King Fahd University of Petroleum & Minerals
Department of Mathematical Sciences

MATH-533: Complex Variables I
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Second Major: Take-Home

Name: **ID:**

Q1. Show that for $n = 2, 3, 4, 5, ...$

$$S_n := \sin\frac{\pi}{n} \cdot \sin\frac{2\pi}{n} \cdot ... \cdot \sin\frac{(n-2)\pi}{n} \cdot \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

Solution: For $k = 1, ...n-1$ and $0 < \theta_k := \frac{k\pi}{n} < \pi$ we have

$$0 < \sin\theta_k = |\sin\theta_k| = \frac{1}{2}\left|e^{i\theta_k} - e^{-i\theta_k}\right| = \frac{1}{2}\left|1 - e^{2i\theta_k}\right|.$$

Notice that

$$p(z) = z^n - 1 = \prod_{k=0}^{n-1}(z - e^{k\cdot\frac{2\pi i}{n}}) = (z-1)\prod_{k=1}^{n-1}(z - e^{k\cdot\frac{2\pi i}{n}}),$$

hence

$$\begin{aligned}
S_n &= \prod_{k=1}^{n-1}\sin\theta_k & &= \tfrac{1}{2^{n-1}}\prod_{k=1}^{n-1}\left|1 - e^{k\cdot\frac{2\pi i}{n}}\right| \\
&= \tfrac{1}{2^{n-1}}\lim_{z\mapsto 1}\prod_{k=0}^{n-1}\left|z - e^{k\cdot\frac{2\pi i}{n}}\right| & &= \tfrac{1}{2^{n-1}}\lim_{z\mapsto 1}\left|\tfrac{z^n-1}{z-1}\right| \\
&= \tfrac{1}{2^{n-1}}\lim_{z\mapsto 1}\left|z^{n-1} + ... + z + 1\right| & &= \tfrac{n}{2^{n-1}}.\blacksquare
\end{aligned}$$

Q2. Let $\Omega \subseteq \mathbb{C}$ be a region and $f : \Omega \to \mathbb{C}$ be such that the differential of f exists and is different from 0 at $z_0 \in \Omega$. Show that f is conformal at z_0 if and only if

$$\lim_{r\mapsto 0} e^{-i\theta}\frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|},\ r > 0$$

exists and is independent of θ.

Solution: Consult (Rudin: Real and Complex Analysis, 2nd. edition, Theorem 14.2).

Q3. Consider the linear fractional transformation

$$f(z) = \frac{z-i}{z+i}.$$

What is the image of the real line $\mathbb{R}$ (respectively $\mathbb{R} \cup \{\infty\}$) under the map $w := f(z)$?

Solution: For $x \in \mathbb{R}$ we have

$$F(x) = \frac{x-i}{x+i} = \frac{x^2-1}{x^2+1} + i\frac{-2x}{x^2+1}.$$

Parametrizing the real line by

$$x(t) = \tan(t), \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$$

we get

$$\begin{array}{lllllllll} \mathrm{Re}(F(x(t))) & = & \frac{\tan^2(t)-1}{\tan^2(t)+1} & = & \frac{\sec^2(t)-2}{\sec^2(t)} & = & \sin^2(t)-\cos^2(t) & = & -\cos(2t), \\ \mathrm{Im}(F(x(t))) & = & \frac{-2\tan(t)}{\tan^2(t)+1} & = & \frac{-2\tan(t)}{\sec^2(t)} & = & -2\sin(t)\cos(t) & = & -2\sin(2t). \end{array}$$

As t movers form $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $F(x)$ traces the unit circle, from which the point $(1,0)$ is removed, counterclockwise. If we consider $\mathbb{R} \cup \{\infty\}$,then $F(\infty) := \lim\limits_{z \mapsto \infty} \frac{z-i}{z+i} = 1$, i.e. the image is the whole unit circle.$\blacksquare$

Q4. Find a linear fractional transformation which carries

$$C_1 := \{z \in \mathbb{C} : |z| = 1\} \text{ and } C_2 := \{z \in \mathbb{C} : \left|z - \frac{1}{4}\right| = \frac{1}{4}\}$$

into cocentric circles. What is the ratio of the radii?

Solution: If a linear fractional transformation $f(z) = \frac{az+b}{cz+d}$ satisfies the desired property, then the same is true for $g(x) = c_1 f(z) + c_2$ with arbitrary constants $c_1 \neq 0$ and c_2. So the problem is reduced to finding a linear fractional transformation $w = f(z)$ with

$$f(C_1) = C_1;\ f(C_2) = C_3 = \{z \in \mathbb{C} : |z| = r\} \text{ for some } r > 0,\ r \neq 1; \text{ and } f(1) = 1.$$

Notice that $C_0 := \mathbb{R}$ (which can be considered as a circle of infinite radius) is orthogonal to both C_1 and C_2. Since $w = f(z)$ is conformal, $f(\mathbb{R})$ is orthogonal to both cocentric circles C_1 and C_3, which is possible only if $f(\mathbb{R})$ is a straight line through the common center of C_1 and C_3. Since $f(1) = 1$, we conclude that $f(\mathbb{R}) = \mathbb{R}$.

By the properties of $w = f(z)$ we have indeed

$$f(1) = 1,\ f(-1) = -1,\ f(0) = \rho = \pm r \text{ and } f(\frac{1}{2}) = -\rho.$$

Taking (w.l.o.g.) $d = 1$ we get $b = c = \rho$ and $a = 1$, so that

$$g(z) = \frac{z + \rho}{\rho z + 1}.$$

From $g(\frac{1}{2}) = -\rho$, we get $\rho = \rho_{1,2} = -2 \pm \sqrt{3} < 0$, $r = -\rho = 2 \pm \sqrt{3} > 0$.

Since $\rho_1 \rho_2 = 1$, the possible transformations are

$$g_1(z) = \frac{z + \rho_1}{\rho_1 z + 1}, \; g_2(z) = \frac{z + \rho_2}{\rho_2 z + 1} = \frac{\rho_1 z + 1}{z + \rho_1} = \frac{1}{g_1(z)}.$$

The ratio of the radii of $g(C_1)$ and $g(C_2)$ has two possible values $2 \pm \sqrt{3}$.$\blacksquare$

Q5. Find a conformal mapping that takes the half plane on and to the left of the line $y = mx$ $(m > 0)$ onto the unit disk.

Solution: Consider $0 < \alpha := \tan^{-1}(m) < \frac{\pi}{2}$. Then $S(z) = e^{-i\alpha} z$ maps the indicated half plane onto the upper half plane (including the real line). Moreover the linear fraction transformation

$$T(z) = \frac{z - i}{z + i}$$

maps the upper half plane onto the unit disk (check!!). So the desired conformal mapping is given by

$$(T \circ S)(z) = \frac{e^{-i\alpha} z - i}{e^{-i\alpha} z + i}.\blacksquare$$

Q6. Show that any conformal mapping of the unit disk onto itself is of the form

$$h(z) = e^{i\theta} \frac{z - \beta}{1 - \overline{\beta} z}, \quad |\beta| < 1.$$

Solution: Consult (Bak & Newman: Complex Analysis, 1982, Lemma 13.14 and Theorem 13.15).$\blacksquare$

GOOD LUCK

14
CONFORMAL MAPPING

Preservation of Angles

14.1 Definition Each complex number $z \neq 0$ determines a *direction* from the origin, defined by the point

$$(1) \qquad A[z] = \frac{z}{|z|}$$

on the unit circle.

Suppose f is a mapping of a region Ω into the plane, $z_0 \in \Omega$, and z_0 has a deleted neighborhood $D'(z_0; r) \subset \Omega$ in which $f(z) \neq f(z_0)$. We say that f *preserves angles at* z_0 if

$$(2) \qquad \lim_{r \to 0} e^{-i\theta} A[f(z_0 + re^{i\theta}) - f(z_0)] \qquad (r > 0)$$

exists and *is independent of* θ.

In less precise language, the requirement is that for any two rays L' and L'', starting at z_0, the angle which their images $f(L')$ and $f(L'')$ make at $f(z_0)$ is the same as that made by L' and L'', in size as well as in orientation.

The property of preserving angles at each point of a region is characteristic of holomorphic functions whose derivative has no zero in that region. This is a corollary of Theorem 14.2 and is the reason for calling holomorphic functions with nonvanishing derivative *conformal mappings*.

14.2 Theorem *Let f map a region Ω into the plane. If $f'(z_0)$ exists at some $z_0 \in \Omega$ and $f'(z_0) \neq 0$, then f preserves angles at z_0. Conversely, if the differential of f exists and is different from 0 at z_0, and if f preserves angles at z_0, then $f'(z_0)$ exists and is different from 0.*

Here $f'(z_0) = \lim[f(z) - f(z_0)]/(z - z_0)$, as usual. The differential of f at z_0 is a linear transformation L of R^2 into R^2 such that, writing $z_0 = (x_0, y_0)$,

$$(1) \qquad f(x_0 + x, y_0 + y) = f(x_0, y_0) + L(x, y) + (x^2 + y^2)^{1/2}\eta(x, y),$$

where $\eta(x, y) \to 0$ as $x \to 0$ and $y \to 0$, as in Definition 8.22.

PROOF Take $z_0 = f(z_0) = 0$, for simplicity. If $f'(0) = a \neq 0$, then it is immediate that

$$(2) \qquad e^{-i\theta} A[f(re^{i\theta})] = \frac{e^{-i\theta} f(re^{i\theta})}{|f(re^{i\theta})|} \to \frac{a}{|a|} \qquad (r \to 0),$$

so f preserves angles at 0. Conversely, if the differential of f exists at 0 and is different from 0, then (1) can be rewritten in the form

$$(3) \qquad f(z) = \alpha z + \beta \bar{z} + |z|\eta(z),$$

where $\eta(z) \to 0$ as $z \to 0$, and α and β are complex numbers, not both 0. If f also preserves angles at 0, then

$$(4) \qquad \lim_{r \to 0} e^{-i\theta} A[f(re^{i\theta})] = \frac{\alpha + \beta e^{-2i\theta}}{|\alpha + \beta e^{-2i\theta}|}$$

exists and is independent of θ. We may exclude those θ for which the denominator in (4) is 0; there are at most two such θ in $[0, 2\pi)$. For all other θ, we conclude that $\alpha + \beta e^{-2i\theta}$ lies on a fixed ray through 0, and this is possible only when $\beta = 0$. Hence $\alpha \neq 0$, and (3) implies that $f'(0) = \alpha$. ////

Note: No holomorphic function preserves angles at any point where its derivative is 0. We omit the easy proof of this. However, the differential of a transformation may be 0 at a point where angles are preserved. Example: $f(z) = |z|z, z_0 = 0.$

(b) If h is an automorphism of D_1, $f \circ h \circ f^{-1}$ is an automorphism of D_2; thus $f \circ h \circ f^{-1} = g$ and $h = f^{-1} \circ g \circ f$. □

We now consider the problem of determining all the automorphisms of the unit disc.

13.14 Lemma. *The only automorphisms of the unit disc with $f(0) = 0$ are given by $f(z) = e^{i\theta}z$.*

Proof. If f maps the unit disc 1-1 onto itself and $f(0) = 0$, then by Schwarz' Lemma (7.2)

$$|f(z)| \leqslant |z| \quad \text{for } |z| < 1. \tag{4}$$

Moreover, since f^{-1} also maps the disc onto itself and $f^{-1}(0) = 0$, by the same argument,

$$|f^{-1}(z)| \leqslant |z| \quad \text{for } |z| < 1. \tag{5}$$

However, (4) and (5) can both be valid only if $|f(z)| = |z|$ and, by Schwarz' Lemma once again, it follows that

$$f(z) = e^{i\theta}z.$$

□

Suppose now that we wish to find an automorphism f of the unit disc with $f(\alpha) = 0$, for a fixed α, $0 < |\alpha| < 1$. If we assume that f is bilinear, then since f is globally 1-1, it must map the unit circle onto itself and we can thus apply the Schwarz Reflection Principle (7.8) (see also Exercise 16, Chapter 7) to conclude that $f(1/\bar{\alpha}) = \infty$. Hence f must be of the form

$$f(z) = c\left(\frac{z - \alpha}{z - 1/\bar{\alpha}}\right).$$

Setting

$$|f(1)| = |c\alpha| = 1$$

we have $|c| = (1/|\alpha|)$, and f may be written in the form

$$f(z) = e^{i\theta}\left(\frac{z - \alpha}{1 - \bar{\alpha}z}\right).$$

This suggests the following theorem.

13.15 Theorem. *The automorphisms of the unit disc are of the form*

$$g(z) = e^{i\theta}\left(\frac{z - \alpha}{1 - \bar{\alpha}z}\right), \qquad |\alpha| < 1.$$

Proof. Let $g(z) = (z - \alpha)/(1 - \bar{\alpha}z)$. Then, as we noted previously (following 7.2), $|g(z)| = 1$ for $|z| = 1$. Since $g(\alpha) = 0$, it follows that g is indeed an automorphism of the unit disc. Now assume that f is an automorphism of the unit disc with $f(\alpha) = 0$. Then $h = f \circ g^{-1}$ is an automorphism with $h(0) = 0$, so that according to the previous lemma

$$h(z) = e^{i\theta}z$$

or

$$f(z) = e^{i\theta}\left(\frac{z - \alpha}{1 - \bar{\alpha}z}\right).$$

□

Suppose next that we wish to determine a conformal mapping h of the upper half-plane onto the unit disc. Again, let us first assume that h is bilinear and $h(\alpha) = 0$, for fixed α with $\operatorname{Im}\alpha > 0$. Then, since the real axis is mapped into the unit circle, it follows by the Schwarz Reflection Principle that $h(\bar{\alpha}) = \infty$ so that

$$h(z) = c\left(\frac{z - \alpha}{z - \bar{\alpha}}\right).$$

13.16 Theorem. *The conformal mappings h of the upper half-plane onto the unit disc are of the form*

$$h(z) = e^{i\theta}\left(\frac{z - \alpha}{z - \bar{\alpha}}\right), \qquad \operatorname{Im}\alpha > 0.$$

Proof. Let $f(z) = (z - \alpha)/(z - \bar{\alpha})$. Since $|z - \alpha| = |z - \bar{\alpha}|$ for real z, f maps the real axis onto the unit circle. Also, since $f(\alpha) = 0$ and $\operatorname{Im}\alpha > 0$, it follows that f maps the upper half-plane onto the unit disc. Suppose then that h is any conformal mapping of the upper half-plane onto the unit disc and $h(\alpha) = 0$. By Lemma 13.13, h is of the form

$$h = g \circ f$$

where g is an automorphism of the disc. However, since $h(\alpha) = g(0) = 0$, it follows that $g(z) = e^{i\theta}z$ (13.14) and

$$h(z) = e^{i\theta}\left(\frac{z - \alpha}{z - \bar{\alpha}}\right).$$

□

13.17 Theorem. *The automorphisms of the upper half-plane are of the form*

$$f(z) = \frac{az + b}{cz + d}$$

with a, b, c, d real and $ad - bc > 0$.

Proof. Let f be as above. Then clearly f maps the real axis onto itself. Also,

$$\operatorname{Im} f(i) = \frac{ad - bc}{c^2 + d^2} > 0,$$

so that i is mapped into the upper half-plane and hence f is an automorphism of the upper half-plane. To show that there are no other automorphisms, we can apply Lemma 13.13 and Theorem 13.15 to show that any