King Fahd University of Petroleum & Minerals

Department of Mathematical Sciences

Semicorings and Semicomodules

Jawad Abuhlail[1]

(abuhlail@kfupm.edu.sa)

May 31, 2008

[1]Supported by Research Committee at KFUPM, Grant Number: FT060010

Contents

Acknowledgments

I would like to thank:

- King Fahd University of Petroleum & Minerals for supporting the project under grant # FT060010.

- Prof. F. Linton (Wesleyan University, USA) and Prof. H. Porst (University of Bremen, Germany) for drawing my attention to the fact that some properties of the category of semimodules over a semiring follow automatically from the fact that they form a (*Birkhoff*) *variety* (see Remark 1.2.16 for more details).

- each of the following mathematicians, for private correspondence and for providing me with several papers/preprints:

  H. Al-Thani (Qatar University, Doha, Qatar);

  Y. Katsov (Hannover College, USA);

  T. Nam (Hue University, Vietnam);

  A. Patchkoria (A. Radmazde Mathematical Institute, Tbilisi, Georgia).

Notation

R	:=	commutative semiring;
A, B	:=	R-semialgebras;
${}_A\mathbb{S}$ (${}_A\mathbb{CS}$)	:=	the category of (cancellative) left A-semimodules;
$\mathbb{S}_A$ ($\mathbb{CS}_A$)	:=	the category of (cancellative) right A-semimodules;
${}_A\mathbb{S}_B$ (${}_A\mathbb{CS}_B$)	:=	the category of (cancellative) (A, B)-bisemimodules;
$\mathcal{C}$ ($\mathcal{D}$)	:=	A-semicoring (B-semicoring);
$\mathbb{S}^{\mathcal{C}}$ ($\mathbb{CS}^{\mathcal{C}}$)	:=	the category of (cancellative) right $\mathcal{C}$-semicomodules;
${}^{\mathcal{C}}\mathbb{S}$ (${}^{\mathcal{C}}\mathbb{CS}$)	:=	the category of (cancellative) left $\mathcal{C}$-semicomodules;
${}^{\mathcal{D}}\mathbb{S}^{\mathcal{C}}$ (${}^{\mathcal{C}}\mathbb{CS}^{\mathcal{C}}$)	:=	the category of (cancellative) $(\mathcal{D}, \mathcal{C})$-bisemicomodules;
${}_A\mathbb{H}$ ($\mathbb{H}_A$)	:=	the category of left (right) half A-semimodules;
${}_A\mathbb{H}_B$	:=	the category of half (A, B)-bisemimodules;
$E_L^M(L')$	:=	$\{m \in M \mid m + l \in L' \text{ for some } l \in L\}$.
$L \overset{f}{\rightarrowtail} M$	:=	The map f is injective;
$M \overset{g}{\twoheadrightarrow} N$	:=	The map g is surjective;

Abstract

Semirings provide a natural generalization of rings, and were shown to have real and significant applications in several areas including Computer Science, Automata Theory and Optimization Theory (for more details see [Gła2002]). Moreover, semimodules over semirings generalize modules over rings (e.g. [Tak1981], [Tak1982a]). In this project we go a further step. We generalize the notions of semirings (semimodules) to *semicorings* (*semicomodules*). The notions we introduce can also be considered as dual to semirings (semimodules). The theory of semicorings and semicomodules will be developed from scratch. In particular, we provide non-trivial examples of semicorings that are not corings and investigate their structure. Particular attention will be paid to the category of semicomodules of a given semicoring satisfying suitable projectivity conditions. Properties of such categories will be studied and results obtained earlier for comodules of corings will be generalized.

2000 Mathematics Subject Classification:
16W30, 16N60, 16A53, 16D80

Keywords:
Semirings; Semialgebras; Semimodules; Semicorings; Semicomodules; Projective Semimodules; Locally Projective Semimodules; Wisbauer's Category of Subgenerated Semimodules.

Introduction

Semirings can be defined (roughly) as “rings not necessarily with subtraction”. Trivial examples of semirings are the set $\mathbb{N}_0$ of non-negative integers, and the (semifield) $\mathbb{R}^+ := [0, \infty)$ of non-negative real numbers. A less trivial example is the semiring of two-sided ideals of a ring as noticed (in the commutative case) by Dedekind [Ded1894]. Semirings were studied by many algebraists, and were shown (since 1960’s) to have real and significant applications in several areas including Computer Science, Automata Theory, Optimization Theory, Baysian Networks and Belief Propagation. The interested reader is referred to the comprehensive literature guide by K. Głazek (in addition to the nice books by J. Golan [Gol1999a], [Gol1999b], [Gol2003] as well as the two monographs [HW1998] and [KS1986]). Moreover, semimodules over semirings generalized modules over rings and were studied also by many algebraists (e.g. Takahashi [Tak1979] - [Tak1985]).

On the other hand, *Hopf Algebras* attracted the attention of many researchers in Mathematics, Mathematical Physics and Theoretical Computer Science, especially after Drinfe’ld [Dri1988] introduced the notion of *Quantum Groups* (which are non-commutative non-cocommutative Hopf algebras). Moreover, with the new discoveries of new families of *corings*, their theory revived and many researchers are working on this topic with a new vision presented in the monograph by Brzeziński and Wisbauer [BW2003].

In this project, we present a notion that generalizes semirings and corings: namely *semicorings.* We show how semirings not only dualize semirings abut also generalize them (in addition to generalizing corings). Examples of semicorings that are not corings will be introduced. We also introduce and study *semicomodules* for semicorings, which generalize semimodules over semirings as well as comodules over corings. The categories of such semicomodules over semicorings will be studied, and previous results we obtained for comodules over rings (e.g. [Abu2003]) will be generalized.

It should be mentioned here, that although several proofs provided may look similar to the ones for comodules and corings over rings, a more careful reading would show that for many of them some extra technical assumptions were needed. Even for those with no extra conditions, very careful proof reading was carried out (since several classical results in Module Theory that we take as granted simply don't hold for general semimodules over semirings).

This report is divided as follows: after this introduction, we introduce a brief literature review on the main topics we considered. The first chapter includes some preliminaries (for convention of the reader, who is not familiar with semirings and semimodules). We also present a notion of semialgebras over commutative semirings that generalize algebras over commutative rings. Semimodules over semirings will be introduced and some basic results will be collected from several resources.

We begin the second chapter with introducing different notions of flatness and projectivity conditions for semimodules. In particular, we consider *local projectivity* (in the sense of Zimmermann-Huisgen [Z-H1976]) and clarify the relation between locally projective semimodules and *α-semimodules* (which we introduce and study as well).

In the third chapter we introduce the basic definitions and results of semicorings. In particular, we provide several examples of semicorings (that are not necessarily corings). We also define semicoalgebras, bisemialgebras and Hopf semialgebras and provide examples of them.

In the fourth chapter, we introduce semicomodules of semicorings and study them. In particular, we show that, in case $\mathcal{C}$ is an A-semicoring, then $\mathcal{C}$ is a half α-module if and only if the category $\mathbb{CS}^{\mathcal{C}}$ of cancellative right $\mathcal{C}$-semicomodules is a full subcategory of the category $\mathbb{CS}_{^*\mathcal{C}}$ of cancellative right $^*\mathcal{C}$-semimodules (where $^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A)$ is considered as an A-semiring with multiplication given by the so called *convolution product*).

Brief Literature Review

First of all, we remark that (to the best of our knowledge) not a single paper is published on the subject of the project and that the notions of semicorings and semicomodules have not been even defined by any other author.

In what follows we provide a brief literature review on "Corings and Comodules" and "semirings and semimodules" separately.

Semirings and Semimodules:

Semirings have numerous applications in Automata Theory, Optimization Theory, Baysian networks and belief propagation, Algebraic Geometry (over the optimization algebra). Many of these applications are documented in the nice books on the subject by J. Golan [Gol1999a], [Gol1999b], [Gol2003], as well as [HW1998] and [Kui1986]. A comprehensive literature review on semirings and their applications is provided by K. Głazek [Gła2002].

- Trivial examples of semirings which are not rings include the set $\mathbb{N}_0$ of non-negative integers and the (semifield) of non-negative real numbers $\mathbb{R}^+$ with the usual addition and multiplication.
- The first non-trivial example of semirings appeared first of the work of the German mathematician R. Dedekind [Ded1894], in connection with the algebra of ideals of a commutative ring.
- Later semiring were studied independently by algebraists, especially by the American mathematician H. S. Vandiver, who worked very hard to get them accepted as a fundamental algebraic structure, being basically the "best" structure which includes both rings and bounded distributive lattices [Van1934].

- Vandiver was not so successful. In fact (with only a few exceptions) semirings had fallen into disuse and were well on their way to mathematical oblivion until they were “rescued” during the late 1960’s when real and significant applications were found for them.

- The theory of semimodules over semirings was developed by several authors. For the foundations of the theory of semimodules over semirings we refer to the fundamental series of papers by M. Takahashi [Tak1979] - [Tak1985] (in addition to Golan’s books [Gol1999a]).

Corings and Comodules:

Hopf algebras appear in many fields of mathematics: number theory (formal groups), algebraic geometry (affine group schemes), Lie algebras (the universal enveloping algebra is a Hopf algebra), graded ring theory (gradings are coactions), Galois Theory, the theory of Azumaya algebras and Brauer groups, etc.

- In 1941, the first example of *Hopf algebras* appeared in algebraic topology in the work of the German mathematician Hopf [Hop1941].

- The first paper to attract the attention of algebraist was on graded Hopf algebras by Milnor and Moore [MM1965].

- During the 1960s and 1970s, Hopf algebras were studied intensively from a purely algebraic point of view. The first book in this direction was that of M. Sweedler [Swe1969]. A subsequent book of this nature, with more flavor of algebraic geometry, is that of E. Abe [Abe1980].

- In 1975, M. Sweedler [Swe1975] introduced the notion of corings. However they did not get much appreciation because of the lack of examples at that time.

- In 1986, Drinfe’ld published his milestone “Quantum Groups” in Russian (translated in [Dri1988]).

- Since then, the subject received a huge impetus because of the discovery of interesting applications in quantum mechanics, statistical mechanics and knot theory. This resulted in a revival of the algebraic theory of

Hopf algebras making it one of the mainstream subjects in mathematics in the 1990's.

- Many books on quantum groups, which are certain non-commutative and non-cocommutative Hopf algebras, have been published since the late 1980's. In addition, there were two books that concentrated on the purely algebraic aspects of the theory of Hopf algebras: one by S. Montgomery [Mon1993] and the other by S. Dascalescu et al. [DNR2001].

- Intensive investigations were conducted by several authors of the so called Doi-Koppinen data and the associated categories of Doi-Koppinen modules (e.g. the monograph by S. Caenepeel et. al. [CMZ2001]). Doi-Koppinen modules were generalized in a work in mathematical physics by T. Brzezinski and S. Majid [BM1998] to the so called *entwining structures* and *entwined modules.* These attracted (and are still attracting) the attention of many researchers.

- In 1999, M. Takeuchi pointed out that entwining structures, give rise to new examples of corings. This resulted in the revival of the theory of corings and their comodules in the recent years.

- With the many new examples discovered, it turned out that corings might have a variety of unexpected and wide-ranging applications, to topics in non-commutative ring theory, category theory, Hopf algebras, differential graded algebras, and non-commutative geometry.

- Since the revival of their theory at the beginning of the current century, "corings and comodules" are gaining the attention of many algebraists. Many aspects of the theory of corings and comodules are dealt with in the recent monograph by T. Brzezinski and R. Wisbauer [BW2003], which is the first book to stress the fact that corings (comodules), over arbitrary ground rings, can be thought of as generalization of rings (modules) and not only as dual to them.

- Although it is impossible to include all (or even almost all) main trends and aspects in the recent and current research of this very hot subject, we mention some samples that lie within the interests of the author:

  - Constructing new classes of corings;

  - Studying the structure of corings;

– Establishing a Galois theory for corings and comodules;
– Studying Morita contexts for corings;
– Studying dualities for categories of comodules of corings;
– Studying functors related to categories of comodules for corings;
– Studying equivalences between categories of comodules for corings;
– Exploring interactions with non-commutative Geometry.

Chapter 1

Semialgebras and Semimodules

In this chapter we introduce a notion of a semialgebra over a commutative semiring. Definitions from the category of semirings and semimodules will be transferred to this new context. We also generalize several notions for modules over rings to semimodules over semirings including the notions of *subgeneration* and *local projectivity*.

1.1 Preliminaries

In this preliminary section we include some basic definitions and results.

Definition 1.1.1. A semigroup $(S,*)$ is called *cancellative*, iff for all $s, s', s'' \in S$:

$$s * s'' = s' * s'' \Rightarrow s = s'.$$

Definition 1.1.2. A *semiring* is an algebraic structure $(S, +, \cdot; 0_S, 1_S)$ consisting of a non-empty set S with two binary operations "+" (addition) and "·" (multiplication), such that

1. $(S, +; 0_S)$ is an Abelian monoid with *zero* 0_S;

2. $(S, \cdot; 1_S)$ is a monoid with neutral element 1_S;

3. $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$ for all $x, y, z \in S$;

4. $0_S \cdot x = 0_S = x \cdot 0_S$ for every $x \in S$ (i.e. 0_S is *absorbing*).

Definition 1.1.3. A semiring $(S, +, \cdot)$ will be called *commutative* (*cancellative*), iff $(S, \cdot)$ is commutative ($(S, +)$ is cancellative).

Definition 1.1.4. A *hemiring* is a semiring $(S, +, \cdot)$ for which $(S, \cdot)$ is a semigroup and not necessarily a monoid. Moreover, a *half-ring* is a cancellative semiring.

Remarks 1.1.5.

1. It should be noted that the notion of a semiring differs from an author to another. For example, in [Gła2002], a semiring is defined (in our terminology) as a hemiring $(S, +, \cdot)$ not necessarily with zero and for which $(S, +)$ is not necessarily Abelian!!. For us, we adopt Golan's terminology [Gol1999a].
2. We always assume $0_S \neq 1_S$ (so that $S \neq \{0\}$, the *zero semiring*).

Examples 1.1.6. Examples of semirings include $(\mathbb{N}_0, +, \cdot)$, $(\mathbb{R}_0^+, +, \cdot)$, as well as $\mathrm{Lat}(R)$, the lattice of two-sided ideals of an arbitrary (*not necessarily commutative*) ring R.

Categories of Semimodules

Definition 1.1.7. Given a semiring S, a *right S-semimodule* is a non-empty set M with a binary operation of addition (usually denoted by "+") and scalar multiplication by elements of S (on the right) defined such that

1. $(M, +; 0_M)$ is an Abelian monoid with neutral element 0_M;
2. $(ms)s' = m(ss')$, $(m + m')s = ms + m's$ and $m(s + s') = ms + ms'$ for all $s, s' \in S$ and $m, m' \in M$.
3. $m1_S = m$ and $m0_S = 0_M = 0_M s$ for all $s \in S$ and $m \in M$.

Remark 1.1.8. Every Abelian monoid is an $\mathbb{N}_0$-semimodule in the obvious way.

Definition 1.1.9. Let M be an S-semimodule. A non-empty subset $L \subseteq M$ is said to be an *S-subsemimodule* (and we write $L \leq_S M$), iff L is closed under "$+_M$" and $sl \in L$ for all $s \in S$ and $l \in L$.

Definition 1.1.10. An S-semimodule M is called *cancellative*, iff the monoid $(M, +)$ is cancellative.

Definition 1.1.11. A non-empty subset $L \subset M$ is said to be *subtractive*, iff for all $m \in M$ and $l \in L$:

$$l + m, \ l \in L \Rightarrow m \in L;$$

strong, iff for all $m, m' \in M$ we have

$$m + m' \in L \Rightarrow m, m' \in N.$$

Notation. Let M be an S-semimodule and $L, L' \subseteq M$. We set

$$E_L^M(L') := \{m \in M \mid m + l \in L' \text{ for some } l \in L\}.$$

Definition 1.1.12. Let M be an S-semimodule and $L \leq_S M$ a subsemimodule.

The *subtractive closure* of L is

$$\bigcap \{L' \supseteq L \mid L' \leq_S M \text{ is a subtractive } S\text{-subsemimodule}\}.$$

The *strong closure* of L is

$$\bigcap \{L' \supseteq L \mid L' \leq_S M \text{ is a strong } S\text{-subsemimodule}\}.$$

Lemma 1.1.13. ([Gol1999a]) *Let M be an S-semimodule and $L \leq_S M$ a subsemimodule.*

1. *The subtractive closure of L is*

$$E_L^M(L) = \{m \in M \mid m + l \in L \text{ for some } l \in L\}.$$

2. *The strong closure of L is*

$$E_M^M(L) = \{m \in M \mid m + m' \in L \text{ for some } m' \in L\}.$$

Definition 1.1.14. An S-semimodule M is said to be *completely subtractive*, iff every S-subsemimodule $N \leq_S M$ is subtractive. The semiring S is said to be *completely subtractive*, iff ${}_SS$ and S_S are completely subtractive.

1.1.15. Let S be a semiring and M, N be right S-semimodules. A map of monoids $f : M \to N$ will be called a morphism of S-semimodules, or (right) S-*semilinear*, iff $f(m + m') = f(m) + f(m')$ and $f(ms) = f(m)s$ for all $m \in M$ and $s \in S$. The set of S-semilinear morphisms from M to N will be denoted by $\mathrm{Hom}_{-S}(M, N)$. The class of right S-semimodules along with the S-semilinear morphisms between them form a category, which we denote with $\mathbb{S}_S$. Analogously one defines the category of left S-semimodules ${}_S\mathbb{S}$. With $\mathbb{CS}_S \leq \mathbb{S}_S$ (resp. ${}_S\mathbb{CS} \leq {}_S\mathbb{S}$) we denote the full subcategory of cancellative right (left) S-semimodules.

1.1.16. Let S, T be semirings. A left S-semimodule M that is also a right T-semimodule is said to be an (S, T)-bisemimodule, iff $(sm)t = s(mt)$ for all $s \in S$ and $t \in T$. Let M, N be (S, T)-bisemimodules. We call an S-semilinear T-semilinear morphism $f : M \to N$ a *morphism of* (S, T)-*bisemimodules*, or (S, T)-*bisemilinear*, and denote with $\mathrm{Hom}_{(S,T)}(M, N)$ the set of (S, T)-bisemilinear morphisms from M to N. The class of (S, T)-bisemimodules along with the (S, T)-bisemilinear morphisms between them build a category which we denote with ${}_S\mathbb{S}_T$. The full subcategory of cancellative (S, T)-bisemimodules will be denoted by ${}_S\mathbb{CS}_T$.

Congruences

1.1.17. Let M be an S-semimodule. An equivalence relation $\equiv$ on M is a said to be an S-*congruence*, iff for any $m, m', m'' \in M$ and $s \in S$ we have

$$m \equiv m' \Rightarrow [m + m'' \equiv m' + m'' \text{ and } ms \equiv m's].$$

The set $S - \mathrm{cong}(M)$ of S-congruences on M is non-empty as it contains the *trivial congruence* $\equiv_t$ and the *universal congruence* $\equiv_u$ given by

$$m \equiv_t m' \Leftrightarrow m = m' \text{ and } m \equiv_u m' \text{ for all } m, m' \in M.$$

1.1.18. The set of equivalence classes corresponding to any $\rho \in S - \mathrm{cong}(M)$ is denoted by M/ρ and is an S-semimodule in the obvious way, and there is a surjective morphism of S-semimodules $\pi_\rho : M \to M/\rho$.

Definition 1.1.19. 1. An S-semimodule $M \neq 0$ will be called *simple*, iff

$$S - \mathrm{cong}(M) = \{\equiv_t, \equiv_u\}.$$

2. An S-semimodule is called *austere*, iff $\{0_M\}$ and M are the only subtractive S-subsemimodules of M.

1.1.20. Every S-subsemimodule $L \leq_S M$ induces two S-congruences on M : the *Bourne relation:*

$$m \equiv_L m' \Leftrightarrow m + l = m' + l' \text{ for some } l, l' \in L;$$

and the *Iizuka relation:*

$$m[\equiv]_L m' \Leftrightarrow m + l + m'' = m' + l' + m'' \text{ for some } l, l' \in L \text{ and } m'' \in M.$$

The S-semimodule $M/L := M/\equiv_L$ is called the *factor S-semimodule of M by L*, and associated to it we have a surjective morphism of S-semimodules

$$\pi_L := M \to M/L, \ m \mapsto [m],$$

with

$$\mathrm{Ker}(\pi_L) = \{m \in M \mid m + l = l' \text{ for some } l, l' \in L\}.$$

Remark 1.1.21. Let M be an S-semimodule and $L \leq_S M$ an S-subsemimodule. The S-semimodule $M/[\equiv]_L$ is cancellative. If M is cancellative, then L and M/L are cancellative.

Definition 1.1.22. Let M be an S-semimodule and consider the S-congruence relation

$$m[\equiv]_{\{0\}} m' \Leftrightarrow m + m'' = m' + m'' \text{ for some } m'' \in M.$$

Then we have a cancellative S-semimodule

$$\mathfrak{c}(M) := M/[\equiv]_{\{0\}} = \{[m]_{\{0\}} : m \in M\}. \tag{1.1}$$

Moreover, we have a canonical surjection $\mathfrak{c}_M : M \to \mathfrak{c}(M)$ with

$$\delta(M) := \mathrm{Ker}(\mathfrak{c}_M) = \{m \in M \mid m + m' = m' \text{ for some } m' \in M\}.$$

Lemma 1.1.23. ([Tak1981]) *Let M be an S-semimodule. The following are equivalent:*

1. *M is cancellative;*

2. $M \overset{\mathfrak{c}_M}{\simeq} \mathfrak{c}(M)$*;*

3. $M \times M$ *is subtractive;*

4. $\delta(M) = 0$ *and* $\mathfrak{c}_M$ *is k-regular*[1]*.*

Proposition 1.1.24. ([Tak1981, Theorem 7.7.]) *We have a functor*

$$\mathfrak{c} : \mathbb{S}_S \to \mathbb{CS}_S, \; M \mapsto \mathfrak{c}(M)$$

that is left adjoint to the forgetful function $\mathcal{F} : \mathbb{CS}_S \to \mathbb{S}_S$, *i.e. for any right* S*-semimodule* M *and any* cancellative S*-semimodule* N *we have*

$$\mathrm{Hom}_S(\mathfrak{c}(M), N) = \mathrm{Hom}_S(M, N).$$

Tensor Products of Semimodules

Tensor products of semimodules over a semiring were defined by Takahashi [Tak1982a]

1.1.25. Let S be a semiring, M a right S-semimodule and N a left S-semimodule and consider the free $\mathbb{N}_0$-semimodule $F := S^{(M\times N)} \times S^{(M\times N)}$. Let $f_{(m,n)}$ be a the canonical basis of $S^{(M\times N)}$ given by

$$f_{(m,n)}(m', n') := \begin{cases} 1_S, & \text{if } (m', n') = (m, n); \\ 0, & \text{otherwise.} \end{cases}$$

Let $F_0 \subseteq F$ be the $\mathbb{N}_0$-subsemimodule generated by all elements of the form

1) $(f_{(m+m',n)}, f_{(m,n)} + f_{(m',n)})$;
2) $(f_{(m,n)} + f_{(m',n)}, f_{(m+m',n)})$;
3) $(f_{(m,n+n')}, f_{(m,n)}, f_{(m,n')})$;
4) $(f_{(m,n)} + f_{(m,n')}, f_{(m,n,n')})$;
5) $(f_{(mr,n)}, f_{(m,rn)})$;
6) $(f_{(m,rn)}, f_{(mr,n)})$.

The *tensor product* of M and N over S is defined as the $\mathbb{N}_0$-semimodule

$$M \otimes_S N := F/F_0,$$

i.e. $M \otimes_S N = S^{(M\times N)}/\tau_{(M,N)}$, where $\tau_{(M,N)}$ is the S-congruence relation on $S^{(M\times N)}$ given by

$$f \; \tau_{(M,N)} \; f' \Leftrightarrow f + g = f' + g' \text{ for some } g, g' \in F_0.$$

[1]i.e. $\mathfrak{c}(m) = \mathfrak{c}(m') \Rightarrow m + k = m' + k'$ for some $k, k' \in \mathrm{Ker}(\mathfrak{c}_M)$.

1.1.26. ([Tak1982a]) Let S be a ring, M a right S-semimodule and N a left S-semimodule. The tensor product of M and N is a cancellative Abelian monoid along with an S-balanced map

$$\tau : M \times N \to M \otimes_S N,\ (m,n) \mapsto m \otimes_S n$$

satisfying the following universal property: for every Abelian monoid G with an S-balanced map $\beta : M \times N \to G$, there exists a unique morphism of monoids $\gamma : M \otimes_S N \to \mathfrak{c}(G)$ that completes the following diagram commutatively

$$\begin{array}{ccc} M \times N & \xrightarrow{\beta} & G \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\mathfrak{c}_G} \\ M \otimes_S N & \xrightarrow[\gamma]{} & \mathfrak{c}(G) \end{array}$$

Remarks 1.1.27. Let S be a semiring, M a right S-semimodule and N be a left S-semimodule. For all $m, m' \in M$, $n, n' \in N$, $s \in S$ and $k \in \mathbb{N}_0$ we have:

1. $(m + m') \otimes_S n = m \otimes_S n + m' \otimes_S n$;
2. $m \otimes_S (n + n') = m \otimes_S n + m \otimes_S n'$;
3. $ms \otimes_S n = m \otimes_S sn$;
4. $k(m \otimes_S n) = km \otimes_S n = m \otimes_S kn$;
5. $0_M \otimes_S n = 0_{M \otimes_S N} = m \otimes_S 0_N$.

Remark 1.1.28. The tensor product of semimodules we adopt is due to Takahashi and is used by many authors. However, we have to mention that there is another notion of tensor product due to Katsov (e.g. [Kat1997]).

Proposition 1.1.29. ([Gol1999a, Proposition 16.16.]) *Let M be a right (respectively left) S-semimodule. Then we have a canonical isomorphism of $\mathbb{N}_0$-semimodules*

$$M \otimes_S S \overset{\vartheta_M^r}{\simeq} \mathfrak{c}(M)\ (S \otimes_S M \overset{\vartheta_M^l}{\simeq} \mathfrak{c}(M)).$$

Moreover, M is cancellative if and only if we have canonical isomorphisms

$$M \otimes_S S \simeq M\ (S \otimes_S M \simeq M).$$

Remark 1.1.30. If M_S or $_SN$ is a module, then $M \otimes_S N$ is a module.

Proposition 1.1.31. ([Gol1999a, Proposition 16.15.]) *Let M be a right S-semimodule and N a left S-semimodule.*

1. *If M is a (T,S)-bisemimodule, then for every* cancellative *left T-semimodule G, we have an isomorphism*

$$\mathrm{Hom}_{T-}(M \otimes_S N, G) \overset{\varsigma^l}{\simeq} \mathrm{Hom}_{S-}(N, \mathrm{Hom}_{T-}(M,G)).$$

2. *If N is an (S,T)-bisemimodule, then for every* cancellative *right T-semimodule G, we have an isomorphism*

$$\mathrm{Hom}_{-T}(M \otimes_S N, G) \overset{\varsigma^r}{\simeq} \mathrm{Hom}_{-S}(M, \mathrm{Hom}_{-T}(N,G)).$$

Remarks 1.1.32. Let M be a right S-semimodule and N a left S-semimodule.

1. If M is a (T,S)-bisemimodule and G is any left T-semimodule, then we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{T-}(M \otimes_S N, \mathfrak{c}(G)) &\simeq \mathrm{Hom}_{S-}(N, \mathrm{Hom}_{T-}(M, \mathfrak{c}(G))\\ &= \mathrm{Hom}_{S-}(N, \mathrm{Hom}_{T-}(\mathfrak{c}(M), \mathfrak{c}(G))\\ &\simeq \mathrm{Hom}_{T-}(\mathfrak{c}(M) \otimes_S N, \mathfrak{c}(G)).\end{aligned}$$

   The image of $f \in \mathrm{Hom}_{T-}(M \otimes_S N, \mathfrak{c}(G))$ under this isomorphism will be denoted by $\ell(f) \in \mathrm{Hom}_{T-}(\mathfrak{c}(M) \otimes_S N, \mathfrak{c}(G))$.

2. If N is an (S,T)-bisemimodule, then for every *cancellative* right T-semimodule G, we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{-T}(M \otimes_S N, \mathfrak{c}(G)) &\simeq \mathrm{Hom}_{-S}(M, \mathrm{Hom}_{-T}(N, \mathfrak{c}(G)))\\ &= \mathrm{Hom}_{-S}(M, \mathrm{Hom}_{-T}(\mathfrak{c}(N), \mathfrak{c}(G)))\\ &\simeq \mathrm{Hom}_{-T}(M \otimes_S \mathfrak{c}(N), \mathfrak{c}(G)).\end{aligned}$$

   The image of $f \in \mathrm{Hom}_{-T}(M \otimes_S N, \mathfrak{c}(G))$ under this isomorphism will be denoted by $\wp(f) \in \mathrm{Hom}_{-T}(M \otimes_S \mathfrak{c}(N), \mathfrak{c}(G))$.

1.2 Semialgebras

In what follows, fix a commutative semiring R with $1_R \neq 0_R$.

1.2.1. An *R-semialgebra* is a triple (A, μ_A, η_A), where A is an R-semimodule and

$$\mu : A \otimes_R A \to \mathfrak{c}(A) \text{ and } \eta : R \to A$$

are R-semilinear morphisms such that

$$\ell(\mu_A)\circ(\mu_A\otimes_R \mathrm{id}_A) = \wp(\mu_A)\circ(\mathrm{id}_A\otimes_R\mu_A),\ \ \mu_A\circ(\eta_A\otimes_R\mathrm{id}_A) = \vartheta^l_A \text{ and } \mu_A\circ(\mathrm{id}_A\otimes_R\eta_A) = \vartheta^r_A.$$

For R-semialgebras A, B we say an R-semilinear morphism $f : A \to B$ is a morphism of R-semialgebras, iff

$$\mathfrak{c}(f) \circ \mu_A = \mu_B \circ (f \otimes_R f) \text{ and } f \circ \eta_A = \eta_B.$$

With $\mathrm{SAlg}_R(A, B)$ we denote the set of R-semialgebra morphisms from A to B. The class of R-semialgebras and morphisms of R-semialgebras form a category, which we denote with $\mathbb{SALG}_R$.

1.2.2. Let (A, μ, η) be an R-semialgebra. A right A-semimodule is an R-semimodule with an R-semilinear morphism $\varphi_M : M \otimes_R A \to \mathfrak{c}(M)$, such that

$$\ell(\varphi_M) \circ (\varphi_M \otimes_A \mathrm{id}_A) = \wp(\varphi_M) \circ (\mathrm{id}_M \otimes_A \mu) \text{ and } \varphi_M \circ (\mathrm{id}_M \otimes_A \eta) = \vartheta^r_M.$$

Let M, N be right A-semimodules. An R-semilinear morphism $f : M \to N$ is a morphism of A-semimodules (or A-semilinear), iff

$$\mathfrak{c}(f) \circ \varphi_M = \varphi_N \circ (f \otimes_A \mathrm{id}_A).$$

The category of right A-semimodules is denoted by $\mathbb{S}_A$. The category ${}_A\mathbb{S}$ of left A-semimodules is analogously defined. For two R-semialgebras A, B the category ${}_A\mathbb{S}_B$ of (A, B)-bisemimodules and (A, B)-bisemilinear morphisms can be defined in the obvious way.

Definition 1.2.3. A *half R-algebra*[2] is a cancellative R-semialgebra.

[2]Recall that a cancellative semiring is called a *half-ring* (e.g. [Gol1999a]).

Remark 1.2.4. Every semiring $(S,+,\cdot)$ is an $\mathbb{N}_0$-semialgebra and every semimodule of the semiring S is a semimodule of the induced $\mathbb{N}_0$-semialgebra. Indeed, (S,μ_A,η_A) is an $\mathbb{N}_0$-semialgebra where

$$\begin{array}{lccccccc}\mu_S & : & S\otimes_{\mathbb{N}_0} S & \rightarrow & \mathfrak{c}(S), & s\otimes_{\mathbb{N}_0} t & \mapsto & \mathfrak{c}_S(st);\\ \eta_S & : & \mathbb{N}_0 & \rightarrow & S, & n & \mapsto & n1_S.\end{array}$$

If M is a right semimodule of the ring $(S,\cdot,1_S)$, then M is a right semimodule of the $\mathbb{N}_0$-semialgebra (S,μ_S,η_S) with semimodule structure map

$$\varphi_M : M\otimes_{\mathbb{N}_0} S \longrightarrow \mathfrak{c}(M),\ m\otimes_{\mathbb{N}_0} s \mapsto \mathfrak{c}_M(ms).$$

On the otherhand, every half $\mathbb{N}_0$-semialgebra (A,μ_A,η_A) is a semiring with $1_A := \eta_A(1)$ and

$$a\cdot b := \mu_A(a\otimes_{\mathbb{N}_0} b) \text{ for all } a,b\in A.$$

Moreover, if (M,φ_M) is a *cancellative* right A-semimodule, then M is a right semimodule of the semiring $(A,\cdot,1_A)$ with

$$ma := \varphi_M(m\otimes_{\mathbb{N}_0} a) \text{ for all } m\in M \text{ and } a\in A.$$

Lemma 1.2.5. *(Compare with [Tak1982a, Proposition 2.2.], [DP2005, Proposition 1]) Let A be a half R-algebra. If M is a cancellative left (right) A-semimodule, then we have a canonical isomorphism of R-semimodules*

$$\mathrm{Hom}_{A-}(A,M)\simeq M\ \ (\mathrm{Hom}_{-A}(A,M)\simeq M).$$

Exact Sequences of Semimodules

Notation. Let A be an R-semialgebra. For a morphisms A-semimodules $f:M\rightarrow N$ we define

$$\begin{array}{lcl}\mathrm{Ker}(f) & := & \{m\in M \mid f(m)=0\};\\ f(M) & := & \{f(m)\mid m\in M\};\\ \mathrm{Coker}(f) & := & N/f(M);\\ \mathrm{Im}(f) & := & \{n\in N\mid n+f(m)=f(m') \text{ for some } m,m'\in M\};\\ \mathrm{Coim}(f) & := & M/\mathrm{Ker}(f);\end{array}$$

We call Ker(f) (resp. Im(f), $f(M)$, Coker(f), Coim(f)) the *kernel* (resp. *image, proper image, cokernel, coimage*) *of* f. With

$$0 \to \mathrm{Ker}(f) \overset{\ker(f)}{\rightarrowtail} M \text{ and } 0 \to \mathrm{Im}(f) \overset{\mathrm{im}(f)}{\rightarrowtail} N$$

we denote the natural embeddings, and with

$$M \overset{\mathrm{coim}(f)}{\twoheadrightarrow} \mathrm{Coim}(f) \to 0 \text{ and } N \overset{\mathrm{coker}(f)}{\twoheadrightarrow} \mathrm{Coker}(f) \to 0$$

the canonical surjections.

Remark 1.2.6. Let A be an R-semialgebra. As in any category with zero object, kernels and cokernels (e.g. [Sch1972, 12.3.7.]), we define for any A-semimodules M, N and $f \in \mathrm{Hom}_A(M, N)$:

$$\begin{aligned} \mathrm{Im}(f) &:= \mathrm{Ker}(\mathrm{coker}(f)) \\ &= \{n \in N \mid n \equiv_{f(M)} 0\} \\ &= \{n \in N \mid n + f(m) = f(m') \text{ for some } m, m' \in M\}; \end{aligned}$$

and

$$\mathrm{Coim}(f) := \mathrm{Coker}(\ker(f)) = N/\mathrm{Ker}(f).$$

Definition 1.2.7. Let A be an R-semialgebra. Let M and N be A-semimodules. We call an A-semilinear morphism $f : M \to N$:

i-regular (*image-regular*), iff $f(M) = \mathrm{Im}(f)$;

k-regular (*kernel-regular*), iff $[f(m) = f(m') \Rightarrow m + k = m' + k'$ for some $k, k' \in \mathrm{Ker}(f)]$;

regular, iff f is i-regular and k-regular.

1.2.8. We call a (possibly infinite) sequence of A-semimodules

$$\ldots \to M_i \overset{f_i}{\to} M_{i+1} \overset{f_{i+1}}{\to} M_{i+2} \to \ldots \qquad (1.2)$$

zero-sequence, iff $f_{i+1} \circ f_i = 0$ for every i;

exact sequence, iff $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$ for every i;

proper exact sequence, iff $f_i(M_i) = \mathrm{Ker}(f_{i+1})$ for every i.

The sequence (1.2) will be called *regular* (resp. *k-regular*, *i-regular*), iff f_i is regular (resp. k-regular, i-regular) for each i.

1.2.9. Let A be an R-semialgebra. A short exact sequence of A-semimodules

$$0 \to L \overset{f}{\to} M \overset{g}{\to} N \to 0, \tag{1.3}$$

is said to be *left P right Q*, iff f has Property P and g has Property Q.

1.2.10. Let A be an R-semialgebra. We call a morphism of A-semimodules $f : M \to N$:

monomorphism, iff for any A-semimodule L and A-semilinear morphisms $g, h : L \to M$ with $f \circ g = f \circ h$ we have $g = h$;

semi-monomorphism, iff $\mathrm{Ker}(f) = 0$;

epimorphism, iff for any A-semimodule U and A-semilinear morphisms $g, h : N \to U$ with $g \circ f = h \circ f$ we have $g = h$;

semi-epimorphism, iff $\mathrm{Im}(f) = N$;

bimorphism, iff f is a monomorphism and an epimorphism;

semi-bimorphism, iff $\mathrm{Ker}(f) = 0$ and $\mathrm{Im}(f) = N$;

retraction, iff there exists $g : N \to M$ such that $f \circ g = \mathrm{id}_N$;

coretraction, iff there exists $g : N \to M$ such that $g \circ f = \mathrm{id}_M$;

isomorphism, iff f is a retraction and a coretraction;

semi-isomorphism, iff $\mathrm{Ker}(f) = 0$ and f is surjective.

Lemma 1.2.11. ([Tak1981, Proposition 4.3.]) *Let A be an R-semialgebra and $f : M \to N$ be a morphism of A-semimodules.*

1. *The sequence of A-semimodules*

$$0 \to \mathrm{Ker}(f) \overset{\ker(f)}{\rightarrowtail} M \overset{f}{\to} N \overset{\mathrm{coker}(f)}{\twoheadrightarrow} \mathrm{Coker}(f) \to 0 \tag{1.4}$$

is exact. Moreover, the canonical embedding $0 \to \mathrm{Ker}(f) \overset{\ker(f)}{\rightarrowtail} M$ and the canonical surjection $N \overset{\mathrm{coker}(f)}{\twoheadrightarrow} \mathrm{Coker}(f) \to 0$ are regular.

2. *We have a regular exact sequence of A-semimodules*

$$0 \to \mathrm{Im}(f) \rightarrowtail N \twoheadrightarrow N/f(M) \to 0.$$

In particular,

$$N/\mathrm{Im}(f) = N/f(M).$$

3. *We have an exact sequence of A-semimodules*

$$0 \longrightarrow \mathrm{Coim}(f) \overset{f_*}{\longrightarrow} \mathrm{Im}(f) \longrightarrow 0,$$

where $f_*([m]) = f(m)$. *Moreover,*

   (a) f *is* k*-regular* $\Leftrightarrow$ f_* *is injective* $\Leftrightarrow$ $M/\mathrm{Ker}(f) \simeq f(M)$;

   (b) f *is* i*-regular* $\Leftrightarrow$ f_* *is surjective* $\Leftrightarrow$ $f(M) = \mathrm{Im}(f)$;

   (c) f *is regular* $\Leftrightarrow$ f_* *is an isomorphism* $\Leftrightarrow$ $M/\mathrm{Ker}(f) \overset{f_*}{\simeq} \mathrm{Im}(f)$.

Definition 1.2.12. Let A be an R-semialgebra and M an A-semimodule.

1. An A-subsemimodule $L \leq_A M$ is said to be a *regular subsemimodule*, iff the embedding $0 \rightarrowtail L \overset{\iota}{\rightarrow} M$ is i-regular (whence regular).

2. Let ρ be an A-congruence on M and consider the exact sequence of A-semimodules

$$0 \longrightarrow \mathrm{Ker}(\pi_\rho) \rightarrowtail M \overset{\pi_\rho}{\twoheadrightarrow} M/\rho \longrightarrow 0.$$

   We say that M/ρ is a *regular quotient of* M, iff the surjection $M \overset{\pi_\rho}{\twoheadrightarrow} M/\rho \rightarrow 0$ is k-regular (whence regular).

Definition 1.2.13. We call a category $\mathfrak{C}$ with zero object:

1. *semi-additive* (*additive*), iff

   − $\mathrm{Hom}_{\mathfrak{C}}(M, N)$ is an Abelian monoid (Abelian group) for any $M, N \in \mathfrak{C}$;

   − composition of maps in $\mathfrak{C}$ is bisemilinear;

   − $\mathfrak{C}$ has finite direct sums and finite direct products.

2. *pre-semiabelian* (*pre-Abelian*), iff $\mathfrak{C}$ is semi-additive (additive) with kernels and cokernels;

3. *semiabelian*[3] (*Abelian*), iff $\mathfrak{C}$ is

[3] This notion is different from other notions of *semiabelian categories* in the literature (e.g. the notion used in [CK1972]).

– pre-semiabelian (pre-Abelian);

– for any $M, N \in \mathfrak{C}$ and $f \in \mathrm{Hom}_{\mathfrak{C}}(M, N)$, the accompanying morphism f_* (below)

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \mathrm{coim}(f) \downarrow & & \uparrow \mathrm{im}(f) \\ \mathrm{Coim}(f) & \xrightarrow{f_*} & \mathrm{Im}(f) \end{array}$$

is a semi-bimorphism (an isomorphism).

4. *semi-Grothendieck* (*Grothendieck*), iff

   – $\mathfrak{C}$ is semiabelian (Abelian);

   – $\mathfrak{C}$ has directs sums;

   – $\mathfrak{C}$ has a generator;

   – the direct limits of short exact sequences in $\mathfrak{C}$ are exact.

Definition 1.2.14. A category $\mathfrak{C}$ is said to be
complete, iff $\mathfrak{C}$ has equalizers and direct products;
cocomplete, iff $\mathfrak{C}$ has coequalizers and coproducts).

Lemma 1.2.15. ([Tak1982c]) *Let A be an A-semialgebra. The categories $\mathbb{S}_A$ (resp. $\mathbb{S}_A$, ${}_A\mathbb{S}_A$) and $\mathbb{CS}_A$ (resp. ${}_A\mathbb{CS}$, ${}_A\mathbb{CS}_A$) are complete and cocomplete.*

Remark 1.2.16. What M. Takahashi really showed in [Tak1982c] was that the category of semimodules over semirings is *complete* and *c-cocomplete* (a property weaker than cocompleteness). However, F. Linton pointed out that these categories should be cocomplete as they are *varieties* (i.e. *equationally defined algebras* in the sense of Universal Algebra[4]). Moreover, according to H. Porst: "your category of semimodules clearly is a *Birkhoff variety* in the sense of universal algebra, or, equivalently, a *finitary monadic category* of the category of sets. So quite a number of the properties you mention is *automatic*: in particular it will have all limits and colimits (only limits and directed colimits created by the underlying functor), and a generator; the monomorphisms are precisely the injective morphisms, and the epis cannot expected to be surjective."[5]

[4] from a message to the author on March 16th, 2008.
[5] from a message to the author on March 17th, 2008.

Lemma 1.2.17. ([Tak1984a, Theorem 4.2.]) *Let A be an R-semialgebra. Let M be an A-semimodule and $K, L \leq_A M$ be A-subsemimodules.*

1. *We have a canonical surjection*

$$L/(K \cap L) \overset{\theta}{\twoheadrightarrow} (K + L)/K \to 0.$$

2. *If $K \leq_A M$ is a regular subsemimodule, then we have an exact sequence of A-semimodules*

$$0 \to L/(K \cap L) \overset{\theta}{\twoheadrightarrow} (K + L)/K \to 0.$$

3. *If K is an A-semimodule and $L \leq_A M$ is a regular subsemimodule, or if L is an A-semimodule and $K \leq_A M$ is a regular A-subsemimodule, then*

$$L/(K \cap L) \overset{\theta}{\simeq} (K + L)/K.$$

Lemma 1.2.18. ([Tak1984a, Theorem 4.3.]) *Let A be an R-semialgebra. Let M be an A-semimodule and $K, L \leq_A M$ be A-subsemimodules with $K \leq_A L$. Then we have an exact sequence of A-semimodules*

$$0 \to L/K \overset{\iota}{\rightarrowtail} M/K \overset{\pi}{\twoheadrightarrow} M/L \twoheadrightarrow 0 \tag{1.5}$$

with ι being injective and π a regular surjection. Moreover,

$$(M/K)/(L/K) \simeq M/L.$$

In particular, if $L \leq_A M$ is a regular subsemimodule, then (1.5) *is regular.*

Lemma 1.2.19. ([Tak1981, Proposition 4.4.], [Gol1999a, Proposition 15.15]) *Let A be an R-semialgebra and $f : M \to N$ be a morphism of A-semimodules.*

1. *The following are equivalent:*

   (a) *f is injective;*

   (b) *f is k-regular and $0 \to M \overset{f}{\to} N$ is exact;*

   (c) *f is k-regular and* $\mathrm{Ker}(f) = 0$;

   (d) *f is a k-regular* (*semi-*)*monomorphism;*

(e) *f is a monomorphism;*

2. *The following are equivalent:*

   (a) *f is surjective;*

   (b) *f is i-regular and $M \xrightarrow{f} N \to 0$ is exact;*

   (c) *f is i-regular and* $\mathrm{Im}(f) = N$;

   (d) *f is i-regular (semi-)epimorphism;*

   (e) *f is an epimorphism and $f(M) \subseteq N$ is subtractive.*

3. *The following are equivalent:*

   (a) *f is an isomorphism;*

   (b) *f is regular and $0 \to M \xrightarrow{f} N \to 0$ is exact;*

   (c) *f is regular,* $\mathrm{Ker}(f) = 0$ *and* $\mathrm{Im}(f) = N$;

   (d) *f is a regular (semi-)bimorphism;*

   (e) *f is injective and surjective.*

Remark 1.2.20. While every monomorphism of semimodules is injective, an epimorphism of semimodules is surjective if and only if it is i-regular ([TW1989]). This should be added to the major differences between the category of semimodules over semirings and the category of modules over rings (in which epimorphisms are surjective).

Lemma 1.2.21. ([Alt]) *Let A be an R-semialgebra. A morphism $f : M \to N$ in $\mathbb{CS}_A$ is an epimorphism if and only if* $\mathrm{Im}(f) = N$.

Example 1.2.22. The map

$$h : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0,\ (m, n) \mapsto (2m + n, m)$$

is an epimorphism of $\mathbb{N}_0$-semimodules (since $\mathrm{Im}(h) = \mathbb{N}_0 \times \mathbb{N}_0$) that is not surjective.

1.2.23. Let A be an R-semialgebra. We call a short sequence of A-semimodules

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \tag{1.6}$$

semi-regular, iff f is injective (denoted $0 \to M \overset{f}{\rightarrowtail} N$) and g is surjective (denoted $M \overset{g}{\twoheadrightarrow} N \to 0$). Notice that, in light of Lemma 1.2.19, the sequence (1.6) is *semi-regular exact* if and only if f is injective, g is surjective and $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Lemma 1.2.24. ([Tak1981, Proposition 4.5.]) *Let A be an R-semialgebra. Consider an exact sequence of A-semimodules*

$$L \xrightarrow{f} M \xrightarrow{g} N$$

so that $\mathrm{Im}(f) = \mathrm{Ker}(g)$ (*and* $\mathrm{Coker}(f) = M/\mathrm{Im}(f) = M/\mathrm{Ker}(g) = \mathrm{Coim}(g)$). *Then we have two exact sequences*

$$0 \to L/\mathrm{Ker}(f) \xrightarrow{f_*} \mathrm{Ker}(g) \to 0 \text{ and } 0 \to M/\mathrm{Im}(f) \xrightarrow{g_*} \mathrm{Im}(g) \to 0.$$

Lemma 1.2.25. ([Tak1981, Proposition 4.6.]) *Let A be an R-semialgebra. Consider a sequence of A-semimodules*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0. \tag{1.7}$$

1. $\mathrm{Ker}(g) \simeq K$ *if and only if f is regular and* $0 \to K \xrightarrow{f} M \xrightarrow{g} N$ *is exact;*

2. $\mathrm{Coker}(f) \simeq N$ *if and only if g is regular and* $K \xrightarrow{f} M \xrightarrow{g} N \to 0$ *is exact;*

3. $K \overset{f_*}{\simeq} \mathrm{Ker}(g)$ *and* $\mathrm{Coker}(f) \overset{g_*}{\simeq} N$ *if and only if* (1.7) *is regular exact.*

Lemma 1.2.26. *Let A be an R-semialgebra and*

$$L \xrightarrow{f} M \xrightarrow{g} N$$

a sequence of A-semimodules.

1. *Let $M \overset{g}{\hookrightarrow} N$ be a regular embedding. Then f is i-regular if and only if $g \circ f$ is i-regular.*

2. *Let f be surjective. Then g is i-regular if and only if $g \circ f$ is i-regular.*

Proof. 1. Let $M \overset{g}{\hookrightarrow} N$ be a regular embedding.

Assume that f is i-regular. Let $n \in \mathrm{Im}(g \circ f)$, so that $n + g(f(l_1)) = g(f(l_2))$ for some $l_1, l_2 \in L$. Since g is i-regular, $n = g(m)$ for some $m \in M$. Since g is injective, we have $m + f(l_1) = f(l_2)$, i.e. $m \in \mathrm{Im}(f) = f(L)$, whence $n \in (g \circ f)(L)$. So $g \circ f$ is i-regular.

On the other hand, assume $g \circ f$ to be i-regular. Let $m \in \mathrm{Im}(f)$, so that $m + f(l_1) = f(l_2)$ for some $l_1, l_2 \in L$. Then $g(m) \in \mathrm{Im}(g \circ f) = (g \circ f)(L)$. Since g is injective, $m \in f(L)$. So f is i-regular.

2. Let f be surjective, so that f is in particular i-regular.

Assume that g is i-regular. Let $n \in \mathrm{Im}(g \circ f)$, so that $n + g(f(l_1)) = g(f(l_2))$ for some $l_1, l_2 \in L$. Since g is i-regular, $n = g(m)$ for some $m \in M$. Since f is surjective, we have $n \in (g \circ f)(L)$. So $g \circ f$ is i-regular.

On the other hand, assume $g \circ f$ to be i-regular. Let $n \in \mathrm{Im}(g)$, so that $n + g(m_1) = g(m_2)$. Sine f is surjective, we have $n + (g \circ f)(l_1) = (g \circ f)(l_2)$ for some $l_1, l_2 \in L$, i.e. $n \in \mathrm{Im}(g \circ f) = (g \circ f)(L) \subseteq g(M)$. So g is i-regular.■

Corollary 1.2.27. *Let A be an R-semialgebra. If $L \leq_A M \leq_A N$ are A-subsemimodules with $M \leq_A N$ regular, then $L \leq_A M$ is regular if and only if $L \leq_A N$ is regular.*

Lemma 1.2.28. ([Tak1981, Proposition 4.8.]) *Let A be an R-semialgebra. Let M be an A-semimodule and $L \leq_A M$ an A-subsemimodule.*

1. *We have a regular exact sequence of A-semimodules*

$$0 \longrightarrow \mathrm{Ker}(\pi_L) \overset{\iota}{\rightarrowtail} M \overset{\pi_L}{\twoheadrightarrow} M/L \longrightarrow 0.$$

2. *We have an exact sequence of A-semimodules*

$$0 \longrightarrow L \overset{\iota}{\rightarrowtail} M \overset{\pi_L}{\twoheadrightarrow} M/L \longrightarrow 0,$$

with π_L regular. This sequence is regular if and only if $L \leq_A M$ is regular.

3. *We have an exact sequence of A-semimodules*

$$0 \to L \to \mathrm{Im}(\iota) = \mathrm{Ker}(\pi_L) \to 0.$$

The A-subsemimodule $L \leq_A M$ is regular if and only if $L = \mathrm{Ker}(\pi_L)$.

Lemma 1.2.29. ([Tak1982a, Theorem 2.6.]) *Let A be an R-semialgebra and*

$$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$$

be a sequence of A-semimodules and X, Y be arbitrary A-semimodules.

1. *If $0 \to K \overset{f}{\rightarrowtail} M$ is k-regular exact (equivalently, f is injective), then*

$$0 \to \mathrm{Hom}_A(X, K) \overset{(X,f)}{\rightarrowtail} \mathrm{Hom}_A(X, M)$$

is k-regular exact (equivalently, (X, f) is injective).

2. *If $M \xrightarrow{g} N \to 0$ is i-regular exact (equivalently, g is surjective), then*

$$0 \to \mathrm{Hom}_A(N, Y) \xrightarrow{(g,Y)} \mathrm{Hom}_A(M, Y)$$

is k-regular exact (equivalently, (g, Y) is injective).

3. *If f is regular and $0 \to K \xrightarrow{f} M \xrightarrow{g} N$ is exact (equivalently, $K \simeq \mathrm{Ker}(g)$), then the following sequence*

$$0 \to \mathrm{Hom}_A(X, K) \xrightarrow{(X,f)} \mathrm{Hom}_A(X, M) \xrightarrow{(X,g)} \mathrm{Hom}_A(X, N)$$

is exact and (X, f) is regular (equivalently, $\mathrm{Hom}_A(X, K) \simeq \mathrm{Ker}(X, g)$).

4. *If g is regular and $K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is exact (equivalently, $N \simeq \mathrm{Coker}(f)$), then the following sequence*

$$0 \to \mathrm{Hom}_A(N, Y) \xrightarrow{(g,Y)} \mathrm{Hom}_A(M, Y) \xrightarrow{(f,Y)} \mathrm{Hom}_A(K, Y)$$

is exact and (g, Y) is regular (equivalently, $\mathrm{Hom}_A(N, Y) \simeq \mathrm{Ker}(g, Y)$).

Base ring extension

1.2.30. Let $\kappa : A \to B$ be a morphism of half R-algebras. Then we have covariant functors

$$\begin{array}{lllllll} (-)^r_\kappa & : & \mathbb{S}_B & \to & \mathbb{S}_A, & (M,\varphi) & \mapsto & (M, \varphi_M \circ (\mathrm{id}_M \otimes_R \kappa); \\ (-)^l_\kappa & : & {}_B\mathbb{S} & \to & {}_A\mathbb{S}, & (N,\psi) & \mapsto & (N, \psi_N \circ (\kappa \otimes_R \mathrm{id}_N). \end{array}$$

On the otherhand, if B is a half R-algebra, then we have covariant functors

$$\begin{array}{lllllll} -\otimes_A B & : & \mathbb{S}_A & \to & \mathbb{CS}_B, & (M,\varphi_M) & \mapsto & (M \otimes_A B, \mathrm{id}_M \otimes_A \mu_B); \\ B \otimes_A - & : & {}_A\mathbb{S} & \to & {}_B\mathbb{CS}, & (M,\varphi_M) & \mapsto & (B \otimes_A M, \mu_B \otimes_A \mathrm{id}_M). \end{array}$$

1.2.31. If M is a right B-semimodule and N is a left A-semimodule, then there exists a canonical morphism of $\mathbb{N}_0$-semimodules

$$\chi^{(M,N)}_{(A,B)} : M \otimes_A N \to M \otimes_B N. \tag{1.8}$$

Proposition 1.2.32. *Let $\kappa : A \to B$ be a morphism of half R-algebras.*

1. *We have an adjoint pair of covariant functors $(-\otimes_A B, (-)_\kappa)$:*

   $$-\otimes_A B : \mathbb{CS}_A \to \mathbb{CS}_B \text{ and } (-)^r_\kappa : \mathbb{CS}_B \to \mathbb{CS}_A,$$

   where the adjunction is given by the canonical isomorphisms

   $$\mathrm{Hom}_{-B}(M \otimes_A B, N) \simeq \mathrm{Hom}_{-A}(M, \mathrm{Hom}_{-B}(B,N)) \simeq \mathrm{Hom}_{-A}(M,N).$$

2. *We have an adjoint pair of covariant functors $(B \otimes_A -, (-)^l_\kappa)$:*

   $$B \otimes_A - : \ {}_A\mathbb{CS}_A \to \ {}_B\mathbb{CS} \text{ and } (-)^l_\kappa : \ {}_B\mathbb{CS} \to \ {}_A\mathbb{CS},$$

   where the adjunction is given by the canonical isomorphism

   $$\mathrm{Hom}_{B-}(B \otimes_A M, N) \simeq \mathrm{Hom}_{A-}(M, \mathrm{Hom}_{B-}(B,N)) \simeq \mathrm{Hom}_{A-}(M,N).$$

Dual Semimodules

1.2.33. Let A be a half R-algebra. If M is a left A-semimodule, then the cancellative *dual* N_0*-semimodule* ${}^*M := \mathrm{Hom}_{A-}(M, A)$ is a right A-semimodule with structure map given by

$$\varphi_{^*M} : \ {}^*M \otimes_R A \to \ {}^*M, \ f \otimes_R a \mapsto [m \mapsto f(m)a].$$

If, moreover, M is an (A, A)-bisemimodule, then ${}^*M$ is an (A, A)-bisemimodule with left A-semimodule structure given by

$$\varphi^l_{^*M} : A \otimes_R \ {}^*M \to \ {}^*M, \ a \otimes_R g \mapsto [m \mapsto g(ma)].$$

On the other hand, if M is a right A-semimodule, then the cancellative $\mathbb{N}_0$-semimodule $M^* := \mathrm{Hom}_{-A}(M, A)$ is a left A-semimodule with stricture map

$$\varphi^l_{M^*} : A \otimes_R M^* \to M^*, \ a \otimes_R f \mapsto [m \mapsto af(m)].$$

If, moreover, M is an (A, A)-bisemimodule, then M^* is an (A, A)-bisemimodule with left A-semimodule structure given by

$$\varphi_{M^*} : M^* \otimes_R A \to M^*, \ g \otimes_R a \mapsto [m \mapsto g(am)].$$

1.3 Generators and Cogenerators

In this section we consider (sub)generators and cogenerators in the category of right semimodules over a semiring. Let A be an arbitrary (not necessarily commutative) R-semialgebra.

Generators

Definition 1.3.1. Let $\mathcal{U} := \{U_\lambda\}_\Lambda$ be a non-empty class of A-semimodules. We say an A-semimodule X is *generated by* $\mathcal{U}$ (or is $\mathcal{U}$*-generated*), iff for any A-semimodule Y and any A-semilinear morphisms $f, g : X \to Y$ with $f \neq g$ there exists $U_\lambda \in \mathcal{U}$ and an A-semilinear morphism $h : U_\lambda \to X$ such that $f \circ h \neq g \circ h$.

Notation. Let $\mathcal{U}$ be a non-empty class of A-semimodule and N an A-semimodule. We set

$$\begin{aligned} \mathrm{Tr}(\mathcal{U}, N) &:= \sum_{\lambda\in\Lambda} \{f(U_\lambda) \mid f \in \mathrm{Hom}_A(U_\lambda, N)\} \\ \mathrm{Gen}(\mathcal{U}) &:= \{X \text{ is an } A\text{-semimodule} \mid V \text{ is } \mathcal{U}\text{-generated}\} \end{aligned}$$

Definition 1.3.2. We say an A-semimodule W is a *generator*, or *coseparator*, iff any A-semimodule is W-generated.

Proposition 1.3.3. *Let W be an A-semimodule. For an A-semimodule X the following are equivalent:*

1. *X is W-generated;*
2. *there exists a surjective A-semilinear morphism $W^{(\Lambda)} \twoheadrightarrow X \to 0$ for some index set Λ.*

Proposition 1.3.4. *Let W be an A-semimodule. For an A-semimodule X the following are equivalent:*

1. $\mathrm{Tr}(W, X) = X$;

2. *there exists a semi-epimorphism*[6] *of A-semimodules $\pi : W^{(\Lambda)} \twoheadrightarrow X$ for some index set Λ.*

Definition 1.3.5. An A-semimodule F is said to be *free*, iff F has a basis over A (equivalently, $F \simeq A^{(\Lambda)}$, a direct sum of copies of A for some index set Λ).

Proposition 1.3.6. *For every A-semimodule M there exists a surjective morphism of A-semimodules*

$$F \overset{\pi}{\twoheadrightarrow} M \longrightarrow 0.$$

So, in particular, A is a generator.

Definition 1.3.7. An A-semimodule M is said to be a *k-semimodule*, iff there exists a free A-semimodule F and a regular epimorphism (i.e. a k-regular surjective morphism) of A-semimodules $F \overset{\pi}{\twoheadrightarrow} M \longrightarrow 0$.

Definition 1.3.8. We say a left A-semimodule M is *finitely generated*, iff there exists some $n \in \mathbb{N}$ and an epimorphism of A-semimodules $A^n \overset{\pi}{\longrightarrow} M$. If, moreover, the epimorphism π is i-regular (resp. k-regular, regular), we say M is *finitely i-generated* (resp. *finitely k-generated*, *finitely r-generated*).

Definition 1.3.9. We say a finitely generated left A-semimodule M is *finitely presented*, iff there exist $k, n \in \mathbb{N}$ and an exact sequence of A-semimodules

$$A^k \overset{f}{\longrightarrow} A^n \overset{\pi}{\longrightarrow} M,$$

such that π is an epimorphism. If, moreover, the epimorphism π is i-regular (resp. k-regular, regular), we say M is *finitely i-presented* (resp. *finitely k-presented*, *finitely r-presented*).

Subgenerators

1.3.10. Let W be a left A-semimodule. We say an A-semimodule N is *W-subgenerated*, iff N is isomorphic to an A-subsemimodule of a W-generated A-semimodule. With $\sigma[{}_AW] \leq {}_A\mathbb{S}$ we denote the *full* subcategory of W-subgenerated A-semimodules. For every A-semimodule N, the A-subsemimodule

[6] i.e. $\mathrm{Im}(\pi) = X$

$\mathrm{Tr}(\sigma[_AW], N)$ is the *largest* A-subsemimodule of N subgenerated by $_AW$. With $\sigma_{reg}[_AW] \leq {}_A\mathbb{S}$ we denote the full subcategory of $_A\mathbb{S}$, whose objects are isomorphic to *regular* A-subsemimodules of W-generated A-semimodules.

Definition 1.3.11. We say that a left A-semimodule $_AW$ is a *subgenerator*, iff $\sigma[_AW] = {}_A\mathbb{S}$.

Remark 1.3.12. For the well-developed theory of *Wisbauer categories* of type $\sigma[M]$, where M is a module over an arbitrary ring, the reader is referred to [Wis1991] and [Wis1996].

Cogenerators

Definition 1.3.13. Let $\mathcal{U} = \{U_\lambda\}_\Lambda$ be a non-empty class of A-semimodule. We say an A-semimodule Z is *cogenerated by* $\mathcal{U}$, or is $\mathcal{U}$*-cogenerated*, iff for any A-semimodule Y and any A-semilinear morphisms $f, g : Y \to Z$ with $f \neq g$ there exists $\lambda \in \Lambda$ and an A-semilinear morphism $h : Z \to U_\lambda$ such that $h \circ f \neq h \circ g$.

Definition 1.3.14. We say an A-semimodule W is a *cogenerator*, or *separator*, iff any A-semimodule is W-cogenerated.

Notation. Let $\mathcal{U}$ be a non-empty class of A-semimodule and L an A-semimodule. We set

$$\begin{aligned} \mathrm{Rej}(L,\mathcal{U}) &:= \sum_{\lambda\in\Lambda} \{f(L) \mid f \in \mathrm{Hom}_A(L, U_\lambda)\} \\ \mathrm{Cogen}(\mathcal{U}) &:= \{Z \text{ is an } A\text{-semimodule} \mid Z \text{ is } \mathcal{U}\text{-cogenerated}\} \end{aligned}$$

Proposition 1.3.15. ([Alt-a, Proposition 3.1.]) *Let W be an A-semimodule. The following are equivalent for an A-semimodule Z :*

1. *Z is W-cogenerated;*
2. *there exists an injective A-semilinear morphism $0 \to Z \rightarrowtail W^\Lambda$ for some index set Λ.*

Proposition 1.3.16. ([Alt-b, Corollary 2.4.]) *Let W be an A-semimodule. The following are equivalent for an A-semimodule Z :*

1. $\mathrm{Rej}(Z, W) = 0$;

2. *there exists a semi-monomorphism*[7] *of A-semimodules* $\theta : Z \to W^{\Lambda}$ *for some index set* Λ.

[7] i.e. $\mathrm{Ker}(\theta) = 0$

Chapter 2

Projectivity Conditions

In this chapter, we consider projectivity conditions for semimodules over semirings. Throughout, R is a commutative semiring and A, B are arbitrary (not necessarily commutative) R-semialgebras. Several notions and results in this chapter will appear in [Abu1].

2.1 Flat Semimodules

Several notions of flatness for semimodules over semirings were introduced and investigated by various authors (e.g. [Alt2004], [Kat2004]).

Definition 2.1.1. ([Alt2004]) Let W be a left A-semimodule and M a right A-semimodule. We say W is *M-flat* (*M-k-flat*), iff for every A-subsemimodule $L \overset{\iota}{\hookrightarrow} M$, the sequence

$$0 \to L \otimes_A W \overset{\iota \otimes_A \mathrm{id}_W}{\to} M \otimes_A W$$

is exact, i.e. $\iota \otimes_A \mathrm{id}_W$ is a semi-monomorphism (k-exact, i.e. $\iota \otimes_A \mathrm{id}_W$ is a monomorphism). We say ${}_AW$ is *flat* (*k-flat*) w.r.t. $\mathfrak{D} \subseteq \mathbb{S}_A$, iff W is M-flat (M-k-flat) for every $M \in \mathfrak{D}$. In particular, we call ${}_AW$ *flat* (*k-flat*), iff W is *flat* (*k-flat*) w.r.t. $\mathbb{S}_A$. Analogously, one can define flat (k-flat) right A-semimodules.

Proposition 2.1.2. ([Tak1982a, Theorem 5.5.])

1. *Let*

$$K \overset{f}{\to} L \overset{g}{\to} M \to 0$$

be an exact sequence of right A-semimodules and W a left A-semimodule. If g is regular, then the following sequence of Abelian monoids

$$K \otimes_A W \overset{f\otimes_A \mathrm{id}_W}{\longrightarrow} L \otimes_A W \overset{g\otimes_A \mathrm{id}_W}{\twoheadrightarrow} M \otimes_A W \to 0$$

is exact and $g \otimes_A \mathrm{id}_W$ is surjective.

2. *Let*

$$K \overset{f}{\to} L \overset{g}{\to} M \to 0$$

be an exact sequence of left A-semimodules and N a right A-semimodule. If g is regular, then the following sequence of Abelian monoids

$$W \otimes_A K \overset{\mathrm{id}_W\otimes_A f}{\longrightarrow} W \otimes_A L \overset{\mathrm{id}_W\otimes_A g}{\twoheadrightarrow} W \otimes_A M \to 0$$

is exact and $\mathrm{id}_W \otimes_A g$ is surjective.

Lemma 2.1.3. ([Alt2004, Propositions 2.4., 2.5.]) *Let W be a left A-semimodule and M a right A-semimodule. Then ${}_AW$ is M-flat, iff $-\otimes_A W$ is exact w.r.t. all left k-regular right regular short exact sequences with middle term M :*

$$0 \to L \rightarrowtail M \to N \twoheadrightarrow 0.$$

Definition 2.1.4. Let W be a left A-semimodule. We say a left A-semimodule W is *regularity-preserving w.r.t.* M_A, iff for any regular A-subsemimodule $L \overset{\iota}{\hookrightarrow} M$ the map $L \otimes_A W \overset{\iota\otimes_A \mathrm{id}_W}{\longrightarrow} M \otimes_A W$ is also regular. We say ${}_AW$ is *regularity-preserving w.r.t.* $\mathfrak{D} \subseteq \mathbb{S}_A$, iff W is regularity-preserving w.r.t. each $M \in \mathfrak{D}$. If ${}_AW$ is regularity-preserving w.r.t. $\mathbb{S}_A$, we say W is *regularity-preserving*.

Definition 2.1.5. Let W be a left A-semimodule and M a right A-semimodule. We say ${}_AW$ is *M-r-flat*, iff for every *regular* A-subsemimodule $L \overset{\iota}{\hookrightarrow} M$ the following sequence is regular exact

$$0 \to L \otimes_A W \overset{\iota\otimes_A \mathrm{id}_W}{\rightarrowtail} M \otimes_A W.$$

We say W is *r-flat* w.r.t. $\mathfrak{D} \subseteq \mathbb{S}_A$, iff W is M-r-flat for every $M \in \mathfrak{D}$. In particular, we call ${}_AW$ *r-flat*, iff W is r-flat w.r.t. $\mathbb{S}_A$. Analogously, one can define r-flat right A-semimodules.

Lemma 2.1.6. *A left A-semimodule ${}_AW$ is r-flat if and only if $- \otimes_A W$ transfers regular short exact sequences in $\mathbb{S}_A$ into semi-regular exact sequences in $\mathbb{S}_{\mathbb{N}_0}$.*

Remarks 2.1.7.

1. While every k-flat semimodule is flat, the notion of r-flatness on one hand and notions of flatness and k-flatness on the otherhand, are not comparable.

2. Every r-flat A-semimodule is regularity preserving. The converse does not hold (in general).

2.2 Projective Semimodules

Throughout this paper R denotes a commutative *semiring* with $1_R \neq 0_R$, A an associative (not necessarily commutative) *R-semialgebra.*

Definition 2.2.1. Let W be a left A-semimodule and M a left A-semimodule. We say ${}_AW$ is (*weakly*) *M-projective*, iff for every epimorphism (surjective morphism) of A-semimodules $M \stackrel{\pi}{\to} N$ and any A-semilinear morphism $f : W \to N$, there exists an A-semilinear morphism $g : W \to M$ such that $\pi \circ g = f$, i.e. that makes the following diagram commutative

$$\begin{array}{ccc} W & & \\ {\scriptstyle g}\,\vdots & \searrow^{f} & \\ M & \xrightarrow[\pi]{} & N \end{array}$$

We say W is (*weakly*) *projective w.r.t.* $\mathfrak{D} \subseteq {}_A\mathbb{S}$, iff W is M-projective (weakly M-projective) for every $M \in \mathfrak{D}$. If ${}_AW$ is (weakly) projective w.r.t. ${}_A\mathbb{S}$, we say ${}_AW$ is (*weakly*) *projective.*

Lemma 2.2.2. ([DP2005]) *The following are equivalent for a left A-semimodule W :*

1. *${}_AW$ is weakly projective;*
2. *there exists a subclass $\beta = \{(f_\lambda, w_\lambda)\}_\Lambda \subseteq {}^*W \times W$ such that for every $w \in W$*

$$w = \sum_{i=1}^{n} f_{\lambda_i}(w) w_{\lambda_i} \text{ for some finite subset } \beta_w \subseteq \beta.$$

Lemma 2.2.3. ([Alt2004, Propositions 2.5.]) *Let W be a left A-semimodule and M a right A-semimodule. If ${}_AW$ is weakly projective and M is cancellative, then W is M-k-flat if and only if W is M-flat.*

By definition, every projective semimodule is weakly projective, while the converse is not true:

Example 2.2.4. ([Alt, Example 3]) $\mathbb{N}_0 \times \mathbb{N}_0$ is a weakly projective (cancellative) $\mathbb{N}_0$-semimodule that is not projective.

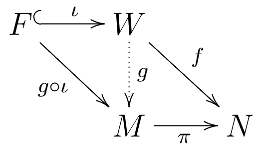
Lemma 2.2.5. ([Tak1983, Proposition 1.16.]) *Let W be a weakly projective A-semimodule and $f : M \to N$ an A-semilinear morphism of right A-semimodules. If N is cancellative and f is k-regular* (*resp. i-regular, regular*), *then $f \otimes_A \mathrm{id}_W$ is k-regular* (*resp. i-regular, regular*).

Example 2.2.6. By Lemma 2.2.5, every (weakly) projective left A-semimodule is regularity-preserving w.r.t. $\mathbb{CS}_A$.

2.3 Locally Projective Semimodules

In this section we generalize the notion of *locally projective* modules (in the sense of Zimmermann-Huisgen [Z-H1976]) to semimodules over semirings. Such modules were investigated by several authors: Garfinkel [Gar1976] called them *universally trorsionless modules* (UTL modules); Ohm and Rush [OR1972] called them *trace modules*, and Raynaud and Gruson [GR1971] called *modules plats et strictement de Mittag Leffler.*

Definition 2.3.1. We say W is (*weakly*) *locally M-projective*, iff for every diagram



with $\pi : M \to N$ an epimorphism (surjective morphism), $F \leq_A W$ a finitely generated A-subsemimodule, and $f : W \to N$ an A-semilinear morphism, there exists an A-semilinear morphism $g : W \to M$, such that the entstanding parallelogram is commutative (i.e. $f \circ \iota = \pi \circ g \circ \iota$). We say W is (*weakly*) *locally projective* w.r.t. $\mathfrak{D} \subseteq {}_A\mathbb{S}$, iff W is (weakly) locally M-projective for every $M \in \mathfrak{D}$. If W is (weakly) locally projective w.r.t. ${}_A\mathbb{S}$, then we say ${}_AW$ is (*weakly*) *locally projective.*

Remarks 2.3.2.
1. Every (weakly) projective A-semimodule is (weakly) locally projective.
2. A finitely generated A-semimodule is (weakly) projective if and only if it is (weakly) locally projective.

Definition 2.3.3. We call a morphism of A-semimodules $\varphi : M \to N$ is called

1-*splitting*, iff for ever $m \in M$ there exists an A-semilinear morphism $\psi_m : N \to M$ such that

$$(\varphi \circ \psi_m)(\varphi(m)) = \varphi(m).$$

finitely splitting, iff for every finite subset $\{m_1, ..., m_n\} \subset M$, there exists an A-semilinear morphism $\psi_E : N \to M$ such that

$$(\varphi \circ \psi)(\varphi(m_i)) = \varphi_E(m_i) \text{ for } i = 1, ..., n;$$

Lemma 2.3.4. *A morphism of A-semimodules $\varphi : M \to N$ is finitely splitting if and only if φ is* 1*-splitting.*

Proof. By induction.■

The proof of the following result is similar to that of [GT2006, Proposition 1.3.11.] (adjusted for semimodules):

Lemma 2.3.5. *Let W be a left A-semimodule. The following are equivalent:*

1. *${}_AW$ is weakly locally projective;*
2. *Every surjective A-semilinear morphism $\varphi : V \to W \twoheadrightarrow 0$ is* finitely splitting.
3. *For each f.g. A-subsemimodule $U \subseteq W$, there exists a f.g. free A-semimodule F_0 and A-semilinear morphisms $\gamma : F \to W$ and $\theta : W \to F$ such that $(\gamma \circ \theta)_{|U} = \mathrm{id}_U$.*
4. *For every finite subset $E \subset W$, there exists $\{(\varphi_1, w_1), ..., (\varphi_k, w_n)\} \subset {}^*W \times W$ such that $\sum_{i=1}^{n} \varphi_i(w)w_i = w$ for every $w \in E$.*
5. *Every surjective A-semilinear morphism $\varphi : V \to W \twoheadrightarrow 0$ is* 1*-splitting.*

Proof. Let W be a left A-semimodule.

(1) $\Rightarrow$ (2) Let V be a left A-semimodule, $\varphi : V \to W \twoheadrightarrow 0$ a surjective morphism of A-semimodules and consider the identity map $\mathrm{id}_W : W \to W$. Let $\{v_1, ..., v_n\} \subset V$ and consider $\{\varphi(v_1), ..., \varphi(v_n)\} \subset W$. Since ${}_AW$ is locally projective, there exists $\psi : W \to V$ such that $(\varphi \circ \psi)(\varphi(v_i)) = \varphi(v_i)$ for $i = 1, ..., n$.

(2) $\Rightarrow$ (1) Let $g : M \to N \twoheadrightarrow 0$ be a surjective morphism of A-semimodules and $f : W \to N$ an A-semilinear morphism. Consider

$$V := \{(m, w) \in M \oplus W \mid g(m) = f(w)\}$$

and the commutative diagram (called *pullback*, or *fibre product*)

$$\begin{array}{ccccc} V & \underset{\psi}{\overset{h}{\rightleftarrows}} & W & \twoheadrightarrow & 0 \\ q\downarrow & \circlearrowleft & \downarrow f & & \\ M & \underset{g}{\longrightarrow} & N & \twoheadrightarrow & 0 \end{array}$$

Notice that since g is surjective, h is also surjective. Let $\{w_1, ..., w_n\} \subset W$ and choose $\{m_1, ..., m_n\} \subset M$ such that $g(m_i) = f(w_i)$. Consider the subset $\{(m_1, w_1), ..., (m_n, w_n)\} \subset V$. By assumption, there exists an A-semilinear morphism $\psi : W \longrightarrow V$ such that $(h \circ \psi)(w_i) = w_i$ for $i = 1, ..., n$. Consider $\widetilde{f} := q \circ \psi : W \to M$. Then we have $(g \circ \widetilde{f} \circ \iota)(w_i) = (g \circ q \circ \psi)(w_i) = g(m_i) = f(w_i)$ for $i = 1, ..., n$. Consequently, ${}_AW$ is locally projective.

(2) $\Rightarrow$ (3) Let $U \leq_A W$ be a f.g. A-subsemimodule with generating set $E := \{w_1, \cdots, w_n\}$. Choose a free A-semimodule F with basis $\{x_\lambda \mid \lambda \in \Lambda\}$ and a surjective morphism of A-semimodules $F \overset{\beta}{\twoheadrightarrow} W \to 0$. For each $i = 1, \cdots, n$ let $x_{\lambda_i} \in F$ be such that $\beta(x_{\lambda_i}) = w_i$ and let $F_0 := \oplus_{i=1}^{k} Ax_{\lambda_i} \leq_A F$ be the free A-subsemimodule of F with basis $\{x_{\lambda_1}, \cdots, x_{\lambda_n}\}$. By assumption, there exists an A-semilinear morphism $\psi : W \to F$ such that $(\beta \circ \psi)(\beta(x_{\lambda_i})) = \beta(x_{\lambda_i})$ for $i = 1, \cdots, n$. Let $\gamma := \beta_{|_{F_0}}$ be the restriction of β to F_0, $\pi : F \to F_0$ be the canonical projection and consider $\theta := \pi \circ \psi : W \to F_0$. Then we have for $i = 1, \cdots, n$:

$$(\gamma \circ \theta)(w_i) = (\gamma \circ \pi \circ \psi)(w_i) = (\beta \circ \psi)(\beta(x_{\lambda_i})) = \beta(x_{\lambda_i}) = w_i.$$

(3) $\Rightarrow$ (4) Fix the notation in the proof of "(2) $\Rightarrow$ (3)" above. For every $x \in F_0$, we have a unique representation $x = \sum_{i=1}^{n} g_i(x) x_{\lambda_i}$ with coordinate functions $g_i \in {}^*F_0$, $i = 1, \cdots, n$. For $i = 1, \cdots, n$, consider $(\varphi_i, w_i) := (g_i \circ \theta, \gamma(x_{\lambda_i})) \in {}^*W \times W$. Then we have for every $w \in E$:

$$w = (\gamma \circ \theta)(w) = \gamma(\sum_{i=1}^{n} g_i(\theta(w)) x_{\lambda_i} = \sum_{i=1}^{n} (g_i \circ \theta)(w) \gamma(x_{\lambda_i}) = \sum_{i=1}^{n} \varphi_i(w) w_i.$$

(4) $\Rightarrow$ (2) Let $\varphi : V \twoheadrightarrow W \to 0$ be a surjective morphism of left A-semimodules. Let $E = \{v_1, ..., v_k\} \subseteq V$ and set $w_i := \varphi(v_i)$ for $i = 1, \cdots, k$. By assumption, there exist $\{(\varphi_1, w_1), \cdots, (\varphi_n, w_n)\} \subseteq {}^*W \times W$ such that $w_j = \sum_{i=1}^{n} \varphi_j(w_j) w_i$ for $j = 1, \cdots, k$. Define

$$\psi : W \to V, \ w \mapsto \sum_{i=1}^{n} \varphi_i(w) v_i.$$

Then we have for $j = 1, \cdots, k$:

$$\begin{array}{rclcl} (\varphi \circ \psi)(\varphi(v_j)) & = & \varphi[\sum_{i=1}^{n} \varphi_i(\varphi(v_j)) v_i] & = & \varphi[\sum_{i=1}^{n} \varphi_i(w_j) v_i] \\ & = & \sum_{i=1}^{n} \varphi_i(w_j) \varphi(v_i) & = & \sum_{i=1}^{n} \varphi_i(w_j) w_i \\ & = & w_j & = & \varphi(v_j). \end{array}$$

(2) $\Leftrightarrow$ (5) follows by Lemma 2.3.4.■

α-Semimodules

In what follows we introduce a class of semimodules over semialgebras that contains all weakly (locally) projective semimodules.

Definition 2.3.6. We say a left A-semimodule W satisfies the *α-condition with respect to* $M \in \mathbb{CS}_A$, iff the following canonical map is injective

$$\alpha_M^W : M \otimes_A W \rightarrowtail \mathrm{Hom}_{\mathbb{N}_0}({}^*W, M),\ m \otimes_A w \mapsto [f \mapsto mf(w)]. \tag{2.1}$$

We say that ${}_AW$ satisfies the α-condition w.r.t. a *subclass* $\mathfrak{D} \subseteq \mathbb{CS}_A$, iff W satisfies the α-condition w.r.t. every $M \in \mathfrak{D}$. In particular, we say that ${}_AW$ is an *α-semimodule*, iff ${}_AW$ satisfies the α-condition w.r.t. $\mathbb{CS}_A$.

Lemma 2.3.7. *Let A, B be half R-semialgebras, W a cancellative (A, B)-bisemimodule and W' a cancellative (B, A)-bisemimodule.*

1. *If ${}_AW$ and ${}_BW'$ are α-semimodules, then for every cancellative right A-semimodule M_A the following map is injective*

$$\begin{array}{rcrcl}\alpha_M^r & : & M \otimes_A W \otimes_B W' & \to & \mathrm{Hom}_{\mathbb{N}_0}({}^*W' \otimes_B {}^*W, M), \\ & & m \otimes_A w \otimes_B w' & \mapsto & [f' \otimes_B f \mapsto mf(wf'(w'))].\end{array}$$

2. *If W_B and W'_A are α-semimodules, then for every cancellative left A-semimodule ${}_AN$, the following map is injective*

$$\begin{array}{rcrcl}\alpha_M^l & : & W \otimes_B W' \otimes_A N & \to & \mathrm{Hom}_{\mathbb{N}_0}(W'^* \otimes_B W^*, N), \\ & & w \otimes_B w' \otimes_A n & \mapsto & [f' \otimes_B f \mapsto f'(f(w)w')n].\end{array}$$

Proof. We prove (1) as the proof of (2) is similar. Let M_A be an arbitrary cancellative right A-semimodule and consider the following commutative diagram

$$\begin{array}{ccc}M \otimes_A W \otimes_B W' & \xrightarrow{\alpha_M^r} & \mathrm{Hom}_{-A}({}^*W' \otimes_B {}^*W, M) \\ \downarrow{\scriptstyle \alpha^{W'}_{M \otimes_A W}} & & \downarrow{\scriptstyle \zeta^r} \\ \mathrm{Hom}_{-B}({}^*W', M \otimes_A W) & \xrightarrow[({}^*W', \alpha_M^W)]{} & \mathrm{Hom}_{-B}({}^*W', \mathrm{Hom}_{-A}({}^*W, M))\end{array}$$

where ζ^r is the canonical isomorphism (see Proposition 1.1.31). Since α_M^W is a monomorphism, $({}^*W', \alpha_M^W)$ is a monomorphism too. So

$$\alpha_M^r = (\zeta^r)^{-1} \circ ({}^*W', \alpha_M^W) \circ \alpha_{M \otimes_A W}^{W'}$$

is injective.■

Lemma 2.3.8. *Let W be a cancellative left A-semimodule. If ${}_AW$ is an α-semimodule, then W is k-flat w.r.t. $\mathbb{CS}_A$.*

Proof. Assume ${}_AW$ is an α-semimodule. Let $M \in \mathbb{CS}_A$ be an arbitrary cancellative right A-semimodule, $L \overset{\iota}{\hookrightarrow} M$ an A-subsemimodule and consider the commutative diagram

$$\begin{array}{ccc} L \otimes_A W & \xrightarrow{\alpha_L^W} & \mathrm{Hom}_{\mathbb{N}_0}({}^*W, L) \\ {\scriptstyle \iota_L \otimes_A \mathrm{id}_W} \downarrow & & \downarrow {\scriptstyle ({}^*W, \iota)} \\ M \otimes_A W & \xrightarrow[\alpha_M^W]{} & \mathrm{Hom}_{\mathbb{N}_0}({}^*W, M) \end{array}$$

Since L is cancellative, α_L^W is injective. Moreover, the morphism $({}^*W, \iota)$ is injective too, whence $\iota_L \otimes_A \mathrm{id}_W$ is injective. Hence, ${}_AW$ is M-k-flat.■

Definition 2.3.9. Let A be a half R-algebra. We say a cancellative left A-semimodule W is a *half A-module*, iff for every $M \in \mathbb{CS}_A$:

$$\alpha_M^W(M \otimes_A W) \subseteq \mathrm{Hom}_{-A}({}^*W, M) \text{ is subtractive.} \tag{2.2}$$

A half A-module that satisfies the α-condition is said to be a *half α-module*.

Definition 2.3.10. Let A, B be half R-algebras. We say an (A, B)-bisemimodule W is a *half (A, B)-bimodule*, iff ${}_AW$ and W_B are half bimodules. If moreover, ${}_AW$ and W_B satisfy the α-condition, we say W is a *half α-bimodule.*

Notation. Let A, B be half R-algebras. The category of left (right) half A-modules will be denoted with ${}_A\mathbb{H}$ ($\mathbb{H}_A$) and the full subcategory of half α-semimodules with ${}_A^\alpha\mathbb{H}$ ($\mathbb{H}_A^\alpha$). The category of half (A, B)-bimodules will be denoted with ${}_A\mathbb{H}_B$. We denote with ${}_A^\alpha\mathbb{H}_B \leq {}_A\mathbb{H}_B$ (resp. ${}_A\mathbb{H}_B^\alpha \leq {}_A\mathbb{H}_B$, ${}_A^\alpha\mathbb{H}_B^\alpha \leq {}_A\mathbb{H}_B$) the full subcategory of (A, B)-bisemimodules which satisfy the left (resp. right, left and right) α-condition.

Remark 2.3.11. If A is cancellative, then ${}_AA_A$ is clearly a half (A,A)-bimodule (i.e. $A \in {}_A\mathbb{H}_A$). So the definition of a half module we introduced above is consistent with the definition of a half-ring (e.g. [Gol1999a]).

Lemma 2.3.12. *Let A be a half R-algebra.*

1. *If ${}_AW$ is a cancellative α-semimodule, then ${}_AW$ is A-cogenerated.*
2. *If ${}_AW$ is a half α-module, then ${}_AW$ is r-flat w.r.t. $\mathbb{CS}_A$.*

Proof. 1. Since A and W are cancellative, we have an embedding of left A-semimodules

$$W = \mathfrak{c}(W) \simeq A \otimes_A W \overset{\alpha_A^W}{\hookrightarrow} \mathrm{Hom}_{-A}({}^*W, A) \subseteq A^{{}^*W},$$

whence ${}_AW$ is A-cogenerated.

2. Assume ${}_AW$ is a half α-module. By "1", it remains to show that W is regularity-preserving w.r.t. $\mathbb{CS}$. Let M be a right A-subsemimodule and $L \leq_A M$ a regular A-semimodule. If $\sum m_i \otimes_A w_i \in \mathrm{Im}(\iota_L \otimes_A \mathrm{id}_W)$, then $\sum m_i \otimes_A w_i + \sum \widetilde{l}_j \otimes_A \widetilde{w}_j = \sum \widehat{l}_k \otimes_A \widehat{w}_k$ for some $\sum \widetilde{l}_j \otimes_A \widetilde{w}_j$, $\sum \widehat{l}_k \otimes_A \widehat{w}_k \in L \otimes_A W$. It follows then that $\alpha_M^W(\sum m_i \otimes_A w_i) + \alpha_M^W(\sum \widetilde{l}_j \otimes_A \widetilde{w}_j) = \alpha_M^W(\sum \widehat{l}_k \otimes_A \widehat{w}_k)$, whence $\sum m_i f(w_i) + \sum \widetilde{l}_j f(\widetilde{w}_j) = \sum \widehat{l}_k f(\widehat{w}_k)$ for every $f \in {}^*W$. Since $L \leq_A M$ is regular and $\sum m_i f(w_i) \in \mathrm{Im}(\iota_L)$, we conclude that $\sum m_i f(w_i) \in L$ for every $f \in {}^*W$, whence $\alpha_M^W(\sum m_i \otimes_A w_i) \subseteq \mathrm{Hom}_{-A}({}^*W, L)$. By assumption $\alpha_L^W(L \otimes_A W) \subseteq \mathrm{Hom}_{-A}({}^*W, L)$ is subtractive, whence $\alpha_M^W(\sum m_i \otimes_A w_i) \in \alpha_M^W(L \otimes_A W)$. Since α_M^W is injective, we conclude that $\sum m_i \otimes_A w_i \in L \otimes_A W$. Consequently, $L \otimes_A W \leq M \otimes_A W$ is a regular subsemimodule.■

Lemma 2.3.13. *Let ${}_AW$ be a half α-module and M a cancellative right A-semimodule. For every regular A-subsemimodule $L \leq_A M$ and any $\sum m_i \otimes_A w_i \in M \otimes_A W$ we have*

$$\sum m_i \otimes_A w_i \in L \otimes_A W \Longleftrightarrow \sum m_i f(w_i) \in L \text{ for all } f \in {}^*W.$$

Proof. Consider the regular exact sequence of A-semimodules

$$0 \to L \overset{\iota}{\rightarrowtail} M \overset{\pi}{\twoheadrightarrow} M/L \to 0.$$

By Lemma 2.3.8, ${}_AW$ is r-flat w.r.t. $\mathbb{CS}_A$ and so we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L \otimes_A W & \xrightarrow{\iota_L \otimes_A \mathrm{id}_W} & M \otimes_A W & \xrightarrow{\pi \otimes_A \mathrm{id}_W} & M/L \otimes_A W & \longrightarrow & 0 \\ & & \downarrow \alpha_L^W & & \downarrow \alpha_M^W & & \downarrow \alpha_{M/L}^W & & \\ 0 & \longrightarrow & \mathrm{Hom}_{-A}({}^*W, L) & \xrightarrow[({}^*W,\iota_L)]{} & \mathrm{Hom}_{-A}({}^*W, M) & \xrightarrow[({}^*W,\pi)]{} & \mathrm{Hom}_{-A}({}^*W, M/L) & & \end{array}$$

with the first row exact, left regular and right i-regular (see Lemma 2.1.6), while the second row is exact and left regular (e.g. Lemma 1.2.29). If $\sum m_i \otimes_A w_i \in M \otimes_A W$, then $\sum m_i f(w_i) \in L$ for every $f \in {}^*W$ if and only if

$$\begin{array}{rcccl} \sum m_i \otimes_A w_i & \in & \mathrm{Ke}(({}^*W, \pi) \circ \alpha_M^W) & = & \mathrm{Ke}(\alpha_{M/L}^W \circ (\pi \otimes_A \mathrm{id}_W)) \\ & = & \mathrm{Ke}(\pi \otimes_A \mathrm{id}_W) & = & L \otimes_A W, \end{array}$$

(the last equality follows by Lemma 1.2.25 and the regularity of $\iota_L \otimes_A \mathrm{id}_W$).■

Proposition 2.3.14. *Let W be a left A-semimodule. If ${}_AW$ is (weakly) locally projective, then ${}_AW$ is an α-semimodule.*

Proof. Assume ${}_AW$ to be weakly locally projective. Let M be a cancellative right A-semimodule and $\sum_{i=1}^{n} m_i \otimes_A w_i \in \mathrm{Ker}(\alpha_M^W)$. By Lemma 2.3.5, there exists for each $i = 1, \cdots, n$ a finite subset $\{(\varphi_{i1}, w_{i1}), ..., (\varphi_{in_i}, w_{in_i})\} \subset {}^*W \times W$ such that $w_i = \sum_{j=1}^{n_i} \varphi_{ij}(w_i) w_{ij}$. It follows then that

$$\sum_{i=1}^{n} m_i \otimes_A w_i = \sum_{i=1}^{n} m_i \otimes_A \sum_{j=1}^{n_i} \varphi_{ij}(w_i) w_{ij} = \sum_{j=1}^{n_i} [\sum_{i=1}^{n} m_i \varphi_{ij}(w_i)] \otimes_A w_i^j = 0.$$

Consequently, ${}_AW$ is an α-semimodule.

Remarks 2.3.15. A module W over a ring is locally projective if and only if W is an α-module. However, this is not the case for half-modules over half-algebras.

Proposition 2.3.16. *Let A be a half R-algebra, ${}_AW$ a cancellative left A-semimodule and for every $w \in W$ set*

$$I_w := \{f(w) \mid f \in {}^*W\}.$$

Assume that

$$A/I_w \otimes_A W \simeq W/I_w W \text{ for every } w \in W \tag{2.3}$$

Then ${}_AW$ is weakly locally projective if and only if ${}_AW$ is an α-semimodule.

Proof. By Proposition 2.3.14 it remains to prove that if ${}_AW$ is an α-semimodule, then - assuming (2.3) - ${}_AW$ is weakly locally projective. Let $w \in W$ and consider the morphism $\delta := \alpha^W_{A/I_w} \circ \varpi^{-1}$ given by

$$\delta : \alpha^W_{A/I_w} \circ \varpi^{-1} : W/I_wW \rightarrow \mathrm{Hom}_{-A}({}^*W, A/I_w),\ \overline{w} \mapsto [f \mapsto \overline{f(w)} = \overline{0}].$$

Since δ is injective, we conclude that $w \in I_wW$. Since $w \in W$ is arbitrary, we have $W = I_wW$ and so ${}_AW$ is weakly locally projective by Lemma 2.3.5.■

Chapter 3

Semicorings

Throughout this chapter, R denotes a *commutative* semiring with $1_R \neq 0_R$ and by A, B arbitrary (not necessarily commutative) R-semialgebras.

3.1 Half-rings over Semialgebras

In this section, we generalize the notion of semialgebras over commutative semirings to semirings over semialgebras.

3.1.1. With a *half A-ring* we mean a cancellative A-bisemimodule with A-bisemilinear morphisms $\mu_{\mathcal{A}} : \mathcal{A} \otimes_A \mathcal{A} \to \mathcal{A}$ and $\eta_{\mathcal{A}} : A \to \mathcal{A}$, such that the following diagrams become commutative

$$\begin{array}{ccc} \mathcal{A} \otimes_A \mathcal{A} \otimes_A \mathcal{A} & \xrightarrow{\mu_{\mathcal{A}} \otimes_A \mathrm{id}_{\mathcal{A}}} & \mathcal{A} \otimes_A \mathcal{A} \\ {\scriptstyle \mathrm{id}_{\mathcal{A}} \otimes_A \mu_{\mathcal{A}}} \downarrow & & \downarrow {\scriptstyle \mu_{\mathcal{A}}} \\ \mathcal{A} \otimes_A \mathcal{A} & \xrightarrow[\mu_{\mathcal{A}}]{} & \mathcal{A} \end{array} \qquad \begin{array}{ccccc} & {\scriptstyle \eta_{\mathcal{A}} \otimes_A \mathrm{id}_{\mathcal{A}}} \nearrow & \mathcal{A} \otimes_A \mathcal{A} & \nwarrow {\scriptstyle \mathrm{id}_{\mathcal{A}} \otimes_A \eta_{\mathcal{A}}} & \\ & & \downarrow {\scriptstyle \mu_{\mathcal{A}}} & & \\ A \otimes_A \mathcal{A} & \xrightarrow[\vartheta^l_{\mathcal{A}}]{} & \mathcal{A} & \xleftarrow[\vartheta^r_{\mathcal{A}}]{} & \mathcal{A} \otimes_A A \end{array}$$

3.1.2. Let $(\mathcal{A} : A)$ and $(\mathcal{B} : B)$ be half rings. A *half-ring morphism* from $\mathcal{A}$ to $\mathcal{B}$ is a pair $(f : \gamma)$, where $\gamma : A \to B$ is a morphism of R-semialgebras and $f : \mathcal{A} \to \mathcal{B}$ is an A-bisemilinear morphism such that

$$f \circ \mu_{\mathcal{A}} = \mu_{\mathcal{B}} \circ \chi^{(\mathcal{A},\mathcal{B})}_{(A,B)} \circ (f \otimes_A f) \text{ and } f \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}} \circ \gamma.$$

The category of half rings will be denoted with **HRing**.

3.1.3. Let $(\mathcal{A}, \mu_{\mathcal{A}}, \eta_{\mathcal{A}})$ be a half A-ring. With a right $\mathcal{A}$-semimodule, we mean an A-semimodule M along with an A-semilinear morphism $\varphi_M : M \otimes_A \mathcal{A} \to \mathfrak{c}(M)$, such that

$$\varphi_M \circ (\mathrm{id}_M \otimes_A \mu_{\mathcal{A}}) = \ell(\varphi_M) \circ (\varphi_M \otimes_A \mathrm{id}_{\mathcal{A}}) \text{ and } \varphi_M \circ (\mathrm{id}_M \otimes_A \eta_{\mathcal{A}}) = \vartheta^r_M.$$

For two right A-semimodules M, N we mean by a morphism of $\mathcal{A}$-semimodules (or an $\mathcal{A}$-semilinear morphism) an A-semilinear morphism $f : M \to N$, such that

$$\mathfrak{c}(f) \circ \varphi_M = \varphi_N \circ (f \otimes_A \mathrm{id}_{\mathcal{A}}).$$

The category of right $\mathcal{A}$-semimodules and $\mathcal{A}$-semilinear morphisms is denoted by $\mathbb{S}_{\mathcal{A}}$. The category ${}_{\mathcal{A}}\mathbb{S}$ of left $\mathcal{A}$-semimodules is analogously defined. For a half A-ring $\mathcal{A}$ and a half B-ring $\mathcal{B}$, we denote with ${}_{\mathcal{A}}\mathbb{S}_{\mathcal{B}}$ the category of $(\mathcal{A}, \mathcal{B})$-bisemimodules that are defined in the obvious way.

3.2 Basic Definitions

3.2.1. An A-*semicoring* is a *cancellative* A-bisemimodule $\mathcal{C}$ associated with A-bilinear morphisms

$$\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C} \text{ and } \varepsilon_{\mathcal{C}} : \mathcal{C} \to A,$$

such that the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \\ {\scriptstyle \Delta_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}} \otimes_A \Delta_{\mathcal{C}}} \\ \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow[\Delta_{\mathcal{C}} \otimes_A \mathrm{id}_{\mathcal{C}}]{} & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array} \qquad \begin{array}{ccccc} \mathcal{C} \otimes_A \mathcal{C} & \xleftarrow{\Delta_{\mathcal{C}}} & \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \\ {\scriptstyle \varepsilon_{\mathcal{C}} \otimes_A \mathrm{id}_{\mathcal{C}}}\downarrow & \nearrow{\scriptstyle \vartheta^l_{\mathcal{C}}} & & \nwarrow{\scriptstyle \vartheta^r_{\mathcal{C}}} & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}} \otimes_A \varepsilon_{\mathcal{C}}} \\ A \otimes_A \mathcal{C} & & & & \mathcal{C} \otimes_A A \end{array} \tag{3.1}$$

The map $\Delta_{\mathcal{C}}$ (respectively $\varepsilon_{\mathcal{C}}$) is called the *comultiplication* (respectively the *counity*) of $\mathcal{C}$.

3.2.2. Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}} : A)$ and $(\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}} : B)$ be semicorings. A *morphism of semicorings* $(f : \gamma) : (\mathcal{C} : A) \to (\mathcal{D} : B)$ consists of a morphism of R-semialgebras $\gamma : A \to B$ and an (A, A)-bilinear morphism $f : \mathcal{C} \to \mathcal{D}$, such that

$$\chi^{(\mathcal{C},\mathcal{D})}_{(A,B)} \circ (f \otimes_A f) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{D}} \circ f \text{ and } \varepsilon_{\mathcal{D}} \circ f = \gamma \circ \varepsilon_{\mathcal{C}}.$$

With **SC** we denote the category of semicorings.

Notation. Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an A-semicoring. For $c \in \mathcal{C}$ we use Sweedler-Heyneman's $\sum$-notation:

$$\Delta(c) = \sum c_1 \otimes_A c_2 \in \mathcal{C} \otimes_A \mathcal{C}.$$

Moreover we define Δ_n inductively as $\Delta_1 := \Delta$ and

$$\Delta_n := (\Delta \otimes_A id^{n-1}) \circ \Delta_{n-1} : \mathcal{C} \to \mathcal{C}^{n+1},\ c \longmapsto \sum c_1 \otimes_A ... \otimes_A c_n \text{ for } n \geq 2.$$

Definition 3.2.3. 1. Let $(\mathcal{C}, \Delta, \varepsilon)$ be an A-semicoring. We call a left (right) A-subsemimodule $K \leq_A \mathcal{C}$ a *left* (*right*) $\mathcal{C}$-*coideal*, iff $\Delta(K) \subseteq \mathcal{C} \otimes_A K$ ($\Delta(K) \subseteq K \otimes_A \mathcal{C}$). Moreover, we call an A-subsemimodule $K \leq_{(A,A)} \mathcal{C}$ a $\mathcal{C}$-*coideal* iff

$$\Delta(K) \subseteq \mathcal{C} \otimes_A K + K \otimes_A \mathcal{C};$$

and a $\mathcal{C}$-*bicoideal*, iff

$$\Delta(K) \subseteq (\iota_K \otimes_A \mathrm{id}_{\mathcal{C}})(K \otimes_A \mathcal{C}) \cap (\mathrm{id}_{\mathcal{C}} \otimes_A \iota_K)(\mathcal{C} \otimes_A K).$$

Dual semirings of semicorings

3.2.4. Let A be a half R-algebra and $(\mathcal{C},\Delta,\varepsilon)$ an A-semicoring.

${}^{*}\mathcal{C} := (\mathrm{Hom}_{A-}(\mathcal{C},A), *^{l})$ is an A-semiring with unity ε and multiplication

$$(f *_{l} \widetilde{f})(c) = \sum \widetilde{f}(c_1 f(c_2)) \text{ for all } f,\widetilde{f} \in {}^{*}\mathcal{C} \text{ and } c \in \mathcal{C};$$

$\mathcal{C}^{*} := (\mathrm{Hom}_{-A}(\mathcal{C},A), *^{r})$ is an A-semiring with unity ε and multiplication

$$(g *_{r} \widetilde{g})(c) = \sum g(\widetilde{g}(c_1)c_2) \text{ for all } g,\widetilde{g} \in \mathcal{C}^{*} \text{ and } c \in \mathcal{C};$$

${}^{*}\mathcal{C}^{*} := (\mathrm{Hom}_{(A,A)}(\mathcal{C},A), *)$ is an A-semiring with unity ε and multiplication

$$(f * g)(c) = \sum f(c_1)g(c_2) \text{ for all } f,g \in {}^{*}\mathcal{C}^{*} \text{ and } c \in \mathcal{C}.$$

Remark 3.2.5. Let A be a half R-algebra and $\mathcal{C}$ an A-coring. Then $\mathcal{C}$ is a $(\mathcal{C}^{*}, {}^{*}\mathcal{C})$-bisemimodule with actions

$$g \rightharpoonup c := \sum g(c_1)c_2 \text{ and } c \leftharpoonup f := \sum c_1 f(c_2) \text{ for } f \in {}^{*}\mathcal{C}, g \in \mathcal{C}^{*}, c \in \mathcal{C}.$$

Notice that for any $f,\widetilde{f} \in {}^{*}\mathcal{C}$ and any $g,\widetilde{g} \in \mathcal{C}^{*}$ we have

$$\begin{array}{lllll} (f *_{l} \widetilde{f})(c) & = & \widetilde{f}(\sum c_1 f(c_2)) & = & \widetilde{f}(c \leftharpoonup f); \\ (g *_{r} \widetilde{g})(c) & = & g(\sum \widetilde{g}(c_1)c_2) & = & g(\widetilde{g} \rightharpoonup c) \end{array}$$

3.3 Examples

In this section we introduce several examples of semicorings that are not (necessarily) corings. These examples are analogous to examples of corings over rings (e.g. [BW2003, 17.2.-17.7.]). With A, B we denote half R-algebras.

Example 3.3.1. (The *trivial semicoring*) The half R-algebra A is a *trivial* A-semicoring, where

$$\begin{array}{llllll}\Delta_A & : & A & \to & A \otimes_A A, & a & \mapsto & a \otimes_A 1_A = 1_A \otimes_A a; \\ \varepsilon_A & : & A & \to & A, & a & \mapsto & a1_A.\end{array}$$

Example 3.3.2. (The *Sweedler semicoring*) Let $\kappa : A \to B$ be a morphism of R-semialgebras, and consider B as an (A, A)-bisemimodule in the canonical way (see 1.2.30). The canonical (A, A)-bisemimodule $\mathcal{C} := B \otimes_A B$ has a B-semicoring structure through

$$\begin{array}{llllllll}\Delta_{\mathcal{C}} & : & \mathcal{C} & \to & \mathcal{C} \otimes_B \mathcal{C}, & b \otimes_A \widetilde{b} & \mapsto & (b \otimes_A 1_B) \otimes_B (1_B \otimes_R \widetilde{b}); \\ \varepsilon_{\mathcal{C}} & : & \mathcal{C} & \to & B, & b \otimes_A \widetilde{b} & \mapsto & b\widetilde{b}.\end{array}$$

Example 3.3.3. (The *base ring extension semicoring*) Let $\kappa : A \to B$ be a morphism of half R-algebras, and consider B as an (A, A)-bisemimodule in the canonical way (see 1.2.30). Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be any A-semicoring. The canonical (B, B)-bisemimodule $\mathcal{D} := B \otimes_A \mathcal{C} \otimes_A B$ is a B-semicoring with

$$\begin{array}{llll}\Delta_{\mathcal{D}} & : \mathcal{D} \to \mathcal{D} \otimes_B \mathcal{D}, & b \otimes_A c \otimes_A \widetilde{b} & \mapsto \sum_{i=1}^{n} (b \otimes_A c_1 \otimes 1_B) \otimes_B (1_B \otimes c_2 \otimes \widetilde{b}); \\ \varepsilon_{\mathcal{D}} & : \mathcal{D} \to B, & b \otimes_A c \otimes_A \widetilde{b} & \mapsto b\kappa(\varepsilon_{\mathcal{C}}(c))\widetilde{b}.\end{array}$$

Example 3.3.4. Let G be a non-empty set an $Q \subseteq G \times G$ be a transitive and reflexive relation such that

$$w(x, z) := \{y \in G \mid (x, y), (y, z) \in Q\}$$

is finite for all $x, z \in G$. The canonical A-bisemimodule $\mathcal{C} := A^{(Q)}$ is an A-semicoring with coproduct and counity defined on the basis as

$$\begin{array}{llllllll}\Delta_{\mathcal{C}} & : & \mathcal{C} & \to & \mathcal{C} \otimes_A \mathcal{C}, & (x, z) & \mapsto & \sum_{y \in w(x,z)} (x, y) \otimes_A (y, z); \\ \varepsilon_{\mathcal{C}} & : & \mathcal{C} & \to & A, & (x, z) & \mapsto & \delta_{x,z}.\end{array}$$

Example 3.3.5. (The *group-like semicoring*) Let G be any set. The direct sum $\mathcal{C} := A^{(G)}$ is an A-semicoring with multiplication and counity defined on the basis G :

$$\begin{array}{lllllll}\Delta_{\mathcal{C}} & : & \mathcal{C} & \rightarrow & \mathcal{C}\otimes_A \mathcal{C}, & g & \mapsto & g\otimes_A g; \\ \varepsilon_{\mathcal{C}} & : & \mathcal{C} & \rightarrow & A, & g & \mapsto & 1_A.\end{array}$$

Example 3.3.6. (The *semicoring of a projective semimodule*) Let A be a half R-algebra, P a (B,A)-bisemimodule, where P_A is cancellative, finitely generated and weakly projective with finite basis $\{(g_1,p_1),\cdots,(g_n,p_n\} \subseteq P^* \times P$. Then $\mathcal{C} := P^* \otimes_B P$ is an A-semicoring with

$$\begin{array}{llllllll}\Delta_{\mathcal{C}} & : & \mathcal{C} & \rightarrow & \mathcal{C}\otimes_A \mathcal{C}, & f\otimes_B p & \mapsto & \sum(f\otimes_B p_i)\otimes_A (g_i\otimes_A p); \\ \varepsilon_{\mathcal{C}} & : & \mathcal{C} & \rightarrow & A, & f\otimes_B p & \mapsto & f(p).\end{array}$$

Example 3.3.7. (The *matrix semicoring*) Let $M_n(A)$ denote the A-bisemimodule of $n\times n$-matrices over A with basis $\{e_{i,j}\}_{i,j=1}^n$. Then $M_n(A)$ is an A-semicoring with

$$\begin{array}{llllllll}\Delta_{M_n(A)} & : & M_n(A) & \rightarrow & M_n(A)\otimes_A M_n(A), & e_{i,j} & \mapsto & \sum_{k=1}^n e_{i,k}\otimes_A e_{k,j}; \\ \varepsilon_{M_n(A)} & : & M_n(A) & \rightarrow & A, & e_{i,j} & \mapsto & \delta_{i,j}.\end{array}$$

3.4 Semicoalgebras, bisemialgebras, Hopf semialgebras

Throughout this section, R is commutative half-ring.

Definition 3.4.1. With an *R-semicoalgebra*, we mean an R-semicoring $(C : R)$ with

$$rc = cr \text{ for all } r \in R \text{ and } c \in C.$$

Example 3.4.2. For any set X, the free R-semimodule $R^{(X)}$ is an R-semicoalgebra with

$$\Delta(x) = x \otimes_R x \text{ and } \varepsilon(x) = 1_R \text{ for all } x \in X.$$

3.4.3. Let C be an R-semicoalgebra. If A is a half R-algebra, then $\mathrm{Hom}_R(C, A)$ is a half R-algebra with

$$(f * g)(c) = \sum f(c_1)g(c_2) \text{ for } c \in C,$$

and unity $\eta_A \circ \varepsilon_C$. In particular, $C^* := \mathrm{Hom}_R(C, R)$ is a half R-algebra and C is a (C^*, C^*)-bisemimodule with

$$f \rightharpoonup c := \sum c_1 f(c_2) \text{ and } c \leftharpoonup f := \sum f(c_1)c_2$$

for all $f \in C^*$ and $c \in C$.

3.4.4. Let A, B be half R-algebras. Then $A \otimes_R B$ is a half R-algebra with multiplication

$$\mu_{A\otimes_R B} : (A \otimes_R B) \otimes_R (A \otimes_R B) \simeq (A \otimes_R A) \otimes_R (B \otimes_R B) \overset{\mu_A \otimes_R \mu_B}{\longrightarrow} A \otimes_R B;$$

and unity map

$$\eta_{A\otimes_R B} : R \simeq R \otimes_R R \overset{\eta_A \otimes_R \eta_B}{\longrightarrow} A \otimes_R B.$$

On the otherhand, let C, D are R-semicoalgebras. Then $C \otimes_R D$ is an R-semicoalgebra with comultiplication

$$\Delta_{C\otimes_R D} : C \otimes_R D \overset{\Delta_C \otimes_R \Delta_D}{\longrightarrow} (C \otimes_R C) \otimes_R (D \otimes_R D) \simeq (C \otimes_R D) \otimes_R (C \otimes_R D),$$

and counity

$$\varepsilon_{C\otimes_R D} : C \otimes_R D \overset{\varepsilon_C \otimes_R \varepsilon_D}{\longrightarrow} R \otimes_R R \simeq R.$$

Definition 3.4.5. With an *R-bisemialgebra*, we mean a datum $(H, \mu, \eta, \Delta, \varepsilon)$, where (H, μ, η) is a half R-algebra and (H, Δ, ε) is an R-semicoalgebra such that

$$\Delta : H \to H \otimes_R H \text{ and } \varepsilon : H \to R$$

are morphisms of R-semialgebras, or equivalently

$$\mu : H \otimes_R H \to H \text{ and } \eta : R \to H$$

are morphisms of R-semicoalgebras.

Example 3.4.6. Let $(G, \cdot, e)$ be a monoid and consider the half R-algebra $R[G]$ (i.e. the free R-semimodule with basis G and multiplication induced by that of G). Then $R[G]$ is an R-bisemialgebra with comultiplication and counity defined on the generators as

$$\begin{array}{llllll} \Delta & : & R[G] & \to & R[G] \otimes_R R[G], & g & \mapsto & g \otimes_R g; \\ \varepsilon & : & R[G] & \to & R, & g & \mapsto & 1_R. \end{array}$$

Example 3.4.7. Consider the half R-algebra $R[x]$. Then $(R[x], \Delta_1, \varepsilon_1)$ is an R-bisemialgebra with the usual multiplication of polynomials and with comultiplication and counity defined on the generators as

$$\Delta_1(x^n) := x^n \otimes_R x^n \text{ and } \varepsilon_1(x^n) = 1_R.$$

Definition 3.4.8. A *Hopf R-semialgebra* is an R-bisemialgebra H, along with an R-semilinear morphism (called the *antipode*)

$$S : H \to H,$$

such that

$$\sum S(h_1)h_2 = \varepsilon(h)1_H = \sum h_1 S(h_2) \text{ for all } h \in H.$$

Example 3.4.9. Let $(G, \cdot, e)$ be a group and consider the half R-algebra $R[G]$ (i.e. the free R-semimodule with basis G with multiplication induced by that of G). Then $R[G]$ is a Hopf R-semialgebra with comultiplication, counity and antipode defined on the generators as

$$\begin{array}{llllllll} \Delta & : & R[G] & \to & R[G] \otimes_R R[G], & g & \mapsto & g \otimes_R g; \\ \varepsilon & : & R[G] & \to & R, & g & \mapsto & 1_R. \\ S & : & R[G] & \to & R[G], & g & \mapsto & g^{-1}. \end{array}$$

Example 3.4.10. Consider the half R-algebra $R[x]$. Then $(R[x], \mu, \eta, \Delta_2, \varepsilon_2)$ is an R-bisemialgebra with the usual multiplication of polynomials and with comultiplication and counity defined on the generators as

$$\Delta_2(x^n) := \sum_{k=0}^{n} \binom{n}{k} x^k \otimes_R x^{n-k} \text{ and } \varepsilon_2(x^n) = \delta_{n,0}.$$

Moreover, $R[x]$ is a Hopf R-semialgebra with antipode

$$S : H \to H,\ S(x^n) = (-1)^n x^n.$$

Example 3.4.11. Consider the half R-algebra $R[x, x^{-1}]$ with the usual multiplication of half R-algebras. Then $(R[x, x^{-1}], \mu, \eta, \Delta, \varepsilon, S)$ is a Hopf R-semialgebra with comultiplication, counity and antipode defined on the generators as

$$\begin{array}{lllllll} \Delta & : & R[x,x^{-1}] & \to & R[x,x^{-1}] \otimes_R R[x,x^{-1}], & x^z & \mapsto & x^z \otimes_R x^z; \\ \varepsilon & : & R[x,x^{-1}] & \to & R, & x^z & \mapsto & 1_R; \\ S & : & R[x,x^{-1}] & \to & R[x,x^{-1}], & x^z & \mapsto & x^{-z}. \end{array}$$

Chapter 4

Semicomodules

Throughout this chapter R denotes a commutative semiring and A, B are arbitrary (not necessarily commutative) half R-algebras. Moreover, $\mathcal{C}$ is an A-semicoring and $\mathcal{D}$ is a B-semicoring.

4.1 Basic Definitions

4.1.1. A right $\mathcal{C}$-*semicomodule* is a right A-semimodule M associated with an A-semilinear morphism ($\mathcal{C}$-*semicoaction*)

$$\varrho_M : M \to M \otimes_A \mathcal{C},\ m \mapsto \sum m_{<0>} \otimes_A m_{<1>},$$

such that the following diagrams are commutative

$$\begin{array}{ccc} M & \xrightarrow{\varrho_M} & M \otimes_A \mathcal{C} \\ {\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_M \otimes_A \Delta} \\ M \otimes_A \mathcal{C} & \xrightarrow[\varrho_M \otimes_A \mathrm{id}_{\mathcal{C}}]{} & M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\varrho_M} & M \otimes_A \mathcal{C} \\ {\scriptstyle \mathfrak{c}_M}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_M \otimes_A \varepsilon} \\ \mathfrak{c}(M) & \xleftarrow[\vartheta_M^r]{} & M \otimes_A A \end{array}$$

Using Sweedler-Heyneman's notation, the commutativity of the left diagram reads (for all $m \in M$):

$$\sum m_{<0>} \otimes_A m_{<1>1} \otimes_A m_{<1>2} = \sum m_{<0><0>} \otimes_A m_{<0><1>} \otimes_A m_{<1>};$$

and the commutativity of the right diagram reads

$$\sum m_{<0>}\varepsilon(m_{<1>})\ [\equiv]_{\{0\}}\ m \text{ for all } m \in M.$$

4.1.2. Let M, N be right $\mathcal{C}$-semicomodules. We call an A-semilinear morphism $f : M \to N$ a *$\mathcal{C}$-semicomodule morphism* (or *$\mathcal{C}$-semicolinear*), iff the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle \varrho_N} \\ M \otimes_A \mathcal{C} & \xrightarrow[f \otimes_A \mathrm{id}_{\mathcal{C}}]{} & N \otimes_A \mathcal{C} \end{array}$$

With $\mathrm{Hom}^{\mathcal{C}}(M, N)$ we denote the set of $\mathcal{C}$-semicolinear morphisms from M to N. The category of (cancellative) right $\mathcal{C}$-semicomodules and $\mathcal{C}$-semicolinear morphisms will be denoted with ($\mathbb{CS}^{\mathcal{C}}$) $\mathbb{S}^{\mathcal{C}}$. For a right $\mathcal{C}$-semicomodule M we call a right A-subsemimodule $L \leq_A M$ a $\mathcal{C}$-subsemicomodule, iff $(L, \varrho_L) \in \mathbb{S}^{\mathcal{C}}$ and the embedding $L \overset{\iota_L}{\hookrightarrow} M$ is $\mathcal{C}$-semicolinear. Analogously we define the category (${}^{\mathcal{C}}\mathbb{CS}$) ${}^{\mathcal{C}}\mathbb{S}$ of (cancellative) left $\mathcal{C}$-semicomodules. For two left $\mathcal{C}$-semicomodules M, N we denote with ${}^{\mathcal{C}}\mathrm{Hom}(M, N)$ the set of $\mathcal{C}$-semicolinear morphisms from M to N.

Proposition 4.1.3. *We have a covariant functor*

$$- \otimes_A \mathcal{C} : \mathbb{S}_A \to \mathbb{CS}^{\mathcal{C}},\ \ N \mapsto (N \otimes_A \mathcal{C}, \mathrm{id}_N \otimes_A \Delta_{\mathcal{C}}).$$

Moreover,

1. $- \otimes_A \mathcal{C} : \mathbb{CS}_A \to \mathbb{CS}^{\mathcal{C}}$ *is right adjoint to the forgetful functor* $\mathcal{F} : \mathbb{CS}^{\mathcal{C}} \to \mathbb{CS}_A$.

2. $- \otimes_A \mathcal{C} : \mathbb{CS}_A \to \mathbb{CS}^{\mathcal{C}}$ *is left adjoint to the functor* $\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, -) : \mathbb{CS}^{\mathcal{C}} \to \mathbb{CS}_A$.

3. $- \otimes_A \mathcal{C} : \mathbb{CS}_A \to \mathbb{CS}^{\mathcal{C}}$ *preserves monomorphisms and epimorphisms.*

4. $- \otimes_A \mathcal{C} : \mathbb{CS}_A \to \mathbb{CS}^{\mathcal{C}}$ *is left-exact and* $\mathcal{F} : \mathbb{CS}^{\mathcal{C}} \to \mathbb{CS}_A$ *is right-exact.*

Proof. For every right A-semimodule L, the R-semimodule $L \otimes_A \mathcal{C}$ is a right A-semimodule (since $\mathcal{C}$ is cancellative). Clearly, $L \otimes_A \mathcal{C}$ is a right $\mathcal{C}$-semicomodule with structure map

$$\mathrm{id}_L \otimes_A \Delta : L \otimes_A \mathcal{C} \to L \otimes_A (\mathcal{C} \otimes_A \mathcal{C}) \simeq (L \otimes_A \mathcal{C}) \otimes_A \mathcal{C}.$$

Moreover, if K, L are right A-semimodules and $f \in \mathrm{Hom}_{-A}(K, L)$, then

$$f \otimes_A \mathrm{id}_{\mathcal{C}} : K \otimes_A \mathcal{C} \to L \otimes_A \mathcal{C}$$

is obviously $\mathcal{C}$-semicolinear.

1. Let $M \in \mathbb{CS}^{\mathcal{C}}$ and $N \in \mathbb{CS}_A$. One checks that we have a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{-A}(M, N) & \simeq & \mathrm{Hom}^{\mathcal{C}}(M, N \otimes_A \mathcal{C}) \\ f & \mapsto & [m \mapsto \sum f(m_{<0>}) \otimes_A m_{<1>}] \\ [m \mapsto (\vartheta^r_N \circ (\mathrm{id}_N \otimes_A \varepsilon))g(m)] & \hookleftarrow & g. \end{array}$$

2. Let $M \in \mathbb{CS}_A$ and $N \in \mathbb{CS}^{\mathcal{C}}$. Since $\mathcal{C}$ is cancellative and N are cancellative, $\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, N)$ is a cancellative right A-semimodule with $(fa)(c) = f(ac)$ for all $a \in A$, $c \in \mathcal{C}$ and $f \in \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, N)$. One checks that we have a natural isomorphism

$$\begin{array}{ccc} \mathrm{Hom}^{\mathcal{C}}(M \otimes_A \mathcal{C}, N) & \simeq & \mathrm{Hom}_{-A}(M, \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, N)) \\ f & \mapsto & [m \mapsto [c \mapsto f(m \otimes_A c)]] \\ [m \otimes_A c \mapsto g(m)(c)] & \hookleftarrow & g. \end{array}$$

3. By [Sch1972, Proposition 16.4.6.], right adjoints preserve all limits (in particular, monomorphisms), and left adjoints preserve colimits (in particular, epimorphisms).

4. This follows from “1” and “2”.■

Remark 4.1.4. In general, monomorphisms in $\mathbb{S}^{\mathcal{C}}$ are not necessarily injective, and epimorphisms are not are not surjective.

Proposition 4.1.5. *1. $\mathbb{S}^{\mathcal{C}}$ has cokernels.*

2. If ${}_A\mathcal{C}$ is r-flat, then $\mathbb{S}^{\mathcal{C}}$ has kernels.

Proof. Let M, N be right $\mathcal{C}$-semicomodules and $f : M \to N$ be $\mathcal{C}$-semicolinear.

1. Consider the following commutative diagram with exact sequences of right A-semimodules

$$\begin{array}{ccccccc} M & \overset{f}{\longrightarrow} & N & \overset{\pi}{\longrightarrow} & N/f(M) & \longrightarrow & 0 \\ \varrho_M \downarrow & & \downarrow \varrho_N & & \vdots & & \\ M \otimes_A \mathcal{C} & \overset{f \otimes_A \mathrm{id}_{\mathcal{C}}}{\longrightarrow} & N \otimes_A \mathcal{C} & \overset{\pi \otimes_A \mathrm{id}_{\mathcal{C}}}{\longrightarrow} & N/f(M) \otimes_A \mathcal{C} & \longrightarrow & 0 \end{array}$$

   Notice that for any $m \in M$:

$$\begin{array}{rcl} ((\pi \otimes_A \mathrm{id}_{\mathcal{C}}) \circ \varrho_N \circ f)(m) & = & (\pi \otimes_A \mathrm{id}_{\mathcal{C}})(\sum f(m)_{<0>} \otimes_A f(m)_{<1>}) \\ & = & (\pi \otimes_A \mathrm{id}_{\mathcal{C}})(\sum f(m_{<0>}) \otimes_A m_{<1>}) \\ & = & 0. \end{array}$$

   Since $N/f(M)$ is the cokernel of f in $\mathbb{S}_A$, there is a well-defined A-semilinear morphism

$$\varrho_{N/f(M)} : N/f(M) \to N/f(M) \otimes_A \mathcal{C},\ \overline{n} \mapsto \sum \overline{n_{<0>}} \otimes_A n_{<1>}$$

   that completes the diagram commutatively. One may easily check, that $(N/f(M).\varrho_{\overline{N}})$ is a right $\mathcal{C}$-semicomodule, and is the cokernel of f in $\mathbb{S}^{\mathcal{C}}$.

2. Consider the following commutative diagram with exact sequence of right A-semimodules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(f) & \overset{\iota}{\longrightarrow} & M & \overset{f}{\longrightarrow} & N \\ & & \vdots & & \downarrow \varrho_M & & \downarrow \varrho_N \\ 0 & \longrightarrow & \mathrm{Ker}(f) \otimes_A \mathcal{C} & \overset{\iota \otimes_A \mathrm{id}_{\mathcal{C}}}{\longrightarrow} & M \otimes_A \mathcal{C} & \overset{f \otimes_A \mathrm{id}_{\mathcal{C}}}{\longrightarrow} & N \otimes_A \mathcal{C}. \end{array}$$

   Notice that for any $m \in \mathrm{Ker}(f)$:

$$\begin{array}{rcl} ((f \otimes_A \mathrm{id}_{\mathcal{C}}) \circ \varrho_M \circ \iota)(m) & = & (f \otimes_A \mathrm{id}_{\mathcal{C}})(\sum \iota(m)_{<0>} \otimes_A \iota(m)_{<1>}) \\ & = & (f \otimes_A \mathrm{id}_{\mathcal{C}})(\sum \iota(m_{<0>}) \otimes_A m_{<1>}) \\ & = & 0. \end{array}$$

   By assumption, ${}_A\mathcal{C}$ is r-flat and so $\mathrm{Ker}(f) \otimes_A \mathcal{C} = \mathrm{Ker}(f \otimes_A \mathrm{id}_{\mathcal{C}})$ (by Lemma 1.2.25). Since $\mathrm{Ker}(f)$ is the kernel of f in $\mathbb{S}_A$, it follows that there is a well-defined A-semilinear morphism

$$\varrho_K : \mathrm{Ker}(f) \to \mathrm{Ker}(f) \otimes_A \mathcal{C},\ m \mapsto \sum \iota(m)_{<0>} \otimes_A m_{<1>}$$

   that completes the diagram commutatively. One checks easily that $(\mathrm{Ker}(f), \varrho_K)$ is a right A-semicomodule, and is the kernel of f in $\mathbb{S}^{\mathcal{C}}$.■

Proposition 4.1.6. *1. If ${}_A\mathcal{C}$ is r-flat, then every monomorphism in $\mathbb{S}^{\mathcal{C}}$ has zero kernel;*

2. If the forgetful functor $\mathcal{F} : \mathbb{CS}^{\mathcal{C}} \rightarrow \mathbb{CS}_A$ preserves monomorphisms, then ${}_A\mathcal{C}$ is k-flat.

Proof. 1. Let $f : L \rightarrow M$ be a monomorphism in $\mathbb{S}^{\mathcal{C}}$. Since ${}_A\mathcal{C}$ is r-flat, $\mathrm{Ker}(f) \leq^{\mathcal{C}} L$ is a $\mathcal{C}$-subsemicomodule by Proposition 4.1.5 "2". Consider the embedding $\mathrm{Ker}(f) \overset{\iota}{\hookrightarrow} L$. Then $f \circ \iota = f \circ 0$, whence $\iota = 0$ (since f is a monomorphism). It follows then that $\mathrm{Ker}(f) = 0$.

2. Let $L \overset{f}{\rightarrowtail} M$ be a monomorphism of right A-semimodules. By Proposition 4.1.3 "3", $L \otimes_A \mathcal{C} \overset{f \otimes_A \mathrm{id}_{\mathcal{C}}}{\longrightarrow} M \otimes_A \mathcal{C}$ is a monomorphism of right $\mathcal{C}$-semicomodules, whence (by assumptions) an injective morphism of cancellative right A-semimodules. It follows then that ${}_A\mathcal{C}$ is k-flat.■

Proposition 4.1.7. *1. $\mathbb{S}^{\mathcal{C}}$ has coproducts.*

2. For any family $\{M_\lambda\}_\Lambda$ of cancellative right A-semimodules, $\prod\limits_{\lambda \in \Lambda} M_\lambda \otimes_A \mathcal{C}$ is the product of $\{M_\lambda \otimes_A \mathcal{C}\}_\Lambda$ in $\mathbb{CS}^{\mathcal{C}}$.

3. Every cancellative right $\mathcal{C}$-semicomodule M is $\mathcal{C}$-subgenerated.

Proof. 1. Let $\{(M_\lambda, \varrho_\lambda)\}_\Lambda$ be a class of right $\mathcal{C}$-semicomodules. Consider the right A-semimodule $M := \bigoplus\limits_\Lambda M_\lambda$ and the canonical injections $\{\iota_\lambda : M_\lambda \rightarrow M\}_\Lambda$. Notice that we have A-semilinear morphisms

$$\begin{array}{ccc} M_\lambda & \overset{\varrho_\lambda}{\longrightarrow} & M_\lambda \otimes_A \mathcal{C} \\ & \ddots & \downarrow \varrho_\lambda \otimes_A \mathrm{id}_{\mathcal{C}} \\ & & M \otimes_A \mathcal{C}, \end{array}$$

and so there exists (by the universal property of coproducts in $\mathbb{S}_A$) an A-semilinear morphism

$$\varrho_M : \bigoplus_\Lambda M_\lambda \longrightarrow \bigoplus_\Lambda M_\lambda \otimes_A \mathcal{C}.$$

One checks that $(\bigoplus\limits_\Lambda M_\lambda, \varrho_M)$ is a right $\mathcal{C}$-semicomodule, and that M is the coproduct of $\{M_\lambda\}_\Lambda$ is $\mathbb{S}^{\mathcal{C}}$.

2. For any right cancellative $\mathcal{C}$-semicomodule L we have

$$\begin{array}{rcll} \mathrm{Hom}^{\mathcal{C}}(L, \prod\limits_{\lambda\in\Lambda} M_\lambda \otimes_A \mathcal{C}) & \simeq & \mathrm{Hom}_{-A}(L, \prod\limits_{\lambda\in\Lambda} M_\lambda) & \text{(Proposition 4.1.3)} \\ & \simeq & \prod\limits_{\lambda\in\Lambda} \mathrm{Hom}_{-A}(L, M_\lambda) & \text{(universal property)} \\ & \simeq & \prod\limits_{\lambda\in\Lambda} \mathrm{Hom}^{\mathcal{C}}(L, M_\lambda \otimes_A \mathcal{C}) & \text{(Proposition 4.1.3).} \end{array}$$

As well known, these isomorphisms characterize the product of $\{M_\lambda \otimes_A \mathcal{C}\}_\Lambda$ in $\mathbb{CS}^{\mathcal{C}}$.

3. Let M be a cancellative right $\mathcal{C}$-semicomodule with a surjective A-semilinear morphism

$$A^{(\Lambda)} \overset{\pi}{\twoheadrightarrow} M \to 0.$$

Tensoring with ${}_A\mathcal{C}$ we obtain a surjective morphism of right $\mathcal{C}$-semicomodules

$$\mathcal{C}^{(\Lambda)} \simeq A^{(\Lambda)} \otimes_A \mathcal{C} \twoheadrightarrow M \otimes_A \mathcal{C} \to 0,$$

whence $M \otimes_A \mathcal{C}$ is $\mathcal{C}$-generated in $\mathbb{CS}^{\mathcal{C}}$. Since M is cancellative, $\varrho_M : M \to M \otimes_A \mathcal{C}$ splits through

$$\vartheta^r_M \circ (\mathrm{id}_M \otimes_A \varepsilon_{\mathcal{C}}) : M \otimes_A \mathcal{C} \to M,$$

and so M is a $\mathcal{C}$-subsemicomodule of $M \otimes_A \mathcal{C}$. Consequently, M is a $\mathcal{C}$-subsemicomodule of a $\mathcal{C}$-generated $\mathcal{C}$-semicomodule, i.e. M is $\mathcal{C}$-subgenerated.■

Corollary 4.1.8. *If* $- \otimes_A \mathcal{C} : \mathbb{CS}_A \to \mathbb{CS}^{\mathcal{C}}$ *is left adjoint to the forgetful functor* $\mathcal{F} : \mathbb{CS}^{\mathcal{C}} \to \mathbb{CS}_A$, *then* ${}_A\mathcal{C}$ *is finitely generated and weakly projective.*

Proof. Assume $- \otimes_A \mathcal{C} : \mathbb{CS}_A \to \mathbb{CS}^{\mathcal{C}}$ is left adjoint to the forgetful functor $\mathcal{F} : \mathbb{CS}^{\mathcal{C}} \to \mathbb{CS}_A$. Then $\mathcal{F}$ (as a right adjoint) preserves monomorphisms and products. Hence ${}_A\mathcal{C}$ is k-flat by Proposition 4.1.6. Moreover, it follows by Proposition 4.1.7 that for any family of cancellative right A-semimodules $\{M_\lambda\}_\Lambda$ we have canonical isomorphisms

$$(\prod_{\lambda\in\Lambda} M_\lambda) \otimes_A \mathcal{C} \simeq \prod_{\lambda\in\Lambda} (M \otimes_A \mathcal{C}).$$

It follows then that ${}_A\mathcal{C}$ is finitely presented. Since ${}_A\mathcal{C}$ is k-flat, we conclude that ${}_A\mathcal{C}$ is weakly projective.■

4.2 Rational modules

Throughout this section, A is a half R-algebra and $\mathcal{C}$ is an A-semicoring. Assuming ${}_A\mathcal{C}$ to be an α-semimodule, we introduce the category $\mathrm{Rat}^{\mathcal{C}}(\mathbb{CS}_{^*\mathcal{C}})$ of $\mathcal{C}$-*rational* cancellative right ${}^*\mathcal{C}$-modules and study its properties.

Remark 4.2.1. For every cancellative right A-semimodule M, $\mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$ is a right A-semimodule through $(\varphi a) := \varphi(a-)$ for all $a \in A$ and $\varphi \in \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$; moreover, $\alpha_M^P : M \otimes_A \mathcal{C} \to \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$ is A-semilinear. If moreover M is a right ${}^*\mathcal{C}$-semimodule, then the canonical map $\rho_M : M \to \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$ is ${}^*\mathcal{C}$-semilinear.

4.2.2. Let ${}_A\mathcal{C}$ satisfy the α-condition and M be a *cancellative* right ${}^*\mathcal{C}$-semimodule. Define

$$\mathrm{Rat}^{\mathcal{C}}(M_{^*\mathcal{C}}) := \rho_M^{-1}(M \otimes_A \mathcal{C}). \tag{4.1}$$

Clearly, $m \in \mathrm{Rat}^{\mathcal{C}}(M_{^*\mathcal{C}})$ if and only if there exists a uniquely determined element $\sum m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$ such that $m \leftharpoonup f = \sum m_i f(c_i)$ for every $f \in {}^*\mathcal{C}$. A cancellative right ${}^*\mathcal{C}$-semimodule M will be called $\mathcal{C}$-*rational*, iff $\mathrm{Rat}^{\mathcal{C}}(M_{^*\mathcal{C}}) = M$; in this case we have an A-semilinear morphism

$$\varrho_M := (\alpha_M^P)^{-1} \circ \rho_M : M \to M \otimes_A \mathcal{C}.$$

Lemma 4.2.3. *Let ${}_A\mathcal{C}$ be a half α-module. If M is a cancellative right ${}^*\mathcal{C}$-semimodule, then $\mathrm{Rat}^{\mathcal{C}}(M_{^*\mathcal{C}}) \leq_A M$ is a regular A-subsemimodule.*

Proof. Consider the ${}^*\mathcal{C}$-semilinear morphism

$$\pi \circ \rho_M : M \overset{\pi\circ\rho_M}{\longrightarrow} \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)/(M \otimes_A \mathcal{C}).$$

Then $m \in \mathrm{Ker}(\pi \circ \rho_M)$ if and only if there exist $\sum m_i \otimes_A c_i$, $\sum m_j \otimes_A c_j \in M \otimes_A \mathcal{C}$ such that

$$\rho_M(m) + \alpha_M^{\mathcal{C}}(\sum m_i \otimes_A c_i) = \alpha_M^{\mathcal{C}}(\sum m_j \otimes_A c_j).$$

By assumption $M \otimes_A \mathcal{C} \overset{\alpha_M^{\mathcal{C}}}{\hookrightarrow} \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$ is subtractive, whence $m \in \mathrm{Ker}(\pi \circ \rho_M)$ if and only if $\rho_M(m) = \alpha_M^{\mathcal{C}}(M \otimes_A \mathcal{C})$. Consequently, $\mathrm{Rat}^{\mathcal{C}}(M_{^*\mathcal{C}}) = \mathrm{Ker}(\pi \circ \rho_M) \overset{\iota}{\hookrightarrow} M$ is a regular A-subsemimodule and the following sequence of A-modules is exact

$$0 \rightarrowtail \mathrm{Rat}^{\mathcal{C}}(M_{^*\mathcal{C}}) \overset{\iota}{\rightarrowtail} M \overset{\pi\circ\rho_M}{\to} \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)/(M \otimes_A \mathcal{C})).\blacksquare \tag{4.2}$$

Lemma 4.2.4. *Let ${}_A\mathcal{C}$ be a half α-module. For every cancellative right ${}^*\mathcal{C}$-semimodule M we have:*

1. $\mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}}) \leq_{{}^*\mathcal{C}} M$ *is a* ${}^*\mathcal{C}$*-subsemimodule.*

2. *For every regular* ${}^*\mathcal{C}$*-subsemimodule* $L \leq_{{}^*\mathcal{C}} M$, *we have*
$$\mathrm{Rat}^{\mathcal{C}}(L_{{}^*\mathcal{C}}) = L \cap \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}}).$$

3. $\mathrm{Rat}^{\mathcal{C}}(\mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})) = \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})$.

4. *For every* $N \in \mathbb{CS}_{{}^*\mathcal{C}}$ *and* $f \in \mathrm{Hom}_{-\ {}^*\mathcal{C}}(M,N)$ *we have* $f(\mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})) \subseteq \mathrm{Rat}^{\mathcal{C}}(N_{{}^*\mathcal{C}})$.

Proof.

1. Let $f \in {}^*\mathcal{C}$ and $m \in \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})$ with $\varrho_M(m) = \sum m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$. For every $g \in {}^*\mathcal{C}$ we have
$$(m \leftharpoondown f) \leftharpoondown g = m \leftharpoondown (f *_l g) = \sum m_i (f *_l g)(c_i) = \sum m_i g(c_i \leftharpoondown f),$$
and so $m \leftharpoondown f \in \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})$ with $\varrho_M(m \leftharpoondown f) = \sum m_i \otimes_A c_i \leftharpoondown f \in M \otimes_A \mathcal{C}$.

2. Clearly $\mathrm{Rat}^{\mathcal{C}}(L_{{}^*\mathcal{C}}) \subseteq L \cap \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})$. On the other hand, if $m \in L \cap \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})$ then there is $\sum m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$ with $\sum m_i f(c_i) = m \leftharpoondown f \in L$ for all $f \in {}^*\mathcal{C}$. Since $L \leq_{{}^*\mathcal{C}} M$ is a regular ${}^*\mathcal{C}$-subsemimodule, it follows by Lemma 2.3.13 that $\sum m_i \otimes_A c_i \in L \otimes_A \mathcal{C}$, i.e. $m \in \mathrm{Rat}^{\mathcal{C}}(L_{{}^*\mathcal{C}})$.

3. By Lemma 4.2.3, $\mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}}) \leq_A M$ is a regular A-subsemimodule; whence the equality follows from (1) and (2) above.

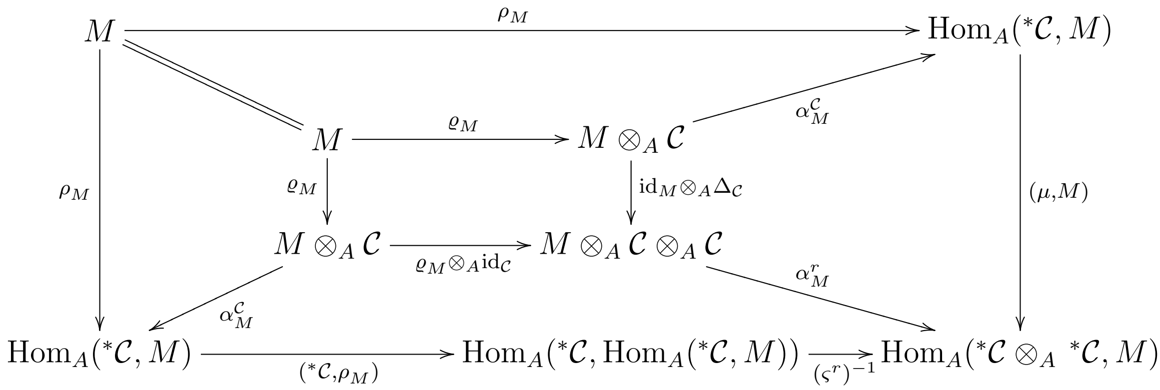
4. Let $N \in \mathbb{CS}_{\mathcal{C}^*}$ and $\psi \in \mathrm{Hom}_{-\ {}^*\mathcal{C}}(M,N)$. If $m \in \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}})$ with $\varrho_M(m) = \sum m_i \otimes_A c_i \in M \otimes_A \mathcal{C}$, then we have for every $f \in {}^*\mathcal{C}$:
$$\psi(m) \leftharpoondown f = \psi(m \leftharpoondown f) = \psi(\sum m_i f(c_i)) = \sum \psi(m_i) f(c_i),$$
i.e. $\psi(m) \in \mathrm{Rat}^{\mathcal{C}}(N_{{}^*\mathcal{C}})$ with $\varrho_N(\psi(m)) = \sum \psi(m_i) \otimes_A c_i \in N \otimes_A \mathcal{C}$.■

Lemma 4.2.5. *1. If (M, ϱ_M) is a cancellative right $\mathcal{C}$-semicomodule, then M is a $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodule through*
$$\rho_M := M \overset{\varrho_M}{\to} M \otimes_A \mathcal{C} \overset{\alpha_M^{\mathcal{C}}}{\to} \mathrm{Hom}_{-A}({}^*\mathcal{C}, M).$$

2. *Let* $(M, \varrho_M), (N, \varrho_N)$ *be cancellative right* $\mathcal{C}$*-semicomodules and consider the induced structures of right* ${}^*\mathcal{C}$*-semimodules* (M, ρ_M)*,* (N, ρ_N)*. If* $f : M \to N$ *is* $\mathcal{C}$*-semicolinear, then* f *is* ${}^*\mathcal{C}$*-semilinear.*

3. *Let* M *be a cancellative right* $\mathcal{C}$*-semicomodule,* $L \leq^{\mathcal{C}} M$ *a* $\mathcal{C}$*-subsemicomodule and consider the induced right* ${}^*\mathcal{C}$*-semimodule structures* (M, ρ_M)*,* (L, ρ_L)*. Then* $L \leq_{{}^*\mathcal{C}} M$ *is a* ${}^*\mathcal{C}$*-subsemimodule.*

Proof. 1. Let M be a cancellative right $\mathcal{C}$-semicomodule and consider the diagram



(4.3)

where

$$\alpha_M^r : M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \longrightarrow \mathrm{Hom}_{-A}({}^*\mathcal{C} \otimes_A {}^*\mathcal{C}, M),$$

and

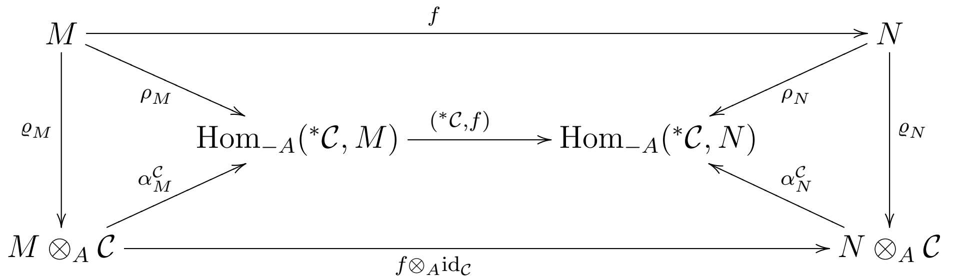
$$\varsigma^r : \mathrm{Hom}_{-A}({}^*\mathcal{C} \otimes_A {}^*\mathcal{C}, M) \to \mathrm{Hom}_{-A}({}^*\mathcal{C}, \mathrm{Hom}_{-A}({}^*\mathcal{C}, M))$$

is the canonical isomorphism (see Proposition 1.1.31). By definition of ρ_M, all trapezoids are commutative. Since (M, ϱ_M) is a right $\mathcal{C}$-semicomodule, the inner rectangle is commutative and so the outer rectangle is commutative, i.e. (M, ρ_M) is a right ${}^*\mathcal{C}$-semimodule. Moreover, we have for every $m \in M$:

$$m \leftharpoonup \varepsilon_{\mathcal{C}} = \sum m_{<0>} \varepsilon_{\mathcal{C}}(m_{<1>}) = m.$$

Clearly $M_{{}^*\mathcal{C}}$ is a $\mathcal{C}$-rational.

2. Consider the diagram



(4.4)

The lower trapezoid is obviously commutative. The triangles are commutative by definition of ρ_M, ρ_N. If f is $\mathcal{C}$-semicolinear, then the outer rectangle is commutative, and so the upper trapezoid is commutative (i.e. f is ${}^*\mathcal{C}$-semilinear).

3. Trivial.■

Lemma 4.2.6. *Let ${}_A\mathcal{C}$ be a half α-module.*

1. *If (M, ρ_M) is a $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodule, then M is a right $\mathcal{C}$-semicomodule with coaction given by*

$$\varrho_M : M \overset{\rho_M}{\to} \mathrm{Hom}_{-A}({}^*\mathcal{C}, M) \overset{(\alpha_M^{\mathcal{C}})^{-1}}{\to} M \otimes_A \mathcal{C}.$$

2. *Let (M, ρ_M), (N, ρ_N) be $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodules and consider the induced structures of right $\mathcal{C}$-semicomodules (M, ϱ_M), (N, ϱ_N). Then $\mathrm{Hom}^{\mathcal{C}}(M, N) = \mathrm{Hom}_{-{}^*\mathcal{C}}(M, N)$.*

3. *Let (M, ρ_N) be a $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodule and consider the induced counital right $\mathcal{C}$-semicomodule (M, ϱ_M). If $L \leq_{{}^*\mathcal{C}} M$ is a regular ${}^*\mathcal{C}$-subsemimodule, then $L \leq^{\mathcal{C}} M$ a $\mathcal{C}$-subsemicomodule. Moreover $\varrho_L = (\varrho_M)_{|L}$.*

Proof. 1. Let (M, ρ_M) be a $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodule, so that (by definition) $\rho_M(M) \subset \alpha_M^{\mathcal{C}}(M \otimes_A \mathcal{C})$, i.e. $\varrho_M := (\alpha_M^{\mathcal{C}})^{-1} \circ \rho_M$ is well defined. By definition of ϱ_M, all trapezoids in Diagram (4.3) are commutative. By assumption M is a right $\mathcal{C}$-semicomodule and so the outer rectangle is commutative. By Lemma 2.3.7, α_M^r is injective and so the inner rectangle is also commutative, i.e. (M, ϱ_M) is a right $\mathcal{C}$-semicomodule. Notice that for each $m \in M$ we have

$$\sum m_{<0>}\varepsilon_{\mathcal{C}}(m_{<1>}) = m \leftharpoonup \varepsilon_{\mathcal{C}} = m.$$

Consequently, M is a right $\mathcal{C}$-semicomodule.

2. Consider Diagram (4.4). The lower trapezoid is obviously commutative and the triangles are commutative by definition of ϱ_M and ϱ_N. Moreover, $\alpha_N^{\mathcal{C}}$ is injective and so the outer rectangle is commutative (f is $\mathcal{C}$-semicolinear) if and only if the upper trapezoid is commutative (f is ${}^*\mathcal{C}$-semilinear). Whence $\mathrm{Hom}^{\mathcal{C}}(M, N) = \mathrm{Hom}_{-{}^*\mathcal{C}}(M, N)$.

3. Let (M, ρ_M) be a $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodule. If $L \leq_{{}^*\mathcal{C}} M$ is a regular ${}^*\mathcal{C}$-subsemimodule, then by Lemma 4.2.4 (2) $\mathrm{Rat}^{\mathcal{C}}(L_{{}^*\mathcal{C}}) = L \cap \mathrm{Rat}^{\mathcal{C}}(M_{{}^*\mathcal{C}}) = L$ and so $L_{{}^*\mathcal{C}}$ is $\mathcal{C}$-rational and L is right $\mathcal{C}$-semicomodule by (1). Moreover $L \overset{\iota_L}{\hookrightarrow} M$ is ${}^*\mathcal{C}$-semilinear and so $\mathcal{C}$-semicolinear by (2), i.e. $L \leq^{\mathcal{C}} M$ is a $\mathcal{C}$-subsemicomodule.■

4.2.7. We have an isomorphism of A-semirings

$$(\mathcal{C}^*, *_r) \simeq \mathrm{End}^{\mathcal{C}}(\mathcal{C}),\ f \mapsto [c \mapsto \sum f(c_1)c_2]$$

with inverse $g \mapsto \varepsilon_{\mathcal{C}} \circ g$ (compare Proposition 4.1.3 (1)). Analogously, we have an isomorphism of A-semirings

$$({}^*\mathcal{C}, \star_l) \simeq {}^{\mathcal{C}}\mathrm{End}(\mathcal{C})^{op}.$$

If ${}_A\mathcal{C}$ is an half α-module, then by Lemma 4.2.6 (2) $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = \mathrm{End}(\mathcal{C}_{{}^*\mathcal{C}})$ and so

$$\begin{array}{rclcl} {}^*\mathcal{C} & \simeq & {}^{\mathcal{C}}\mathrm{End}(\mathcal{C})^{op} & \subseteq & \mathrm{End}({}_{\mathcal{C}^*}\mathcal{C})^{op} = \mathrm{End}({}_{\mathrm{End}^{\mathcal{C}}(\mathcal{C})}\mathcal{C})^{op} \\ & = & \mathrm{End}({}_{\mathrm{End}(\mathcal{C}_{{}^*\mathcal{C}})}\mathcal{C})^{op} & := & \mathrm{Biend}(\mathcal{C}_{{}^*\mathcal{C}}), \end{array}$$

i.e. $({}^*\mathcal{C}, *_l)$ is isomorphic to an A-subsemiring of $\mathrm{Biend}(\mathcal{C}_{{}^*\mathcal{C}})$. If moreover $\mathcal{C}_A$ is an half α-module, then

$${}^{\mathcal{C}}\mathrm{End}(\mathcal{C}) = \mathrm{End}({}_{\mathcal{C}^*}\mathcal{C}) \text{ and } {}^*\mathcal{C} \simeq \mathrm{Biend}(\mathcal{C}_{{}^*\mathcal{C}}).$$

On the other hand, if $\mathcal{C}_A$ is a half α-module, then we have analogously ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C}) = \mathrm{End}({}_{{}^*\mathcal{C}}\mathcal{C})$ and so

$$\begin{array}{rclcl} \mathcal{C}^* & \simeq & \mathrm{End}^{\mathcal{C}}(\mathcal{C}) & \subseteq & \mathrm{End}(\mathcal{C}_{{}^*\mathcal{C}}) = \mathrm{End}(\mathcal{C}_{{}^{\mathcal{C}}\mathrm{End}(\mathcal{C})^{op}}) \\ & = & \mathrm{End}(\mathcal{C}_{\mathrm{End}({}_{{}^*\mathcal{C}}\mathcal{C})^{op}}) & := & \mathrm{Biend}({}_{{}^*\mathcal{C}}\mathcal{C}), \end{array}$$

i.e. $(\mathcal{C}^*, *_A)$ is isomorphic to an A-subsemiring of $\mathrm{Biend}({}_{^*\mathcal{C}}\mathcal{C})$. If moreover, ${}_A\mathcal{C}$ is a half α-module, then $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = \mathrm{End}(\mathcal{C}_{^*\mathcal{C}})$, whence $\mathcal{C}^* \simeq \mathrm{Biend}({}_{^*\mathcal{C}}\mathcal{C})$.

Note that it follows from above, that in case ${}_A\mathcal{C}$ and $\mathcal{C}_A$ are half α-modules, then we have isomorphism of A-semirings

$$^*\mathcal{C} \simeq \mathrm{Biend}(\mathcal{C}_{^*\mathcal{C}}) \text{ and } \mathcal{C}^* \simeq \mathrm{Biend}({}_{\mathcal{C}^*}\mathcal{C}).$$

Remark 4.2.8. Let M be a cancellative right A-semimodule. Then $\mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$ is a right ${}^*\mathcal{C}$-semimodule with

$$(\gamma \leftharpoonup g)(f) := \gamma(g *_l f).$$

Moreover, the canonical map

$$\alpha_M^{\mathcal{C}} : M \otimes_A \mathcal{C} \to \mathrm{Hom}_{-A}({}^*\mathcal{C}, M)$$

is ${}^*\mathcal{C}$-semilinear. Indeed, for every $g \in {}^*\mathcal{C}$, we have

$$\begin{array}{rcl} \alpha_M^{\mathcal{C}}(\sum m_i \otimes_A c_i \leftharpoonup g)(f) & = & \sum m_i f(c_i \leftharpoonup g) \\ & = & \sum m_i f(c_{i1} g(c_{i2})) \\ & = & \sum m_i (g *_l f)(c_i) \\ & = & \alpha_M^{\mathcal{C}}(\sum m_i \otimes_A c_i)(g *_l f) \\ & = & (\alpha_M^{\mathcal{C}}(\sum m_i \otimes_A c_i) \leftharpoonup g)(f). \end{array}$$

We are now ready to prove the main theorem in this chapter:

Theorem 4.2.9. *Let ${}_A\mathcal{C}$ be a half A-module. The following are equivalent:*

1. *${}_A\mathcal{C}$ is a half α-module;*
2. *$\mathbb{CS}^{\mathcal{C}} \simeq \mathrm{Rat}^{\mathcal{C}}(\mathbb{CS}_{^*\mathcal{C}})$;*
3. *$\mathbb{CS}^{\mathcal{C}} \leq \mathbb{CS}_{^*\mathcal{C}}$ is a full subcategory.*

Proof. $(1 \Rightarrow 2)$ By Lemmata 4.2.5 and 4.2.6 we have covariant functors

$$\begin{array}{rcccl} (-)_{^*\mathcal{C}} & : & \mathbb{CS}^{\mathcal{C}} & \to & \mathrm{Rat}^{\mathcal{C}}(\mathbb{CS}_{^*\mathcal{C}}), \\ & & (M, \varrho_M) & \mapsto & (M, \alpha_M \circ \varrho_M), \end{array}$$

$$\begin{array}{rcccl} (-)^{\mathcal{C}} & : & \mathrm{Rat}^{\mathcal{C}}(\mathbb{CS}_{^*\mathcal{C}}) & \to & \mathbb{CS}^{\mathcal{C}}, \\ & & (M, \rho_M) & \mapsto & (M, \alpha_M^{-1} \circ \rho_M), \end{array}$$

that act as the identity on morphisms. Clearly we have

$$(-)^{\mathcal{C}} \circ\ (-)_{^*\mathcal{C}} = \mathrm{id}_{\mathbb{CS}^{\mathcal{C}}},\ (-)_{^*\mathcal{C}} \circ (-)^{\mathcal{C}} = \mathrm{id}_{\mathrm{Rat}^{\mathcal{C}}(\mathbb{CS}_{^*\mathcal{C}})},$$

i.e. $\mathrm{Rat}^{\mathcal{C}}(\mathbb{CS}_{^*\mathcal{C}}) \simeq \mathbb{CS}^{\mathcal{C}}$.

(2 ⇒ 3) Clear.

(3 ⇒ 1) By assumption monomorphisms in $\mathbb{CS}^{\mathcal{C}}$ are injective, whence ${}_A\mathcal{C}$ is k-flat by Proposition 4.1.6. Let M be any cancellative right A-semimodule, and consider the morphism of left $^*\mathcal{C}$-semimodules

$$\alpha_M^{\mathcal{C}} : M \otimes_A \mathcal{C} \to \mathrm{Hom}_{-A}(^*\mathcal{C}, M).$$

For any cancellative right $\mathcal{C}$-semicomodule L we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{-\ ^*\mathcal{C}}(L, M \otimes_A \mathcal{C}) & \overset{(L,\alpha_M^{\mathcal{C}})}{\longrightarrow} & \mathrm{Hom}_{^*\mathcal{C}-}(L, \mathrm{Hom}_{-A}(^*\mathcal{C}, M)) \\ \| & & \downarrow\simeq \\ \mathrm{Hom}^{\mathcal{C}}(L, M \otimes_A \mathcal{C}) & \simeq & \mathrm{Hom}_{-A}(L \otimes_{^*\mathcal{C}}\ ^*\mathcal{C}, M) \\ \downarrow\simeq & & \downarrow\simeq \\ \mathrm{Hom}_{-A}(L, M) & = & \mathrm{Hom}_{-A}(\mathfrak{c}(L), M) \end{array}$$

This shows that $(L, \alpha_M^{\mathcal{C}})$ is injective. It follows then that the corestriction of $\alpha_M^{\mathcal{C}}$ is a monomorphism in $\mathbb{CS}^{\mathcal{C}}$, whence injective.■

Theorem 4.2.10. *Let ${}_A\mathcal{C}$ be a half α-module. If M is a cancellative right $\mathcal{C}$-semicomodule, then every finite subset of M is contained in a $\mathcal{C}$-subsemicomodule $L \leq^{\mathcal{C}} M$ such that L_A is finitely generated.*

Proof. Let $E := \{m_1, \cdots, m_k\} \subseteq M$ be a finite subset and choose

$$\{(m_{11}, c_{11}), \cdots, (m_{1n_1}, c_{1n_1}), \cdots, (m_{k1}, c_{k1}), \cdots, (m_{kn_k}, c_{kn_k}\}\} \subseteq M \times \mathcal{C},$$

such that

$$\varrho_M^{\mathcal{C}}(m_i) = \sum_{j=1}^{n_j} m_{ij} \otimes_A c_{ij},\ \text{for } i = 1, \cdots, k.$$

Notice that $L = \sum_{i=1}^{k} m_i\ ^*\mathcal{C}$ is a $\mathcal{C}$-subsemicomodule of M by Lemma 4.2.6. Moreover, it is clear that L_A has a finite generating set, namely

$$\{m_{11}, \cdots, m_{1n_1}, \cdots, m_{k1}, \cdots, m_{kn_k}\}.$$

Moreover, $E \subseteq L$ since

$$m_i := m_i \leftharpoonup \varepsilon \in L \text{ for } i = 1, \cdots, k.■$$

Analogous to Theorem 4.2.9, we have

Theorem 4.2.11. *Let $\mathcal{C}_A$ be a half A-module. The following are equivalent:*

1. *$\mathcal{C}_A$ is a half α-module;*
2. *${}^{\mathcal{C}}\mathbb{CS} \simeq {}^{\mathcal{C}}\mathrm{Rat}({}_{\mathcal{C}^*}\mathbb{CS})$;*
3. *${}^{\mathcal{C}}\mathbb{CS} \leq {}_{\mathcal{C}^*}\mathbb{CS}$ is a full subcategory.*

Theorem 4.2.12. *Let $\mathcal{C}_A$ be a half α-module. If M is a cancellative left $\mathcal{C}$-semicomodule, then every finite subset of M is contained in a $\mathcal{C}$-subsemicomodule $L \leq^{\mathcal{C}} M$ such that ${}_AL$ is finitely generated.*

4.2.13. Birational modules. Let $\mathcal{C}$ be an A-semicoring and $\mathcal{D}$ a B-semicoring. For every cancellative $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodule $(M, \rho_M^{\mathcal{D}^*}, \rho_M^{{}^*\mathcal{C}})$ one can see easily that ${}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{D}^*}M)$ is a right ${}^*\mathcal{C}$-semimodule, $\mathrm{Rat}^{\mathcal{C}}(M_{*\mathcal{C}})$ is a left $\mathcal{D}^*$-semimodule, and

$$\mathrm{Rat}^{\mathcal{C}}(({}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{D}^*}M))_{*\mathcal{C}}) = {}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{D}^*}M) \cap \mathrm{Rat}^{\mathcal{C}}(M_{*\mathcal{C}}) = {}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{D}^*}(\mathrm{Rat}^{\mathcal{C}}(M_{*\mathcal{C}}))) \tag{4.5}$$

is a $(\mathcal{D}^*,{}^*\mathcal{C})$-subbisemimodule of M, which we call the $(\mathcal{D},\mathcal{C})$-*birational* $(\mathcal{D}^*,{}^*\mathcal{C})$-*subbisemimodule* of M. If $M = \mathrm{Rat}^{\mathcal{C}}(({}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{B}}M))_{*\mathcal{C}})$, then we call ${}_{\mathcal{D}^*}M_{*\mathcal{C}}$ $(\mathcal{D},\mathcal{C})$-*birational*.

With ${}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{D}^*}\mathbb{CS}_{*\mathcal{C}}) \leq {}_{\mathcal{D}^*}\mathbb{CS}_{\mathcal{C}^*}$, we denote the full subcategory of $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodules. The subcategory of unital $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodules is denoted with ${}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{D}^*}\mathbb{CS}_{*\mathcal{C}})$.

Theorem 4.2.14. *Let $(\mathcal{C} : A)$ and $(\mathcal{D} : B)$ be semicorings, such that ${}_A\mathcal{C}$ and $\mathcal{D}_B$ are half α-modules. Then there are isomorphisms of categories*

$$ {}^{\mathcal{D}}\mathbb{CS}^{\mathcal{C}} \simeq {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{D}^*}\mathbb{CS}_{*\mathcal{C}})$$

Proof. Let M be an arbitrary cancellative (B, A)-bisemimodule. In view of the previous results in this section, it is enough to show that M is a $(\mathcal{D},\mathcal{C})$-bisemicomodules iff M is a $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodule. If M is a $(\mathcal{D},\mathcal{C})$-bisemicomodules, then M is by Lemma 4.2.5 (1) a $\mathcal{C}$-rational right ${}^*\mathcal{C}$-semimodule and analogously a $\mathcal{D}$-rational left $\mathcal{D}^*$-semimodule. Moreover $\alpha_M^{\mathcal{D}}$ is obviously ${}^*\mathcal{C}$-semilinear, $\varrho_M^{\mathcal{D}}$ is by assumption $\mathcal{C}$-semicolinear, hence ${}^*\mathcal{C}$-semilinear by Lemma 4.2.5 (2). Consequently $\rho_M^{\mathcal{D}^*} = \alpha_M^{\mathcal{D}} \circ \varrho_M^{\mathcal{D}}$ is ${}^*\mathcal{C}$-semilinear, i.e. M is a $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodule.

On the other hand, let M be a $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodule. By Lemma 4.2.6 M is a counital right $\mathcal{C}$-semicomodule and analogously a left $\mathcal{D}$-semicomodule. Since M is a $(\mathcal{D}^*,{}^*\mathcal{C})$-bisemimodule, $\rho_M^{\mathcal{D}^*}$ is ${}^*\mathcal{C}$-semilinear and so we have for all $f \in {}^*\mathcal{C}$ and $m \in M$:

$$\begin{array}{rclcl} \alpha_M^{\mathcal{D}^*}(\varrho_M^{\mathcal{D}}(m \leftharpoonup f)) & = & \rho_M^{\mathcal{D}^*}(m \leftharpoonup f) & = & \rho_M^{\mathcal{D}^*}(m) \leftharpoonup f \\ & = & (\alpha_M^{\mathcal{D}^*}(\varrho_M^{\mathcal{D}}(m)) \leftharpoonup f & = & \alpha_M^{\mathcal{D}^*}(\varrho_M^{\mathcal{D}}(m) \leftharpoonup f), \end{array}$$

hence $\varrho_M^{\mathcal{D}}$ is ${}^*\mathcal{C}$-semilinear by the injectivity of $\alpha_M^{\mathcal{D}^*}$. By Lemma 4.2.6 (2), $\varrho_M^{\mathcal{D}}$ is $\mathcal{C}$-semicolinear, i.e. M is a $(\mathcal{D},\mathcal{C})$-bisemicomodule.■

Theorem 4.2.15. *Let ${}_A\mathcal{C}$ and $\mathcal{D}_B$ be half α-modules. If M is a cancellative $(\mathcal{D},\mathcal{C})$-bisemicomodule, then every finite subset of M is contained in a $(\mathcal{D},\mathcal{C})$-subbisemicomodule $L \leq^{(\mathcal{D},\mathcal{C})} M$ such that L is finitely generated as a (B,A)-bisemimodule.*

Notation. For any non-empty subset $K \subseteq \mathcal{C}$ we set

$$\begin{array}{rcl} {}^{\perp}K & := & \{f \in {}^*\mathcal{C} \mid f(K) = 0\}; \\ K^{\perp} & := & \{f \in \mathcal{C}^* \mid f(K) = 0\}. \end{array}$$

For any non-empty subset $I \subseteq {}^*\mathcal{C}$ or $I \subseteq \mathcal{C}^*$, we set

$$\mathrm{Ke}(I) := \{c \in \mathcal{C} \mid f(c) = 0 \text{ for every } f \in I\}.$$

As a consequence of Theorems 4.2.9 and 4.2.11 we get

Proposition 4.2.16. *Let $\mathcal{C}$ be an A-semicoring.*

1. *If $K \subset \mathcal{C}$ is a right $\mathcal{C}$-coideal (resp. a left $\mathcal{C}$-coideal, a $\mathcal{C}$-bicoideal), then ${}^{\perp}K$ is a left ${}^*\mathcal{C}$-ideal (resp. a right ${}^*\mathcal{C}$-ideal, a ${}^*\mathcal{C}$-ideal). If K is a $\mathcal{C}$-coideal, then ${}^{\perp}K \subset {}^*\mathcal{C}$ is a half A-subring with unity $1_{^*\mathcal{C}}$. If ${}_A\mathcal{C}$ is a half α-module and $I \subset {}^*\mathcal{C}$ is a left ${}^*\mathcal{C}$-ideal, then $\mathrm{Ke}(I) \subset \mathcal{C}$ is a right $\mathcal{C}$-coideal.*

2. *If $K \subset \mathcal{C}$ is a left $\mathcal{C}$-coideal (resp. a right $\mathcal{C}$-coideal, a $\mathcal{C}$-bicoideal), then $K^{\perp}$ is a right $\mathcal{C}^*$-ideal (resp. a left $\mathcal{C}^*$-ideal, an $\mathcal{C}^*$-ideal). If K is a $\mathcal{C}$-coideal, then $K^{\perp} \subset \mathcal{C}^*$ is an A-subsemiring with unity $1_{^*\mathcal{C}}$. If $\mathcal{C}_A$ is a half α-module and $I \subset \mathcal{C}^*$ is a right $\mathcal{C}^*$-ideal, then $\mathrm{Ke}(I) \subset \mathcal{C}$ is a left $\mathcal{C}$-coideal.*

Bibliography

[Abu1] J. Abuhlail, *Locally projective semimodules,* under preparation.

[Abu2] J. Abuhlail, *Semicorings and Semicomodules,* preprint.

[Abe1980] E. Abe, *Hopf algebras*, Cambridge University Press (1980).

[Abu2003] J. Abuhlail, *Rational modules for corings*, Commun. Algebra **31(12)** (2003), 5793-5840.

[Abu2005] J.Y. Abuhlail, *On the linear weak topology and dual pairings over rings*, Topology Appl. **149(1-3)**, 161-175 (2005).

[Alt] H. Al-Thani, *Projective semimodules*, preprint.

[Alt-a] H. Al-Thani, *Cogenerators of Semimodules and the notion of rejects I,* preprint.

[Alt-b] H. Al-Thani, *Cogenerators of Semimodules and the notion of rejects II,* preprint.

[Alt-c] H. Al-Thani, *The rejects and the SemiCogenerators,* preprint.

[Alt2004] H. Al-Thani, *Flat semimodules*, Int. J. Math. Math. Sci., **2004(17)** (2004), 873-880.

[BEG-T2006] T. Brzeziński, L. El Kaoutit and J. Gómez-Torrecillas, *The bicategories of corings*, J. Pure Appl. Algebra. Vol. **205** (2006), 510-541.

[BG-T2003] T. Brzeziński and J. Gómez-Torrecillas, *On comatrix corings and bimodules*, K-Theory **29** (2003), 101-115.

[BM1998] T. Brzeziński and S. Majid, *Coalgebra bundles*, Comm. Math. Phys. **191(2)** (1998), 467-492.

[Bou1974] N. Bourbaki, *Elements of Mathematics, Algebra I, Chapters* **1-3**, Hermann (1974).

[BW2003] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, Lond. Math. Soc. Lec. Not. Ser. **Vol.** 309, Cambridge University Press (2003).

[CK1972] V. I. Kuźminov and A. Ju. Ĉerevikin. *Semiabelian categories*, Sib. Math. J. **13(6)** (1972), 895-902.

[CMZ2001] S. Caenepeel, G. Militaru and S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations*, Springer-Verlag (2002).

[Ded1894] R. Dedekind, *Über die Theorie der ganzen algebraiscen Zahlen.* Supplement XI to P. G. Dirichlet, L.: *Vorlesung über Zahlentheorie.* 4 Aufl., Druck and Verlag, Braunschweig (1894).

[DNR2001] S. Dăscălescu, C. Năstăsescu and Ş. Raianu, *Hopf Algebras: an Introduction*, Pure and Applied Mathematics **235**, Marcel Dekker, New York (2001).

[DP2005] R. P. Deore and K. B. Patil, *On the dual basis of projective semimodules and its applications*, Sarajevo J. Math. **1(14)** (2005), 161-169.

[Dri1988] V. G. Drinfel'd, *Quantum groups*, J. Soviet Math. **41(2)** (1988), 898-915.

[EG-T2003] L. El Kaoutit and J. Gómez-Torrecillas, *Comatrix corings: Galois corings, descent theory, and a structure theorem for cosemisimple corings*, Math. Z. **244** (2003), 887-906.

[EG-T2005a] L. El Kaoutit and J. Gómez-Torrecillas, *Morita duality for corings over quasi-Frobenius rings*, Hopf algebras in noncommutative geometry and physics, 137-153 (2005).

[EG-T2005b] L. El Kaoutit and J. Gómez-Torrecillas, *Comatrix corings and invertible bimodules*, Annali dell Universita di Ferrara, sez. VII, Scienze Matematiche, vol. **LI** (2005), 263-280.

[EGTL2004] L. El Kaoutit, J. Gómez-Torrecillas and F.J. Lobillo, *Semisimple corings*, Algebra Colloq. **11(2)** (2004), 427-442.

[Gar1976] G. Garfinkel, *Universally torsionless and trace modules*, Trans. Amer. Math. Soc. **215** (1976), 119-144.

[Gła2002] K. Głazek, *A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences. With Complete Bibliography,* Kluwer Academic Publishers, Dordrecht (2002).

[Gol1999a] J. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht (1999).

[Gol1999b] J. Golan, *Power Algebras over Semirings. With Applications in Mathematics and Computer Science.* Kluwer Academic Publishers, Dordrecht (1999).

[Gol2003] J. Golan, *Semirings and Affine Equations over Them.* Kluwer, Dordrecht (2003).

[GR1971] M. Raynaud and L. Gruson, *Critéres de platitude et de projectivité, Techniques de "platification" d'un module.* Invent. Math. **13** (1971), 1-89.

[G-T1998] J. Gómez-Torrecillas, *Coalgebras and comodules over a commutative ring*, Rom. J. Pure Appl. Math. **43**, 591-603 (1998).

[GT2006] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules,* De Gruyter (2006).

[G-TV2007] J. Gómez-Torrecillas and J. Vercruysse, *Comatrix Corings and Galois Comodules over Firm Rings,* Algebr. Represent. Theory **10(3)** (2007), 271-306.

[Hop1941] H. Hopf, *Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen,* Ann. of Math. (2) **42**, (1941) 22-52.

[HW1998] U. Hebisch and H. J. Weinert, *Semirings: algebraic theory and applications in computer science*, World Scientific Publishing Co., Inc., River Edge, NJ (1998).

[Kat1997] Y. Katsov, *Tensor products and injective envelopes of semimodules over additively regular semirings*, Algebra Colloq. **4(2)** (1997), 121-131.

[Kat2004] Y. Katsov, *On flat semimodules over semirings.* Algebra Universalis **51(2-3)** (2004), 287-299.

[Kat2004b] Y. Katsov, *Toward homological characterization of semirings: Serre's conjecture and Bass's perfectness in a semiring context*, Algebra Universalis **52(2-3)** (2004), 197-214.

[KS1986] W. Kuich and A. Salomaa, *Semirings, Automata, Languages*, Springer-Verlag, Berlin (1986).

[Kui1986] W. Kuich, *Semirings, automata, languages*, Springer-Verlag (1986).

[LMS1999] G. L. Litvinov, V.P. Maslov and G.B. Shpiz, *Tensor products of idempotent semimodules. An algebraic approach.* Math. Notes **65(3-4)** (1999), 479-489.

[MM1965] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211-264.

[Mon1993] S. Montgomery, *Hopf Algebras and their Actions on Rings*, AMS (1993).

[OR1972] J. Ohm and D. Rush, *Content modules and algebras*, Math. Scand. 31 (1972), 49-68.

[Rad1973] D. Radford, *Coreflexive coalgebras,* J. Algebra **26** (1973), 512-535.

[Sch1972] H. Schubert, *Categories*, Springer-Verlag (1972).

[Sok2002a] O. Sokratova, *Projective semimodules.* Algebra Universalis, **4(4)** (2002) 389-398.

[Sok2002b] O. Sokratova, *On semimodules over commutative, additively idempotent semirings*. Semigroup Forum **64(1)** (2002), 1-11.

[Swe1975] M. Sweedler, *The predual theorem to the Jacobson-Bourbaki theorem*. Trans. Amer. Math. Soc. **213** (1975), 391-406.

[Swe1969] M. Sweedler, *Hopf Algebras*, New York: Benjamin, (1969).

[Tak1979] Takahashi, M., *A bordism category for the ordinary homology theory*, Math. Sem. Notes Kobe Univ. **7(3)** (1979), 547-572.

[Tak1980] M. Takahashi, *On the bordism categories. I.* Math. Sem. Notes Kobe Univ. **8(3)** (1980), 527-546.

[Tak1981] M. Takahashi, *On the bordism categories. II. Elementary properties of semimodules.* Math. Sem. Notes Kobe Univ. **9(2)** (1981), 495-530.

[Tak1982a] M. Takahashi, *On the bordism categories. III. Functors Hom and for semimodules.* Math. Sem. Notes Kobe Univ. **10(2)** (1982), 551-562.

[Tak1982b] M. Takahashi, *Extensions of semimodules. I.* Math. Sem. Notes Kobe Univ. **10(2)** (1982), 563-592.

[Tak1982c] M. Takahashi, *Completeness and c-cocompleteness of the category of semimodules,* Math. Sem. Notes Kobe Univ. **10(2)** (1982), 551-562.

[Tak1983] M. Takahashi, *Extensions of semimodules. II.* Math. Sem. Notes Kobe Univ. **11(1)** (1983), 83-118.

[Tak1984a] M. Takahashi, *On semimodules. I,* Kobe J. Math. **1(1)** (1984), 67-97.

[Tak1984b] M. Takahashi, *On semimodules. II. System of generators.* Kobe J. Math. **1(2)** (1984), 177-190.

[Tak1985] Takahashi, M., *On semimodules. III. Cyclic semimodules.* Kobe J. Math. **2(2)** (1985), 131-141.

[TW1989] M. Takahashi and H.X. Wang, *On epimorphisms of semimodules.* Kobe J. Math. **6(2)** (1989), 297-298.

[Van1934] H.S. Vandiver, *Note on a simple type of algebra in which cancellation law of addition does not hold*, Bull. Amer. Math. Soc. **40** (1934), 914-920.

[Wis1991] R. Wisbauer, *Foundations of Module and Ring Theory. A Handbook for Study and Research*, Gordon and Breach Science Publishers (1991).

[Wis1996] R. Wisbauer, *Modules and Algebras: Bimodule Structure and Group Action on Algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics 81, Addison Wesely Longman Limited (1996).

[Wis1999] R. Wisbauer, *Semiperfect coalgebras over rings*, Algebra and Combinatorics. Papers from the International Congress ICAC'97 Hongkong, K.-P. Shum et al. (ed.), Singapore: Springer-Verlag, 487-512 (1999).

[Z-H1976] B. Zimmermann-Huisgen, *Pure submodules of direct products of free modules*, Math. Ann. **224** (1976), 233-245.

Index