King Fahd University of Petroleum & Minerals
Deanship of Scientific Research

FAST TRACK/RESEARCH PROJECT NO: ....

Research Project Entitled

"Zariski-like Topologies for Modules over Commutative Rings"

Duration of Project:	**18 months**
Proposed Starting Date:	01.12.2007
Ending Date:	01.06.2009
Total Project Cost:	34,600 SR

Submitted (on 26.10.2007) by:

Principal Investigator: **Dr. Jawad Abuihlail** (7030290):
Associate Professor, Department of Mathematics and Statistics

Approvals:

Chairman: .... **Date:** ....
Department of Mathematics and Statistics

Chairman: .... **Date:** ....
Research Committee

Vice Rector: .... **Date:** ....
for Graduate Studies and Scientific Research

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Name of the Principal Investigator: **Dr. Jawad Abuihlail** (7030290)

Date: 26.10.2007

Contents

1. Abstract

In this project, we consider several primeness (coprimeness) and semiprimeness (cosemiprimeness) properties of submodules for a given non-zero module M over a commutative ground ring R. Assuming suitable conditions on R and M, *Zariski-like topologies* are introduced on the spectrum of prime (coprime) submodules in each case. Such topologies are studied and investigated; and the interplay between their properties and the algebraic properties of the modules under consideration is clarified. Moreover, is case the spectrum of prime (coprime) submodules of M attains a Zariski-like topology, we investigate to what extent the R-module structure of M is determined by a suitable semimodule structure of $\xi(M)$ (the set of *closed* varieties of the submodules of M) over $\xi(R)$ (the set of *closed* varieties of ideals of R) considered with a suitable structure of a semiring.

2000 Mathematics Subject Classification:

13: Commutative rings and algebras
13C: Theory of modules and ideals
13C05: Structure, classification theorems
13C13: Other special types

Keywords: Zariski Topology; Zariski-like Topology; Prime Submodules; Semiprime Submodules; Coprime Submodules; Cosemiprime Submodules; Fully Prime Modules; Fully Coprime Submodules; Top Modules; Semirings; Semimodules.

2. Introduction

The classical Zariski topology on the spectrum of prime ideals of a (commutative) ring is one of the main tools in Algebraic Geometry. Moreover, it proved to be a very useful in understanding the structure of the module under consideration and recovering its properties. The notion of prime ideals was generalized to that of *prime submodules* of modules (over commutative rings) by several authors (e.g. [Dau1978], [BJKN80]). The prime spectrum of some classes of modules (e.g. finitely generated multiplication modules) proved to attain a Zariski-like topology, and were called *top modules*.

In this project, and instead of restricting ourselves to one notion of prime submodules as the case is in most of the papers on Zariski topologies for modules, we consider several primeness (coprimeness) properties and concentrate in particular on modules, for which the corresponding spectrum of prime (coprime) submodules attains a Zariski-like topology.

The main spectra we consider and investigate, in addition to the classical spectrum $\mathrm{Spec}^{\mathrm{P}}(M)$ of prime submodules in the sense of [Dau1978], are the spectrum $\mathrm{Spec}^{\mathrm{C}}(M)$ of *coprime submodules* in the sense of [KNR] in addition to the spectrum $\mathrm{Spec}^{\mathrm{FP}}(M)$ of *fully prime submodules* in the sense of [Wij2006] and the spectrum $\mathrm{Spec}^{\mathrm{FC}}(M)$ of *fully coprime submodules* in the sense of [RRW2005]. We investigate conditions under which such spectra attain a Zariski-like topology, and seek conditions on M and R that allow such spectra to satisfy various finite generating properties.

In case a spectrum of these attains a Zariski-like topology, we study several properties of this topological space and their interplay with the module structure of M. We also define a suitable structure of a semiring on $\xi(R)$ (the set of closed varieties of ideals of R) and a suitable $\xi(R)$-semimodule structure on $\xi(M)$ (the set of closed varieties of submodule of M) in the Zariski-like topology under consideration; and investigate to what extent the algebraic structures of $\xi(R)$ and $\xi(M)$ determine the R-module structure of M and recover its properties.

3. Literature Review

Prime submodules of modules were introduced as a generalization of prime ideals of rings by J. Dauns [Dau1978] and have been studied intensively since then (e.g. [MMS1992], [MMS1993], [Lu1995], [MMS1997], [Lu1997], [MMS1998], [HST1999], [MMS2000], [MMS2002], [MS2006]).

Throughout R is a commutative ring and M is a non-zero R-module. A *proper* submodule $K \lneqq_R M$ is called *prime in* M (or a *prime submodule*), iff

$$rm \in K \Rightarrow rM \subseteq K \text{ or } m \in K \text{ (for any } r \in R \text{ and } m \in M).$$

The prime spectrum $\mathrm{Spec}^{\mathrm{P}}(M)$ is defined to be the set of all prime submodules of M (if any). If $N \leq_R M$ is an R-submodule, denote by $V(N)$ the variety of N, which is the set consisting of all prime submodules of M that contain N. The R-module M is called a *top*$^{\mathrm{P}}$ *module,* iff the spectrum of M has the property (true for the usual spectrum $\mathrm{Spec}^{\mathrm{P}}(R)$) that set of all varieties

$$\xi^{\mathrm{P}}(M) := \{V(N) \mid N \leq_R M \text{ an } R\text{-submodule}\}$$

is closed under finite unions, whence constitute the closed sets in a Zariski-like topology on $\mathrm{Spec}^{\mathrm{P}}(M)$.

Beginning with [Lu1984] (where prime and primary *extended submodules* of the form IM, $I \lhd R$ are studied), C-P. Lu investigated prime submodules (of Noetherian modules) and top$^{\mathrm{P}}$ modules. In [Lu1995][1], he investigated when $\mathrm{Spec}^{\mathrm{P}}(M) \neq \varnothing$), and in [Lu1997] he extended to modules the *Prime Avoidance Theorem*; and investigated the S-closed subsets of modules (where $S \subseteq R$ is a multiplicatively closed subset).

In [Dur1994], Duraivel generalized to modules the notions of a prime ideal and a spectral topology. Moreover, he investigated the interplay between properties of the topological space $\mathbf{Z}_m^{\mathrm{P}}(M) := (\mathrm{Spec}^{\mathrm{P}}(M), \tau_{\mathrm{m}}^{\mathrm{P}}(M))$, where

$$\tau_{\mathrm{m}}^{\mathrm{P}}(M) := \{\mathrm{Spec}^{\mathrm{P}}(M) \backslash V(IM) \mid I \lhd R\}.$$

[1] In this paper, the assumption that $_RM$ is f.g. is missing, which leads to several mistakes (as indicated by Roger A. Wiegand in his review of the paper MR1348262 (96i:13015)).

and the algebraic properties of ${}_RM$ (for example: $\mathbf{Z}_m^{\mathrm{P}}(M)$ is quasi-compact, if ${}_RM$ is finitely generated; and is a Noetherian space, if ${}_RM$ Noetherian. In case ${}_RM$ is finitely generated, if the quotient ring $R/\mathrm{ann}_R(M)$ is decomposable, then $\mathbf{Z}_m^{\mathrm{P}}(M)$ is a disconnected space; and the converse is true if M is R-flat or admits a primary decomposition for its submodules). The investigation of $\mathbf{Z}_m^{\mathrm{P}}(M)$ was continued in [Lu1999], where C-P. Lu gave a group of conditions for $\mathbf{Z}_m^{\mathrm{P}}(M)$ to be a *spectral space* for various types of modules M (a topological space is called a *spectral space,* iff it is homeomorphic to $\mathrm{Spec}^{\mathrm{P}}(T)$ for some ring T).

In a series of paper (e.g. [MMS1992], [MMS1993], [MMS1997], [MMS1998], [MMS2002]), a group of algebraists including mainly R. McCasland, M. Moore and P. Smith carried out an intensive and systematic study of the spectrum of prime submodules. For example, they showed in [MMS1997] that in case ${}_RM$ is finitely generated, M is a *top*$^{\mathrm{P}}$ *module* if and only if M is a multiplication module (i.e. any R-submodule $N <_R M$ is of the form $N = IM$ for a suitable ideal $I \lhd R$); the paper included also discussions of when $\mathrm{Spec}^{\mathrm{P}}(M) = \varnothing$ and of spectra of direct sums. In [MMS2000], conditions on a finitely generated R-module M and on the ground ring R are determined under which the $\mathrm{Sepc}^{\mathrm{P}}(M)$ satisfies various finite generation conditions. In [MMS1998], it is shown that $\xi^{\mathrm{P}}(R) := \{V(I) \mid I \lhd R\}$ is a *semiring*, where

$$V(I) + V(J) = V(I + J) \ \&\ V(I) \bullet V(J) = V(IJ)$$

and that $\xi^{\mathrm{P}}(M)$ is a $\xi^{\mathrm{P}}(R)$-*semimodule,* where

$$V(I)V(L) = V(IL) \text{ for } I \lhd R \text{ and } L <_R M$$

(see [Gol1992] for definitions). In [MS2006] it is investigated, to what extent the module structure of ${}_RM$ is determined by the semimodule structure of ${}_{\xi^{\mathrm{P}}(R)}\xi^{\mathrm{P}}(M)$.

Coprime modules over commutative rings were introduced and named *second modules* by S. Yassemi [Yas2001] by his study of *coassociated modules* in [Yas1995] and [Yas1997]. The definition was transferred to modules over arbitrary (not necessarily commutative) rings by S. Annin, who named them *coprime modules* in [Ann2002]. Yassemi defined a *second submodule* of ${}_RM$ as a non-zero R-submodule $0 \neq L \subseteq M$, such that ${}_RL$ is a second module.

We adopt the definition of A. Kazemifard et. al. in [KNR], who defined *coprime submodule* of M as a proper R-submodule $L \subsetneqq M$ such that M/L is coprime.

The *internal product* (*internal coproduct*) of submodules of a given module over an associative - not necessarily commutative - ring was first introduced by Bican et. al. [BJKN80] to present the notion of *prime* (*coprime*) *modules.* The definitions were modified in [Wij2006] ([RRW2005]), where arbitrary submodules are replaced by fully invariant ones. To avoid any possible confusion, such modules are referred to as *fully coprime* (*fully coprime*) *modules.*

Remark: To the best of our knowledge, there is no single paper which defined Zariski-like topologies on the spectra of coprime submodules, fully prime submodules or fully coprime submodules of modules (over commutative or associative rings). Such topologies will be introduced here for the 1st time in the literature. However, we should mention that the idea was inspired by [NT2001], in which the authors introduced a topology on the spectrum of *coprime subcoalgebras* of a coalgebra over base fields (in a way dual to the classical Zariski topology of) and which led us to introduce and investigate several primeness (coprimeness) notions for corings and comodules in [Abu2006], and to introduce a Zariski-like topology for *bicomodules* and *corings* in [Abu2007].

4. Project Objectives and Proposed Research

Throughout, R is a commutative ring with $1_R \neq 0_R$, M is a non-zero R-module, $S := \mathrm{End}(_RM)^{op}$ (the ring of R-linear endomorphisms of M with multiplication the opposite composition of maps) and we consider M as a (R,S)-bimodule in the canonical way. For an R-submodule $K \subseteq M$, we denote with $\pi_K : M \to M/K$ the canonical R-linear map.

- With $\mathcal{L}(M)$ ($\mathcal{L}_{f.i.}(M)$) we denote the lattice of (fully invariant) R-submodules of M and with $\mathcal{I}_r(S)$ ($\mathcal{I}(S)$) the lattice of right (two-sided) ideals of S. With $\mathcal{I}_r^{f.g.}(S) \subseteq \mathcal{I}_r(S)$ ($\mathcal{L}^{f.g.}(M) \subseteq \mathcal{L}(M)$) we denote the subclass of finitely generated right ideals of S (finitely generated R-submodules of M). Moreover, we set $\mathcal{L}_m(M) := \{IM \mid I \lhd R$ is an ideal$\} \subseteq \mathcal{L}(M)$.

- For each $r \in R$, we have an R-*linear* map (called *homothety*) $\overset{\bullet}{r} : M \to M$, $m \mapsto rm$. Notice that we have a morphism of rings

$$\phi_M : R \to \mathrm{End}({}_RM), r \mapsto \overset{\bullet}{r}.$$

Clearly, ${}_RM$ is faithful if and only if $\overset{\bullet}{r}$ is injective for every $r \in R$.

Top$^{\mathrm{P}}$ modules

Notation. Set

$$\begin{array}{lcl} Z(M) & := & \{r \in R \mid \overset{\bullet}{r} : M \to M \text{ is not injective}\} & = & \{r \in R \mid rm = 0, 0 \neq m \in M\}; \\ W(M) & := & \{r \in R \mid \overset{\bullet}{r} : M \to M \text{ is not surjective}\} & = & \{r \in R \mid rM \neq M\}; \end{array}$$

Definition 1. 1. We call a proper R-submodule $K \subsetneqq M$ *prime in* M, iff $\mathrm{ann}(M/K) = Z(M/K)$ (equivalently, iff for every $r \in R$, the homothety $\overset{\bullet}{r} : M/K \to M/K$ is either injective or zero, i.e. whenever $rm \in K$ for some $r \in R$ and $m \in M$, we have $m \in K$ or $rM \subseteq K$;

2. We call M *prime,* iff $\mathrm{ann}(M) = Z(M)$ (equivalently, iff the R-submodule $(0_M) \subsetneqq M$ is prime in M).

Remark 2. A proper R-submodule $N \subsetneqq M$ is prime in M if and only if M/N is a prime module if and only if $\mathrm{ann}_R(M/N) = \mathrm{ann}_R(L/N)$ for every R-submodule $N \subsetneqq L \subseteq M$.

3. We define the P-*Spectrum* of M as

$$\mathrm{Spec}^{\mathrm{P}}(M) := \{K \in \mathcal{L}(M) \mid K \text{ is prime in } M\}.$$

Notation. For every R-submodule $L \subseteq M$ set

$$\mathcal{V}^{\mathrm{P}}(L) := \{K \in \mathrm{Spec}^{\mathrm{P}}(M) \mid L \subseteq K\} \text{ and } \mathcal{X}^{\mathrm{P}}(L) := \{K \in \mathrm{Spec}^{\mathrm{P}}(M) \mid L \nsubseteq K\}$$

and

$$\begin{array}{rclrcl} \xi^{\mathrm{P}}(M) & := & \{\mathcal{V}^{\mathrm{P}}(L) \mid L \in \mathcal{L}(M)\}; & \xi^{\mathrm{P}}_m(M) & := & \{\mathcal{V}^{\mathrm{P}}(L) \mid L \in \mathcal{L}_m(M)\}; \\ \tau^{\mathrm{P}}(M) & := & \{\mathcal{X}^{\mathrm{P}}(L) \mid L \in \mathcal{L}(M)\}; & \tau^{\mathrm{P}}_m(M) & := & \{\mathcal{X}^{\mathrm{P}}(L) \mid L \in \mathcal{L}_m(M)\}; \\ \mathbf{Z}^{\mathrm{P}}(M) & := & (\mathrm{Spec}(M), \tau^{\mathrm{P}}(M)); & \mathbf{Z}^{\mathrm{P}}_m(M) & := & (\mathrm{Spec}(M), \tau^{\mathrm{P}}_m(M)). \end{array}$$

Moreover, set

$$\begin{array}{rcl} \mathcal{V}^{\mathrm{P}}_w(L) & := & \{K \in \mathrm{Spec}^{\mathrm{P}}(M) \mid \mathrm{ann}_R(M/L) \subseteq \mathrm{ann}_R(M/K)\} \\ \mathcal{X}^{\mathrm{P}}_w(L) & := & \{K \in \mathrm{Spec}^{\mathrm{P}}(M) \mid \mathrm{ann}_R(M/L) \subseteq \mathrm{ann}_R(M/K)\} \end{array}$$

and

$$\xi^{\mathrm{P}}_w(M) := \{\mathcal{V}^{\mathrm{P}}_w(L) \mid L \in \mathcal{L}(M)\},\ \tau^{\mathrm{P}}_w(M) := \{\mathcal{X}^{\mathrm{P}}_w(L) \mid L \in \mathcal{L}(M)\}.$$

It is not difficult to see that $\mathbf{Z}^{\mathrm{P}}_m(M) := (\mathrm{Spec}^{\mathrm{P}}(M), \tau^{\mathrm{P}}_m(M)) = \mathbf{Z}^{\mathrm{P}}_w(M)$ is a topological space and coincides with $\mathbf{Z}^{\mathrm{P}}_w(M) := (\mathrm{Spec}^{\mathrm{P}}(M), \tau^{\mathrm{P}}_w(M))$. However, in general, $\xi^{\mathrm{P}}(M)$ is not closed under finite unions. This inspires the following definition

Definition 4. We call M a *top*$^{\mathrm{P}}$ *module*, iff $\xi^{\mathrm{P}}(M)$ is closed under finite unions.

Theorem 5. ([MMS1997]) *Let $_RM$ be finitely generated. Then M is a top*$^{\mathrm{P}}$ *module if and only if M is a multiplication module.*

Top$^{\mathrm{C}}$ modules

Definition 6. 1. We call a proper R-submodule $K \subsetneqq M$ *coprime in* M, iff $\mathrm{ann}(M/K) = W(M/K)$ (equivalently, iff for every $r \in R$, the homothety $\overset{\bullet}{r} : M/K \to M/K$ is either surjective or zero, i.e. whenever $r(M/K) \subsetneqq M/K$ for some $r \in R$, we have $rM \subseteq K$).

2. We call M *a coprime module,* iff $\mathrm{ann}(M) = W(M)$ (equivalently, iff the R-submodule $(0_M) \subsetneqq M$ is coprime in M).

7. We define the C-*Spectrum* of M as

$$\mathrm{Spec}^{\mathrm{C}}(M) := \{K \in \mathcal{L}(M) \mid K \text{ is coprime in } M\}.$$

Notation. For every R-submodule $L \subseteq M$ set

$\mathcal{V}^{\mathrm{C}}(L) := \{K \in \mathrm{Spec}^{\mathrm{C}}(M) \mid L \subseteq K\}$ and $\mathcal{X}^{\mathrm{C}}(L) := \{K \in \mathrm{Spec}^{\mathrm{C}}(M) \mid L \nsubseteq K\}$.

Moreover, set

$$\begin{array}{lll lll} \xi^{\mathrm{C}}(M) & := & \{\mathcal{V}^{\mathrm{C}}(L) \mid L \in \mathcal{L}(M)\}; & \xi^{\mathrm{C}}_m(M) & := & \{\mathcal{V}^{\mathrm{C}}(L) \mid L \in \mathcal{L}_m(M)\}; \\ \tau^{\mathrm{C}}(M) & := & \{\mathcal{X}^{\mathrm{C}}(L) \mid L \in \mathcal{L}(M)\}; & \tau^{\mathrm{C}}_m(M) & := & \{\mathcal{X}^{\mathrm{C}}(L) \mid L \in \mathcal{L}_m(M)\}; \\ \mathbf{Z}^{\mathrm{C}}(M) & := & (\mathrm{Spec}^{\mathrm{C}}(M), \tau^{\mathrm{C}}(M)); & \mathbf{Z}^{\mathrm{C}}_m(M) & := & (\mathrm{Spec}^{\mathrm{C}}(M), \tau^{\mathrm{C}}_m(M)). \end{array}$$

One can show that $\mathbf{Z}^{\mathrm{C}}_m(M) := (\mathrm{Spec}^{\mathrm{C}}(M), \tau^{\mathrm{C}}_m(M))$ is a topological space. In general, $\xi^{\mathrm{C}}(M)$ is not closed under finite unions. This inspires the following

Definition 8. We call M a *top*$^{\mathrm{C}}$ *module*, iff $\xi^{\mathrm{C}}(M)$ is closed under finite unions.

Top$^{\mathrm{FP}}$ modules

9. For R-submodules $X, Y \subseteq M$, consider the R-submodule of Y :

$$X *_M Y := \sum\{f(X) \mid f \in \mathrm{Hom}_R(M, Y)\}.$$

Notice that, if $Y \subseteq M$ is *fully invariant*, then $X *_M Y \subseteq M$ is also fully invariant; and if $X \subseteq M$ is fully invariant, then $X *_M Y \subseteq X \cap Y$.

Definition 10. We call a proper *fully invariant* R-submodule $W \subsetneqq M$:

fully prime in M, iff whenever $X *_M Y \subseteq W$ for some fully invariant R-submodules $X, Y \subseteq M$, it follows that $X \subseteq W$ or $Y \subseteq W$;

fully semiprime in M, iff whenever $X *_M X \subseteq W$ for some fully invariant R-submodule $X \subseteq M$, it follows that $X \subseteq W$;

We call M *fully prime module* (*fully semiprime module*), iff whenever $X *_M Y = 0$ for some two (equal) fully invariant R-submodules $X, Y \subseteq M$, it follows that $X = 0 = Y$, equivalently, iff $0 \subsetneqq M$ is fully prime in M (fully semiprime in M).

11. We define the FP-spectrum of M as

$$\mathrm{Spec}^{\mathrm{FP}}(M) := \{K \in \mathcal{L}(M) \mid K \text{ is fully prime in } M\}.$$

Notation. For every R-submodule $L \subseteq M$ consider the varieties

$$\mathcal{V}^{\mathrm{FP}}(L) := \{K \in \mathrm{Spec}^{\mathrm{FP}}(M) \mid L \subseteq K\} \text{ and } \mathcal{X}^{\mathrm{FP}}(L) := \{K \in \mathrm{Spec}^{\mathrm{FP}}(M) \mid L \nsubseteq K\}.$$

Notation. Set

$$\begin{array}{lcllcl} \xi^{\mathrm{FP}}(M) & := & \{\mathcal{V}^{\mathrm{FP}}(L) \mid L \in \mathcal{L}(M)\}; & \xi^{\mathrm{FP}}_{f.i.}(M) & := & \{\mathcal{V}^{\mathrm{FP}}(L) \mid L \in \mathcal{L}_{f.i.}(M)\}; \\ \tau^{\mathrm{FP}}(M) & := & \{\mathcal{X}^{\mathrm{FP}}(L) \mid L \in \mathcal{L}(M)\}; & \tau^{\mathrm{FP}}_{f.i.}(M) & := & \{\mathcal{X}^{\mathrm{FP}}(L) \mid L \in \mathcal{L}_{f.i.}(M)\}; \\ \mathbf{Z}^{\mathrm{FP}}(M) & := & (\mathrm{Spec}^{\mathrm{FP}}(M), \tau^{\mathrm{FP}}(M)); & \mathbf{Z}^{\mathrm{FP}}_{f.i.}(M) & := & (\mathrm{FP\text{-}Spec}(M), \tau^{\mathrm{FP}}_{f.i.}(M)). \end{array}$$

We noticed that $\mathbf{Z}^{\mathrm{FP}}_{f.i.}(M) := (\mathrm{FP\text{-}Spec}(M), \tau^{\mathrm{FP}}_{f.i.}(M))$ is a topological space. However, in general, $\xi^{\mathrm{FP}}(M)$ is not closed under finite unions. This inspires the following

Definition 12. We call M an *FP-top R-module*, iff $\xi^{\mathrm{FP}}(M)$ is closed under finite unions.

Top$^{\mathrm{FC}}$ modules

13. For any R-submodules $X, Y \subseteq M$ we set

$$X :_M Y := \bigcap\{f^{-1}(Y) \mid f \in \mathrm{An}_S(X)\} = \bigcap_{f \in \mathrm{An}(X)} \{\mathrm{Ker}(\pi_Y \circ f : M \to M/Y)\}.$$

If $X \subseteq M$ is fully invariant, then $X :_M Y \subseteq M$ is also fully invariant; and if $Y \subseteq M$ is fully invariant, then $X + Y \subseteq X :_M Y$.

Definition 14. We call a non-zero fully invariant R-submodule $0 \neq K \subseteq M$:

fully coprime in M, iff for any fully invariant R-submodules $X, Y \subseteq M$ with $K \subseteq X :_M Y$, it follows that $K \subseteq X$ or $K \subseteq Y$;

fully cosemiprime in M, iff for any fully invariant R-submodule $X \subseteq M$ with $K \subseteq X :_M X$, it follows that $K \subseteq X$;

In particular, we call M *fully coprime* (*fully cosemiprime*), iff M is fully coprime in M (fully cosemiprime in M).

15. We define the FC-*Spectrum* of M as

$$\mathrm{Spec}^{\mathrm{FC}}(M) := \{K \in \mathcal{L}(M) \mid K \text{ is fully coprime in } M\}$$

Notation. For every R-submodule $L \subseteq M$ set

$\mathcal{V}^{\mathrm{FC}}(L) := \{K \in \mathrm{Spec}^{\mathrm{FC}}(M) \mid K \subseteq L\},\ \mathcal{X}^{\mathrm{FC}}(L) := \{K \in \mathrm{Spec}^{\mathrm{FC}}(M) \mid K \nsubseteq L\}.$

Moreover, we set

$$\begin{array}{lcllcl} \xi^{\mathrm{FC}}(M) & := & \{\mathcal{V}^{\mathrm{FC}}(L) \mid L \in \mathcal{L}(M)\}; & \xi^{\mathrm{FC}}_{f.i.}(M) & := & \{\mathcal{V}^{\mathrm{FC}}(L) \mid L \in \mathcal{L}_{f.i.}(M)\}; \\ \tau^{\mathrm{FC}}(M) & := & \{\mathcal{X}^{\mathrm{FC}}(L) \mid L \in \mathcal{L}(M)\}; & \tau^{\mathrm{FC}}_{f.i.}(M) & := & \{\mathcal{X}^{\mathrm{FC}}(L) \mid L \in \mathcal{L}_{f.i.}(M)\}; \\ \mathbf{Z}^{\mathrm{FC}}(M) & := & (\mathrm{Spec}^{\mathrm{FC}}(M), \tau^{\mathrm{FC}}(M)); & \mathbf{Z}^{\mathrm{FC}}_{f.i.}(M) & := & (\mathrm{Spec}^{\mathrm{FC}}(M), \tau^{\mathrm{FC}}_{f.i.}(m)). \end{array}$$

It can be shown that $\mathbf{Z}^{\mathrm{FC}}_{f.i.}(M) := (\mathrm{Spec}^{\mathrm{FC}}(M), \tau^{\mathrm{FC}}_{f.i.})$ is a topological space. However, for an arbitrary R-module M, the set $\xi^{\mathrm{FC}}(M)$ is not necessarily closed under finite unions; which suggests the following

Definition 16. We call M an *top*$^{\mathrm{FC}}$ *module*, iff $\xi^{\mathrm{FC}}(M)$ is closed under finite unions.

OBJECTIVES AND TASKS:

The **main objectives** will be:

1. **transferring** results in the literature on spectra of *prime modules* ($\mathrm{top}^{\mathrm{P}}$ modules) to spectra of *fully prime modules* ($\mathrm{top}^{\mathrm{FP}}$ modules).
2. **dualizing** results on spectra of (*fully*) *prime modules*, and $\mathrm{top}^{\mathrm{P}}$ modules, to spectra of (*fully*) *coprime modules*, and $\mathrm{top}^{\mathrm{FP}}$ modules.

The main tasks will include - but will not be limited to - :

- **Task I:** We
  - investigate conditions, under which M is a top^{C} module (resp. a top^{FP} module, a top^{FC} module), whence $\mathbf{Z}^{\text{C}}(M) := (\text{Spec}^{\text{C}}(M), \tau^{\text{C}}(M))$ (resp. $\mathbf{Z}^{\text{FP}}(M) := (\text{Spec}^{\text{FP}}(M), \tau^{\text{FP}}(M)), \mathbf{Z}^{\text{FC}}(M) := (\text{Spec}^{\text{FC}}(M), \tau^{\text{FC}}(M)))$ is a topological space;
  - characterize top^{C} modules (resp. top^{FP} modules, top^{FC} modules) among several special classes of modules.

- **Task II:** We investigate, which conditions should be assumed on M and R, so that $\text{Spec}^{\text{C}}(M)$ (resp. $\text{Spec}^{\text{FP}}(M)$, $\text{Spec}^{\text{FC}}(M)$) is
  - (non-)empty;
  - satisfies various finite generating properties.

- **Task III:** Given a top^{P} module (resp. a top^{FP} module, a top^{FC} module), we
  - investigate the properties of the corresponding Zariski-like topologies, e.g. when such a topology is T_1, T_2, discrete, (countably) compact, etc.;
  - we characterizing special subspaces (e.g. the connected subspaces, the locally finite subspaces, etc.).

- **Task IV:** We investigate whether $\xi^{\text{C}}(R)$ (resp. $\xi^{\text{FP}}(R)$, $\xi^{\text{FC}}(R)$) can be given a suitable semiring structure, so that $\xi^{\text{C}}(M)$ (resp. $\xi^{\text{FP}}(M)$, $\xi^{\text{FC}}(M)$) has a suitable semimodule structure, as the case is for $\xi^{\text{P}}(R)$ and $\xi^{\text{P}}(M)$. Once the suitable semi-algebraic structures are defined, we study to what extent these structures determine the R-module structure of M and recover its properties.

5. Rough Time Schedule

Task I	1st - 6th month			
Task II:		7th -9th month		
Task III:			10th - 15th month	
Task VI:				16th - 18th month

6. Personnel Requirements

1 Secretary (18 months)

7. Monitoring and Evaluation

The best way to evaluate the results of project is to have them published in reputed refereed journals and/or proceedings of international conferences. I highly expect to publish the main results of the project in **two papers.**

8. Utilizing the Results

The results of the project are expected to be very useful in:

- shedding more light on the role of Zariski-like topologies in understanding the structure or the modules under consideration.
- although restricted to *modules over commutative rings*, the expected results will open the door for further generalizations to :
  - modules over arbitrary - not necessarily commutative - rings;
  - (bi)comodules and corings (the applicant has already published 2 papers [Abu2006] & [Abu2007] in this direction, which in fact inspired the current project).

9. Budget

Principal Investigator (18 months)		21,600 SR
Conferences/Scientific Visits		10,000 SR
Secretary		01,000 SR
Books		01,000 SR
Stationary		01,000 SR
Total		**34,600 SR**

10. Resume

(See the attached CV)

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