1=3B \Longrightarrow \boxed{B=\frac{1}{3}}$$

$$u=-2;\ 1=-3A \Longrightarrow A=\frac{-1}{3}$$

$$\int\frac{du}{(u+2)(u-1)} = \frac{1}{3}\int\frac{du}{u-1}-\frac{1}{3}\int\frac{du}{u+2} = \frac{1}{3}\ln\left|\frac{u-1}{u+2}\right|+C$$

$$=\frac{1}{3}\ln\left|\frac{\sin x-1}{\sin x+2}\right|+C$$

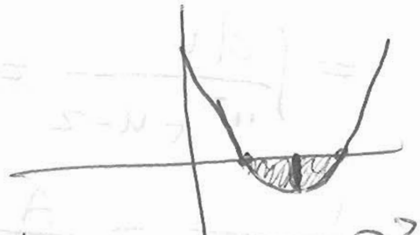
9. Set up the integral (do not evaluate) to find the volume of the solid (**by shell method**) genarated if the region bounded by $y=\ln x$, $y=0$, and $x=e$ is revolving about $x-axis$. (3 *points*)

$$V=2\pi\int_0^1 y\,(e-e^{y})\,dy$$

10. Find the volume of the solid genarated if the region bounded by $y = x^2 - 3x + 2$, and $y = 0$ is revolving about $y-axis.$ (4 *points*)

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0 \Rightarrow x = 1 \text{ or } x = 2$

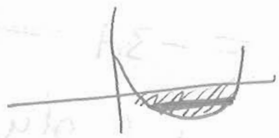


By shell Method:

$$V = 2\pi \int_1^2 x \cdot (-x^2 + 3x - 2)\, dx = 2\pi \left[\frac{-x^4}{4} + \frac{3x^3}{3} - \frac{2x^2}{2} \right]_1^2$$

$$= 2\pi \left[\frac{-1}{4}(16-1) + (8-1) - (4-1) \right]$$

$$= 2\pi \left[\frac{-15+16}{4} \right] = \frac{\pi}{2}$$



By Disc Method

$$y = \left(x^2 - 3x + \frac{9}{4}\right) + 2 - \frac{9}{4} = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$$

$$V = \pi \int_{-1}^{0} \left[\left(\sqrt{y + \tfrac{1}{4}} + \tfrac{3}{2}\right)^2 - \left(-\sqrt{y + \tfrac{1}{4}} + \tfrac{3}{2}\right)^2 \right] dy$$

$x - \frac{3}{2} = \pm \sqrt{y + \frac{1}{4}}$

$x = \sqrt{y + \frac{1}{4}} + \frac{3}{2}$

$x = -\sqrt{y + \frac{1}{4}} + \frac{3}{2}$

11. $\displaystyle\int \frac{x}{1 - x^2 + \sqrt{1 - x^2}}\, dx$ (4 *points*)

$t^2 = (1 - x^2), \quad 2t\, dt = -2x\, dx$

$$= \int \frac{\frac{dt}{-2} \cdot 2t}{t^2 + t} = -\int \frac{dt}{t+1} = -\ln|t+1| + C$$

$$= -\ln\left|\sqrt{1 - x^2} + 1\right| + C$$

12. $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$ (4 points)

$t = \sec\theta$; $dt = \sec\theta\tan\theta\,d\theta$

$t=\sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$, $t = 2 \Rightarrow \theta = \frac{\pi}{3}$

$$= \int_{\pi/4}^{\pi/3} \frac{\sec\theta\cancel{\tan\theta}\,d\theta}{\sec^3\theta\cdot\cancel{\tan\theta}} = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\Big]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) + \frac{1}{4}\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}\cdot 1 = \frac{\pi}{24} + \frac{\sqrt{3}-2}{8}$$

13. Find the arclength of the curve $y = e^x$ from $x = 0$ to $x = 1$ (4 *points*)

$$L = \int_0^1 \sqrt{1+e^{2x}}\cdot dx, \quad \text{let } \tan\theta = e^x, \quad \sec^2\theta\, d\theta = e^x dx$$

$$dx = \frac{\sec^2\theta\, d\theta}{\tan\theta}$$

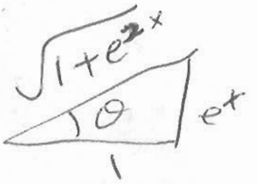
$$\int \sqrt{1+e^{2x}}\, dx = \int \sqrt{1+\tan^2\theta}\cdot\frac{\sec^2\theta\, d\theta}{\tan\theta} = \int \frac{\sec^3\theta\, d\theta}{\tan\theta}$$

$$\int \sec^2\theta\cdot\csc\theta\, d\theta, \quad \text{by Parts; let } u = \csc\theta, \ \frac{dv}{d\theta} = \sec^2\theta\, d\theta$$

$$\cancel{du = \ldots}, \ v = \tan\theta$$

$$= \csc\theta\tan\theta + \int \csc\theta\cot\theta\tan\theta\, d\theta \quad \left(du = -\csc\theta\cot\theta\, d\theta\right.$$

$$= \csc\theta\tan\theta + \ln|\csc\theta - \cot\theta| + \cancel{C}$$



$$= \frac{\sqrt{1+e^{2x}}}{e^x}\cdot\frac{e^x}{1} + \ln\left|\frac{\sqrt{1+e^{2x}}}{e^x} - \frac{1}{e^x}\right| = \sqrt{1+e^{2x}} + \ln\left|\frac{\sqrt{1+e^{2x}}-1}{e^x}\right|$$

$$\int_0^1 \sqrt{1+e^{2x}}\, dx = \sqrt{1+e^{2x}} + \ln\left|\frac{\sqrt{1+e^{2x}}-1}{e^x}\right|\Bigg]_0^1$$

$$= \sqrt{1+e^2} + \ln\left|\frac{\sqrt{1+e^2}-1}{e}\right| - \sqrt{2} - \ln\left|\frac{\sqrt{2}-1}{1}\right|$$