Morita Contexts for Corings and Equivalences

Jawad Y. Abuhlail
Mathematics Department
Birzeit University
P.O.Box 14, Birzeit - Palestine
jabuhlail@birzeit.edu

Abstract

In this note we study Morita contexts and Galois extensions for corings. For a coring $\mathcal{C}$ over a (not necessarily commutative) ground ring A we give equivalent conditions for $\mathcal{M}^{\mathcal{C}}$ to satisfy the weak. resp. the strong structure theorem. We also characterize the so called *cleft C-Galois extensions* over commutative rings. Our approach is similar to that of Y. Doi and A. Masuoka in their work on (cleft) H-Galois extensions (e.g. [Doi94], [DM92]).

Introduction

Let $\mathcal{C}$ be a coring over a not necessarily commutative ring A and assume A to be a right $\mathcal{C}$-comodule through $\varrho_A : A \longrightarrow A\otimes_A\mathcal{C} \simeq \mathcal{C}$, $a \mapsto \mathbf{x}a$ for some group-like element $\mathbf{x} \in \mathcal{C}$ (see [Brz02, Lemma 5.1]). In the first section we study from the viewpoint of Morita theory the relationship between A and its subring of coinvariants $B := A^{co\mathcal{C}} := \{b \in A \mid \varrho(b) = b\mathbf{x}\}$. We consider the A-ring ${}^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A)$ and its left ideal $Q := \{q \in {}^*\mathcal{C} \mid \sum c_1 q(c_2) = q(c)\mathbf{x}$ for all $c \in \mathcal{C}\}$ and show that B and ${}^*\mathcal{C}$ are connected via a Morita context using ${}_BA_{^*\mathcal{C}}$ and ${}_{^*\mathcal{C}}Q_B$ as connecting bimodules. Our Morita context is in fact a generalization of Doi's Morita context presented in [Doi94].

In the second section we introduce the weak (resp. the strong) structure theorem for $\mathcal{M}^{\mathcal{C}}$. For the case ${}_A\mathcal{C}$ is locally projective, in the sense of B. Zimmermann-Huignes, we characterize A being a generator (a progenerator) in the category of right $\mathcal{C}$-comodules by $\mathcal{M}^{\mathcal{C}}$ satisfying the weak (resp. the strong) structure theorem. Here the notion of Galois corings introduced by T. Brzeziński [Brz02] plays an important role. The results and proofs are essentially module theoretic and similar to those of [MZ97] for the catgeory $\mathcal{M}(H)_A^C$ of Doi-Koppinen modules corresponding to a right-right Doi-Koppinen structure (H, A, C) (see also [MSTW01] for the case $C = H$).

The notion of a C-Galois extension A of a ring B was introduced by T. Brzeziński and S. Majid in [BM98] and is related to the so called entwining structures introduced in the same paper. In the third section we give equivalent conditions for a C-Galois extension A/B to be cleft. Our results generalize results of [Brz99] from the case of a base field to the case of a commutative ground ring. In the special case $\varrho(a) = \sum a_\psi \otimes x^\psi$, for some group-like element $x \in C$, we get a complete generalization of [DM92, Theorem 1.5] (and [Doi94, Theorem 2.5]).

With A we denote a not necessarily commutative ring with $1_A \neq 0_A$ and with $\mathcal{M}_A$ (resp. ${}_A\mathcal{M}$, ${}_A\mathcal{M}_A$) the category of *unital* right A-modules (resp. left A-modules, A-bimodules). For every right A-module W we denote by $\mathrm{Gen}(W_A)$ (resp. $\sigma[W_A]$) the class of W-generated (resp. W-subgenerated) right A-modules. For the well developed theory of categories of type $\sigma[W]$ the reader is referred to [Wis88, Section 15].

An A-module W is called **locally projective** (in the sense of B. Zimmermann-Huignes [Z-H76]), if for every diagram

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r, "\iota"] \arrow[dr, dotted, "g'\circ\iota"'] & W \arrow[d, dotted, "g'"] \arrow[dr, "g"] & & \\
& & L \arrow[r, "\pi"'] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows and F f.g.: for every A-linear map $g : W \longrightarrow N$, there exists an A-linear map $g' : W \longrightarrow L$, such that the entstanding parallelogram is commutative. Note that every projective A-module is locally projective. By [Z-H76, Theorem 2.1] a left A-module W is locally projective, iff for every right A-module M the following map is injective

$$\alpha_M^W : M \otimes_A W \longrightarrow \mathrm{Hom}_{-A}({}^*W, M),\ m \otimes_A w \mapsto [f \mapsto mf(w)].$$

It's easy then to see that every locally projective A-module is flat and A-cogenerated.

Let $\mathcal{C}$ be an A-coring. We consider the canonical A-bimodule ${}^*\mathcal{C} := \mathrm{Hom}_{A-}(\mathcal{C}, A)$ as an A-ring with the canonical A-bimodule structure, multiplication $(f \cdot g)(c) := \sum g(c_1 f(c_2))$ and unity $\varepsilon_{\mathcal{C}}$. If ${}_A\mathcal{C}$ is locally projective, then we have an isomorphism of categories $\mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{{}^*\mathcal{C}}]$ (in particular $\mathcal{M}^{\mathcal{C}} \subseteq \mathcal{M}_{{}^*\mathcal{C}}$ is a full subcategory) and we have a left exact functor $\mathrm{Rat}^{\mathcal{C}}(-) : \mathcal{M}_{{}^*\mathcal{C}} \to \mathcal{M}^{\mathcal{C}}$ assigning to every right ${}^*\mathcal{C}$-module its maximum $\mathcal{C}$*-rational* ${}^*\mathcal{C}$-submodule, which turns to be a right $\mathcal{C}$-comodule. Moreover $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_{{}^*\mathcal{C}}$ iff ${}_A\mathcal{C}$ is f.g. and projective. For more investigation of the $\mathcal{C}$-rational ${}^*\mathcal{C}$-modules see [Abu].

After this paper was finished, it turned out that some results in this paper were discovered independently by S. Caenepeel, J. Vercruysse and S. Wang [CVW].

1 Morita Contexts

In this section we fix the following: $\mathcal{C}$ is an A-coring with group-like element $\mathbf{x}$ and A is a right $\mathcal{C}$-comodule with structure map

$$\varrho_A : A \longrightarrow A \otimes_A \mathcal{C} \simeq \mathcal{C},\ a \mapsto \mathbf{x}a$$

(e.g. [Brz02, Lemma 5.1]), hence $A \in \mathcal{M}_{{}^*\mathcal{C}}$ with $a \leftharpoonup g := \sum a_{<0>} g(a_{<1>}) = g(\mathbf{x}a)$ for all $a \in A$ and $g \in {}^*\mathcal{C}$. For $M \in \mathcal{M}_{{}^*\mathcal{C}}$ put

$$M^{\mathbf{x}} := \{m \in M \mid mg = mg(\mathbf{x}) \text{ for all } g \in\ {}^*\mathcal{C}\}.$$

In particular $A^{\mathbf{x}} := \{a \in A \mid a \leftharpoonup g = ag(\mathbf{x}) \text{ for all } g \in {}^*\mathcal{C}\} \subset A$ is a subring. For $M \in \mathcal{M}^{\mathcal{C}}$ we set

$$M^{co\mathcal{C}} := \{m \in M \mid \varrho(m) = m \otimes_A \mathbf{x}\} \subseteq M^{\mathbf{x}}.$$

Obviously $B := A^{co\mathcal{C}} = \{b \in A \mid b\mathbf{x} = \mathbf{x}b\} \subseteq A^{\mathbf{x}}$ is a subring and ϱ_A is (B, A)-bilinear. For $M \in \mathcal{M}^{\mathcal{C}}$ we have $M^{co\mathcal{C}} \in \mathcal{M}_B$. Moreover we set

$$Q := \{q \in {}^{*}\mathcal{C} \mid \sum c_1 q(c_2) = q(c)\mathbf{x} \text{ for all } c \in \mathcal{C}\} \subseteq ({}^{*}\mathcal{C})^{\mathbf{x}}.$$

Lemma 1.1. *1. For every right ${}^{*}\mathcal{C}$-module M we have an isomorphism of right B-modules*

$$\omega_M : \mathrm{Hom}_{-{}^{*}\mathcal{C}}(A, M) \longrightarrow M^{\mathbf{x}},\ f \mapsto f(1_A)$$

with inverse $m \mapsto [a \mapsto ma]$.

2. Let ${}_A\mathcal{C}$ be locally projective. If $M \in \mathcal{M}^{\mathcal{C}}$, then $M^{co\mathcal{C}} = M^{\mathbf{x}} \simeq \mathrm{Hom}_{-{}^{*}\mathcal{C}}(A, M) = \mathrm{Hom}^{\mathcal{C}}(A, M)$. Hence

$$\Psi_M : M^{co\mathcal{C}} \otimes_B A \longrightarrow M,\ m \otimes_B a \mapsto ma$$

is surjective (resp. injective, bijective), iff

$$\Psi'_M : \mathrm{Hom}^{\mathcal{C}}(A, M) \otimes_B A \longrightarrow M,\ f \otimes_B a \mapsto f(a)$$

is surjective (resp. injective, bijective).

3. We have $\mathrm{Hom}_{-{}^{*}\mathcal{C}}(A, {}^{*}\mathcal{C}) \simeq ({}^{*}\mathcal{C})^{\mathbf{x}}$. If moreover ${}_A\mathcal{C}$ is A-cogenerated (resp. locally projective and ${}^{\Box}\mathcal{C} := \mathrm{Rat}^{\mathcal{C}}({}^{*}\mathcal{C}_{{}^{*}\mathcal{C}})$), then $Q = ({}^{*}\mathcal{C})^{\mathbf{x}}$ (resp. $Q = ({}^{\Box}\mathcal{C})^{co\mathcal{C}}$).

4. For every $M \in \mathcal{M}_{{}^{*}\mathcal{C}}$ (resp. $M \in \mathcal{M}^{\mathcal{C}}$) and all $m \in M$, $q \in Q$ we have $mq \in M^{\mathbf{x}}$ (resp. $mq \in M^{co\mathcal{C}}$).

Proof. 1. Obvious.

2. Trivial.

3. Considering ${}^{*}\mathcal{C}$ as a right ${}^{*}\mathcal{C}$-module via right multiplication we get $\mathrm{Hom}_{-{}^{*}\mathcal{C}}(A, {}^{*}\mathcal{C}) \simeq ({}^{*}\mathcal{C})^{\mathbf{x}}$ by (1). If $q \in ({}^{*}\mathcal{C})^{\mathbf{x}}$, then we have for all $g \in {}^{*}\mathcal{C}$ and $c \in \mathcal{C}$:

$$g(\sum c_1 q(c_2)) = \sum g(c_1 q(c_2)) = (q \cdot g)(c) = (qg(\mathbf{x}))(c) = q(c)g(\mathbf{x}) = g(q(c)\mathbf{x}),$$

i.e. $\sum c_1 q(c_2) - q(c)\mathbf{x} \in \mathrm{Re}(\mathcal{C}, A) := \bigcap\{\mathrm{Ke}(g) \mid g \in \mathrm{Hom}_{A-}(\mathcal{C}, A)\}$. If ${}_A\mathcal{C}$ is A-cogenerated, then $\mathrm{Re}(\mathcal{C}, A) = 0$, hence $Q = ({}^{*}\mathcal{C})^{\mathbf{x}}$.

Assume ${}_A\mathcal{C}$ to be locally projective. Then we have for all $q \in Q, g \in {}^{*}\mathcal{C}$ and $c \in \mathcal{C}$:

$$(q \cdot g)(c) = \sum g(c_1 q(c_2)) = g(q(c)\mathbf{x}) = q(c)g(\mathbf{x}) = (qg(\mathbf{x}))(c),$$

hence $q \in {}^{\Box}\mathcal{C}$, with $\varrho(q) = q \otimes_A \mathbf{x}$, i.e. $q \in ({}^{\Box}\mathcal{C})^{co\mathcal{C}}$. On the other hand, if $q \in ({}^{\Box}\mathcal{C})^{co\mathcal{C}}$, then for all $g \in {}^{*}\mathcal{C}$ we have $q \cdot g = qg(\mathbf{x})$, i.e. $q \in ({}^{*}\mathcal{C})^{\mathbf{x}} = Q$.

4. Let $M \in \mathcal{M}_{^*\mathcal{C}}$. Then we have for all $q \in Q$, $g \in {}^*\mathcal{C}$ and $m \in M$:

$$(mq)g = m(q \cdot g) = m(qg(\mathbf{x})) = (mq)g(\mathbf{x}),$$

i.e. $mq \in M^{\mathbf{x}}$. If $M \in \mathcal{M}^{\mathcal{C}}$, then we have for all $m \in M$ and $q \in Q$:

$$\begin{array}{rcl} \varrho_M(mq) & = & \varrho_M(\sum m_{<0>}q(m_{<1>})) \\ & = & \sum m_{<0><0>} \otimes_A m_{<0><1>}q(m_{<1>}) \\ & = & \sum m_{<0>} \otimes_A m_{<1>1}q(m_{<1>2}) \\ & = & \sum m_{<0>} \otimes_A q(m_{<1>})\mathbf{x} \\ & = & \sum m_{<0>}q(m_{<1>}) \otimes_A \mathbf{x} \\ & = & \sum mq \otimes_A \mathbf{x}, \end{array}$$

i.e. $mq \in M^{co\mathcal{C}}$.■

Lemma 1.2. *1. With the canonical actions A is a $(B, {}^*\mathcal{C})$-bimodule.*

2. Q is a $({}^*\mathcal{C}, B)$-bimodule.

Proof. 1. By assumption $A \in \mathcal{M}^{\mathcal{C}} \subseteq \mathcal{M}_{^*\mathcal{C}}$. For all $b \in B$, $a \in A$ and $g \in {}^*\mathcal{C}$ we have

$$b(a \leftharpoonup g) = bg(\mathbf{x}a) = g(b(\mathbf{x}a)) = g(\mathbf{x}(ba)) = (ba) \leftharpoonup g.$$

2. For all $a \in A$, $q \in Q$ and $c \in \mathcal{C}$ we have

$$\sum c_1(aq)(c_2) = \sum c_1q(c_2a) = \sum (ca)_1q((ca)_2) = q(ca)\mathbf{x} = (aq)(c)\mathbf{x}.$$

For all $q \in Q$, $b \in B$ and $c \in \mathcal{C}$ we have

$$\sum c_1(qb)(c_2) = \sum c_1q(c_2)b = q(c)\mathbf{x}b = q(c)b\mathbf{x} = (qb)(c)\mathbf{x}.$$

On the other hand we have for all $q \in Q$, $g \in {}^*\mathcal{C}$ and $c \in \mathcal{C}$:

$$\begin{array}{rclcl} \sum c_1(g \cdot q)(c_2) & = & \sum c_1q(c_{21}g(c_{22})) & = & \sum c_{11}q(c_{12}g(c_2)) \\ & = & \sum c_{11}(g(c_2)q)(c_{12}) & = & \sum (g(c_2)q)(c_1)\mathbf{x} \\ & = & \sum q(c_1g(c_2))\mathbf{x} & = & (g \cdot q)(c)\mathbf{x}. \end{array}$$

Moreover we have for all $b \in B$, $q \in Q$, $g \in {}^*\mathcal{C}$ and $c \in \mathcal{C}$:

$$\begin{array}{rclcl} ((g \cdot q)b)(c) & = & (g \cdot q)(c)b & = & \sum q(c_1g(c_2))b \\ & = & \sum (qb)(c_1g(c_2)) & = & (g \cdot qb)(c). \blacksquare \end{array}$$

Theorem 1.3. *Keep the notation above fixed.*

1. $(A^{\mathbf{x}}, {}^*\mathcal{C}, A, ({}^*\mathcal{C})^{\mathbf{x}}, \widetilde{F}, \widetilde{G})$ is a Morita context derived form $A_{^*\mathcal{C}}$, where

$$\begin{array}{rcccccccc} \widetilde{F} & : & ({}^*\mathcal{C})^{\mathbf{x}} \otimes_{A^{\mathbf{x}}} A & \longrightarrow & {}^*\mathcal{C}, & q \otimes_{A^{\mathbf{x}}} a & \mapsto & qa, \\ \widetilde{G} & : & A \otimes_{^*\mathcal{C}} ({}^*\mathcal{C})^{\mathbf{x}} & \longrightarrow & A^{\mathbf{x}}, & a \otimes_{^*\mathcal{C}} q & \mapsto & a \leftharpoonup q. \end{array}$$

2. *$(B,{}^*\mathcal{C},A,Q,F,G)$ is a Morita context, where*

$$\begin{array}{lllllll} F & : & Q\otimes_B A & \longrightarrow & {}^*\mathcal{C}, & q\otimes_B a & \mapsto & qa, \\ G & : & A\otimes_{{}^*\mathcal{C}} Q & \longrightarrow & B, & a\otimes_{{}^*\mathcal{C}} q & \mapsto & a \leftharpoonup q. \end{array}$$

If moreover ${}_A\mathcal{C}$ is locally projective, then the two Morita contexts coincide.

Proof. 1. By Lemma 1.1 we have $\mathrm{End}(A_{{}^*\mathcal{C}}) \simeq A^{\mathbf{x}}$, $({}^*\mathcal{C})^{\mathbf{x}} \simeq \mathrm{Hom}_{-{}^*\mathcal{C}}(A,{}^*\mathcal{C})$ and the result follows by [Fai81, Proposition 12.6].

2. By Lemma 1.2 A is a $(B,{}^*\mathcal{C})$-bimodule and Q is a $({}^*\mathcal{C},B)$-bimodule. For all $q\in Q, g\in {}^*\mathcal{C}$, $a\in A$ and $c\in\mathcal{C}$ we have

$$F(g\cdot q\otimes_B a)(c) = \sum q(c_2 g(c_1))a = (g\cdot qa)(c) = (g\cdot F(q\otimes_B a))(c)$$

and

$$\begin{array}{rclcl} F(q\otimes_B a\leftharpoonup g)(c) & = & q(c)(a\leftharpoonup g) & = & q(c)g(\mathbf{x}a) \\ & = & g(q(c)\mathbf{x}a) & = & \sum g(c_1 q(c_2)a) \\ & = & \sum g(c_1(qa)(c_2)) & = & (F(q\otimes_B a)\cdot g)(c), \end{array}$$

hence F is ${}^*\mathcal{C}$-bilinear. Note that by Lemma 1.1 G is well defined and is obviously B-bilinear. Moreover we have for all $a,\widetilde{a}\in A$ and $q,\widetilde{q}\in Q$ the following associativity relations:

$$\begin{array}{rclcl} (F(q\otimes_B a)\cdot\widetilde{q})(c) & = & \sum\widetilde{q}(c_1 q(c_2)a) & = & \widetilde{q}(q(c)\mathbf{x}a) \\ & = & q(c)\widetilde{q}(\mathbf{x}a) & = & (qG(a\otimes_{{}^*\mathcal{C}}\widetilde{q}))(c), \\ G(a\otimes_{{}^*\mathcal{C}} q)\widetilde{a} & = & q(\mathbf{x}a)\widetilde{a} & = & (q\widetilde{a})(\mathbf{x}a) \\ & = & F(q\otimes_B\widetilde{a})(\mathbf{x}a) & = & a\leftharpoonup F(q\otimes_B\widetilde{a}). \end{array}$$

If ${}_A\mathcal{C}$ is locally projective, then $A^{\mathbf{x}} = A^{co\mathcal{C}}$, $({}^*\mathcal{C})^{\mathbf{x}} = Q$ by Lemma 1.1 and the two contexts coincide.■

1.4. [Brz02, Definition 5.3] An A-coring $\mathcal{C}$ is said to be **Galois**, if there exists an A-coring isomorphism $\chi : A\otimes_B A \longrightarrow \mathcal{C}$ such that $\chi(1_A\otimes_B 1_A) = \mathbf{x}$. Recall that $A\otimes_B A$ is an A-coring with the canonical A-bimodule structure, comultiplication

$$\Delta : A\otimes_B A \longrightarrow (A\otimes_B A)\otimes_A (A\otimes_B A),\ \widetilde{a}\otimes_B a \mapsto (\widetilde{a}\otimes_B 1_A)\otimes_A (1_A\otimes_B a)$$

and counity $\varepsilon_{A\otimes_B A} : A\otimes_B A \longrightarrow A$, $\widetilde{a}\otimes_B a\mapsto \widetilde{a}a$.

1.5. Consider the functors

$$(-)^{co\mathcal{C}} : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}_B \text{ and } -\otimes_B A : \mathcal{M}_B \longrightarrow \mathcal{M}^{\mathcal{C}}.$$

By [Brz02, Proposition 5.2] $(-\otimes_B A, (-)^{co\mathcal{C}})$ is an adjoint pair of covariant functors, where the adjunctions are given by

$$\Phi_N : N \longrightarrow (N\otimes_B A)^{co\mathcal{C}},\ n\mapsto n\otimes_B 1_A \tag{1}$$

and

$$\Psi_M : M^{co\mathcal{C}} \otimes_B A \longrightarrow M,\ m \otimes_B a \mapsto ma. \tag{2}$$

If Ψ_M is an isomorphism for all $M \in \mathcal{M}^{\mathcal{C}}$, then we say $\mathcal{M}^{\mathcal{C}}$ satisfies the **weak structure theorem.** If in addition Φ_N is an isomorphism for all $N \in \mathcal{M}_B$, then we say $\mathcal{M}^{\mathcal{C}}$ satisfies the **strong structure theorem** (in this case $(-)^{co\mathcal{C}}$ and $- \otimes_B A$ give an equivalence of categories $\mathcal{M}^{\mathcal{C}} \simeq \mathcal{M}_B$).

1.6. Let $W \in \mathcal{M}_A$ and consider the canonical right $\mathcal{C}$-comodule $W \otimes_A \mathcal{C}$. Then $W \simeq (W \otimes_A \mathcal{C})^{co\mathcal{C}}$ via $w \mapsto w \otimes_A \mathbf{x}$ with inverse $w \otimes_A c \mapsto w\varepsilon_{\mathcal{C}}(c)$ and we define

$$\beta_W := \Psi_{W\otimes_A\mathcal{C}} : W \otimes_B A \longrightarrow W \otimes_A \mathcal{C},\ w \otimes_B a \mapsto w \otimes_A \mathbf{x}a. \tag{3}$$

In particular we have for $W = A$ the morphism of A-corings

$$\beta := \Psi_{A\otimes_A\mathcal{C}} : A \otimes_B A \longrightarrow A \otimes_A \mathcal{C} \simeq \mathcal{C},\ \widetilde{a} \otimes_B a \mapsto \widetilde{a}\mathbf{x}a. \tag{4}$$

If β is bijective, then $\mathcal{C}$ is a Galois A-coring and we call the ring extension A/B $\mathcal{C}$-**Galois**.

Theorem 1.7. *For the Morita context $(B, {}^*\mathcal{C}, A, Q, F, G)$ the following statements are equivalent:*

1. *$G : A \otimes_{^*\mathcal{C}} Q \longrightarrow B$ is surjective (bijective and $B = A^{\mathbf{x}}$);*
2. *there exists $\widehat{q} \in Q$, such that $\widehat{q}(\mathbf{x}) = 1_A$;*
3. *for every right ${}^*\mathcal{C}$-module M we have a B-module isomorphism $M \otimes_{^*\mathcal{C}} Q \simeq M^{\mathbf{x}}$.*
4. *for every right $\mathcal{C}$-comodule M we have $M \otimes_{^*\mathcal{C}} Q \simeq M^{\mathbf{co}\mathcal{C}}$ as B-modules.*

   If moreover ${}_A\mathcal{C}$ is locally projective, then (1)-(4) are moreover equivalent to:
5. *$A_{^*\mathcal{C}}$ is (f.g.) projective.*

Proof. (1) ⇒ (2). Assume G to be surjective. Then there exist $a_1, ..., a_k$ and $q_1, ..., q_k \in Q$, such that $G(\sum_{i=1}^{k} a_i \otimes_{^*\mathcal{C}} q_i) = 1_A$. Set $\widehat{q} := \sum_{i=1}^{k} a_i q_i \in Q$. Then we have

$$\widehat{q}(\mathbf{x}) = (\sum_{i=1}^{k} a_i q_i)(\mathbf{x}) = \sum_{i=1}^{k} q_i(\mathbf{x}a_i) = \sum_{i=1}^{k} (a_i \leftharpoonup q_i) = G(\sum_{i=1}^{k} a_i \otimes_{^*\mathcal{C}} q_i) = 1_A.$$

(2) ⇒ (3). Consider the B-module morphism

$$\xi_M : M \otimes_{^*\mathcal{C}} Q \longrightarrow M^{\mathbf{x}},\ m \otimes_{^*\mathcal{C}} q \mapsto mq.$$

Let $\widehat{q} \in Q$ with $\widehat{q}(\mathbf{x}) = 1_A$ and define $\widetilde{\xi}_M : M^{\mathbf{x}} \longrightarrow M \otimes_{^*\mathcal{C}} Q$, $m \mapsto m \otimes_{^*\mathcal{C}} \widehat{q}$. For every $n \in M^{\mathbf{x}}$ we have

$$(\xi_M \circ \widetilde{\xi}_M)(n) = \xi_M(n \otimes_{^*\mathcal{C}} \widehat{q}) = n \leftharpoonup \widehat{q} = n\widehat{q}(\mathbf{x}) = n.$$

On the other hand we have for all $m \in M$ and $q \in Q$:

$$\begin{array}{rclclcl}(\widetilde{\xi}_M \circ \xi_M)(m \otimes_{^*\mathcal{C}} q) & = & \widetilde{\xi}_M(m \leftharpoonup q) & = & m \leftharpoonup q \otimes_{^*\mathcal{C}} \widehat{q} & = & m \otimes_{^*\mathcal{C}} q \cdot \widehat{q} \\ & = & m \otimes_{^*\mathcal{C}} q\widehat{q}(\mathbf{x}) & = & m \otimes_{^*\mathcal{C}} q, & & \end{array}$$

i.e. ξ_M is bijective with inverse $\widetilde{\xi}_M$.

(3) $\Rightarrow$ (4). Let $M \in \mathcal{M}^{\mathcal{C}}$. By Lemma 1.1 we have $\xi_M(M \otimes_{^*\mathcal{C}} Q) \subseteq M^{co\mathcal{C}} \subseteq M^{\mathbf{x}}$. By assumption $\xi_M : A \otimes_{^*\mathcal{C}} Q \longrightarrow M^{\mathbf{x}}$ is bijective. Hence $M^{\mathbf{x}} = M^{co\mathcal{C}}$ and $M \otimes_{^*\mathcal{C}} Q \overset{\xi_M}{\simeq} M^{co\mathcal{C}}$.

(4) $\Rightarrow$ (1) We are done since $G = \xi_A$.

Assume ${}_A\mathcal{C}$ to be locally projective.

Then $B \simeq \mathrm{End}(A_{^*\mathcal{C}})$, $Q \simeq \mathrm{Hom}_{-^*\mathcal{C}}(A, {}^*\mathcal{C})$ and we get (1) $\Longleftrightarrow$ (5) by [Fai81, Corollary 12.8].■

Corollary 1.8. *For the Morita context* $(B, {}^*\mathcal{C}, A, Q, F, G)$ *assume there exists* $\widehat{q} \in Q$ *with* $\widehat{q}(\mathbf{x}) = 1_A$ *(equivalently* $G : Q \otimes_B A \longrightarrow {}^*\mathcal{C}$ *is surjective). Then:*

1. *For every* $N \in \mathcal{M}_B$, Φ_N *is an isomorphism.*
2. B *is a left* B*-direct summand of* A.

Proof. 1. Let $N \in \mathcal{M}_B$. Then we have by Theorem 1.7 the isomorphisms $G : A \otimes_{^*\mathcal{C}} Q \longrightarrow B$ and $\xi_{N \otimes_B A} : (N \otimes_B A) \otimes_{^*\mathcal{C}} Q \longrightarrow (N \otimes_B A)^{co\mathcal{C}}$. Moreover Φ_N is given by the canonical isomorphisms

$$N \simeq N \otimes_B B \simeq N \otimes_B (A \otimes_{^*\mathcal{C}} Q) \simeq (N \otimes_B A) \otimes_{^*\mathcal{C}} Q \simeq (N \otimes_B A)^{co\mathcal{C}}.$$

2. The map $\mathrm{tr}_A : A \longrightarrow B$, $a \mapsto a \leftharpoonup \widehat{q}$ is left B-linear with $\mathrm{tr}_A(b) = b$ for all $b \in B$.■

Corollary 1.9. *For the Morita context* $(B, {}^*\mathcal{C}, A, Q, F, G)$ *assume there exists* $\widehat{q} \in Q$ *with* $\widehat{q}(\mathbf{x}) = 1_A$ *(equivalently* $G : Q \otimes_B A \longrightarrow B$ *is surjective). Then:*

1. ${}_BA$ *and* Q_B *are generators.*
2. $A_{^*\mathcal{C}}$ *and* ${}_{^*\mathcal{C}}Q$ *are f.g. and projective.*
3. $F : Q \otimes_B A \longrightarrow {}^*\mathcal{C}$ *induces bimodule isomorphisms*

$$A \simeq \mathrm{Hom}_{^*\mathcal{C}-}(Q, {}^*\mathcal{C}) \text{ and } Q \simeq \mathrm{Hom}_{-^*\mathcal{C}}(A, {}^*\mathcal{C}).$$

4. *The bimodule structures above induce ring isomorphisms*

$$B \simeq \mathrm{End}(A_{^*\mathcal{C}}) \text{ and } B \simeq \mathrm{End}({}_{^*\mathcal{C}}Q)^{op}.$$

Proof. The result follows by standard argument of Morita Theory (e.g. [Fai81, Proposition 12.7]).■

Proposition 1.10. *Consider the Morita context* $(B, {}^*\mathcal{C}, A, Q, F, G)$ *and assume that* $F : Q \otimes_{^*\mathcal{C}} A \longrightarrow {}^*\mathcal{C}$ *is surjective. Then:*

1. *$A_{^*\mathcal{C}}$ is a generator, $Q \simeq \mathrm{Hom}_{B-}(A,B)$ as bimodules and ${}^*\mathcal{C} \simeq \mathrm{End}(Q_B)$.*

2. *$\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem (in particular A/B is $\mathcal{C}$-Galois).*

Proof. 1. The result follows by standard argument of Morita Theory (e.g. [Fai81, Proposition 12.7]).

2. By assumption $\varepsilon_{\mathcal{C}} = F(\sum\limits_{i=1}^{k} q_i \otimes_B a_i)$ for some $\{(q_i,a_i)\}_{i=1}^{k} \subseteq Q \times A$. In this case $\Psi_M : M^{co\mathcal{C}} \otimes_B A \longrightarrow M$ is bijective with inverse $\widetilde{\Psi}_M : M \longrightarrow M^{co\mathcal{C}} \otimes_B A$, $m \mapsto \sum\limits_{i=1}^{k} mq_i \otimes_B a_i$. In fact, we have for all $m \in M$, $n \in M^{co\mathcal{C}}$ and $a \in A$:

$$\begin{aligned} (\Psi_M \circ \widetilde{\Psi}_M)(m) &= \sum_{i=1}^{k}(mq_i)a_i &&= \sum_{i=1}^{k}(m_{<0>}q_i(m_{<1>})a_i \\ &= \sum_{i=1}^{k} m_{<0>}(q_ia_i)(m_{<1>}) &&= \sum_{i=1}^{k} m_{<0>}\varepsilon_{\mathcal{C}}(m_{<1>}) \\ &= m \end{aligned}$$

and

$$\begin{aligned} (\widetilde{\Psi}_M \circ \Psi_M)(n \otimes_B a) &= \sum_{i=1}^{k}(na)q_i \otimes_B a_i &&= \sum_{i=1}^{k} nq_i(\mathbf{x}a) \otimes_B a_i \\ &= \sum_{i=1}^{k} n \otimes_B q_i(\mathbf{x}a)a_i &&= \sum_{i=1}^{k} n \otimes_B (q_ia_i)(\mathbf{x}a) \\ &= n \otimes_B \varepsilon_{\mathcal{C}}(\mathbf{x}a) &&= n \otimes_B a. \blacksquare \end{aligned}$$

Theorem 1.11. *For the Morita context $(B, {}^*\mathcal{C}, A, Q, F, G)$ the following are equivalent:*

1. *$F : Q \otimes_B A \longrightarrow {}^*\mathcal{C}$ is surjective (bijective);*

2. *(a) Q_B is f.g. and projective;*

   (b) $\Omega : A \longrightarrow \mathrm{Hom}_{-B}(Q,B)$, $a \mapsto [q \mapsto a \leftharpoondown q]$ is a bimodule isomorphism;

   (c) ${}_{^*\mathcal{C}}Q$ is faithful.

   If ${}_A\mathcal{C}$ is A-cogenerated, then (1) & (2) are moreover equivalent to:

3. *(a) ${}_BA$ is f.g. and projective;*

   (b) $\Lambda : {}^*\mathcal{C} \longrightarrow \mathrm{End}({}_BA)^{op}$, $g \mapsto [a \mapsto a \leftharpoondown g]$ is a ring isomorphism.

4. *$A_{^*\mathcal{C}}$ is a generator.*

   If moreover ${}_A\mathcal{C}$ is f.g. and projective, then (1)-(4) are equivalent to:

5. *$\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem.*

Proof. The implications (1) $\Rightarrow$ (2), (3), (4) follow without any finiteness conditions on $\mathcal{C}$ by standard argument of Morita Theory (e.g. [Fai81, Proposition 12.7]). Note that ${}_{*\mathcal{C}}Q$ is faithful by the embedding ${}^*\mathcal{C} \hookrightarrow \mathrm{End}(Q_B)$ (see Proposition 1.10 (1)).

(2) $\Rightarrow$ (1). Let $\{(q_i, p_i)\}_{i=1}^k \subset Q \times \mathrm{Hom}_{-B}(Q, B)$ be a dual basis for Q_B. By (b) there exist $a_1, ..., a_k \in A$, such that $\Omega(a_i) = q_i$ for $i = 1, ..., k$. For every $q \in Q$ we have then $(\sum\limits_{i=1}^k q_i a_i) \cdot q = \sum\limits_{i=1}^k q_i(a_i \leftharpoondown q) = \sum\limits_{i=1}^k q_i p_i(q) = q$, hence $\sum\limits_{i=1}^k q_i a_i = \varepsilon_{\mathcal{C}}$ by (c) and the ${}^*\mathcal{C}$-bilinear morphism $F : Q \otimes_{{}^*\mathcal{C}} A \longrightarrow {}^*\mathcal{C}$ is surjective.

Assume ${}_A\mathcal{C}$ to be A-cogenerated.

(3) $\Rightarrow$ (1). Let $\{(a_i, p_i)\}_{i=1}^k \subset A \times \mathrm{Hom}_{B-}(A, B)$ be a dual basis of ${}_BA$. By (b), there exist $g_1, ..., g_k \in {}^*\mathcal{C}$, such that $\Lambda(g_i) = p_i$ for $i = 1, ..., k$. **Claim**: $g_1, ..., g_k \in Q$. For all $f \in {}^*\mathcal{C}$ and $i = 1, ..., k$ we have

$$\begin{array}{rclcl} \Lambda(g_i \cdot f)(a) & = & a \leftharpoondown (g_i \cdot f) & = & (a \leftharpoondown g_i) \leftharpoondown f) \\ & = & p_i(a) \leftharpoondown f & = & f(\mathbf{x} p_i(a)) \\ & = & f(p_i(a)\mathbf{x}) & = & p_i(a) f(\mathbf{x}) \\ & = & (p_i f(\mathbf{x}))(a) & = & \Lambda(g_i f(\mathbf{x}))(a), \end{array}$$

hence $g_i \cdot f = g_i f(\mathbf{x})$, i.e. $g_i \in ({}^*\mathcal{C})^{\mathrm{x}} = Q$ (by Lemma 1.1 (2)). Moreover for every $a \in A$ we have: $\Lambda(\sum\limits_{i=1}^k g_i a_i)(a) = \sum\limits_{i=1}^k a \leftharpoondown g_i a_i = \sum\limits_{i=1}^k p_i(a) a_i = a$, i.e. $\sum\limits_{i=1}^k g_i a_i = \varepsilon_{\mathcal{C}}$ and the ${}^*\mathcal{C}$-bilinear morphism F is surjective.

(4) $\Rightarrow$ (1). Since $Q \simeq \mathrm{Hom}_{-\,{}^*\mathcal{C}}(A, {}^*\mathcal{C})$, we have $\mathrm{Im}(F) = \mathrm{tr}(A, {}^*\mathcal{C}) := \sum\{\mathrm{Im}(h) : h \in \mathrm{Hom}_{-\,{}^*\mathcal{C}}(A, {}^*\mathcal{C})\}$, hence $\mathrm{Im}(F) = {}^*\mathcal{C}$ iff $A_{{}^*\mathcal{C}}$ is a generator (e.g. [Wis88, Page 154]).

Assume ${}_A\mathcal{C}$ to be f.g. and projective.

(1) $\Rightarrow$ (5) follows without any finiteness conditions on $\mathcal{C}$ by Proposition 1.10 (2).

(5) $\Rightarrow$ (1). Since ${}_A\mathcal{C}$ is f.g. and projective, we have $\mathcal{M}^{\mathcal{C}} \simeq \mathcal{M}_{{}^*\mathcal{C}}$ (e.g. [Brz02, Lemma 4.3]), hence ${}^*\mathcal{C} \in \mathcal{M}^{\mathcal{C}}$, $Q = ({}^*\mathcal{C})^{co\mathcal{C}}$ and $F = \Psi_{{}^*\mathcal{C}}$.■

2 Galois Extensions and Equivalences

The notation of the first section remains fixed. For every $M \in \mathcal{M}^{\mathcal{C}}$ we have the $\mathcal{C}$-colinear morphism

$$\Psi'_M : \mathrm{Hom}^{\mathcal{C}}(A, M) \otimes_B A \longrightarrow M, \ f \otimes_B a \mapsto f(a).$$

In this section we characterize A being a generator (resp. a progenerator) in $\mathcal{M}^{\mathcal{C}}$ under the assumption that ${}_A\mathcal{C}$ is locally projective. Our approach is similar to that of [MZ97] and our results generalize those obtained there for the special case of the category of Doi-Koppinen modules $\mathcal{M}(H)_A^C$.

Lemma 2.1. *Assume ${}_A\mathcal{C}$ to be locally projective. If ${}_BA$ is flat and A/B is $\mathcal{C}$-Galois, then:*

1. *A is a subgenerator in $\mathcal{M}^{\mathcal{C}}$, i.e. $\sigma[A_{{}^*\mathcal{C}}] = \sigma[\mathcal{C}_{{}^*\mathcal{C}}]$.*
2. *for each $M \in \mathcal{M}^{\mathcal{C}}$, Ψ'_M is injective.*
3. *for every A-generated $M \in \mathcal{M}^{\mathcal{C}}$, Ψ'_M is an isomorphism.*

Proof. Assume ${}_A\mathcal{C}$ to be locally projective.

1. Since A/B is $\mathcal{C}$-Galois, $\beta' := \Psi'_{\mathcal{C}}$ is an isomorphism, hence $\mathcal{C}$ is A-generated. Consequently $\sigma[A_{^*\mathcal{C}}] \subseteq \sigma[\mathcal{C}_{^*\mathcal{C}}] \subseteq \sigma[A_{^*\mathcal{C}}]$, i.e. $\sigma[A_{^*\mathcal{C}}] = \sigma[\mathcal{C}_{^*\mathcal{C}}]$.
2. With slight modifications, the proof of [MZ97, Lemma 3.22] applies.
3. If $M \in \mathcal{M}^{\mathcal{C}}$ is A-generated, then Ψ'_M is surjective, hence bijective by (2).■

The following result is a generalization of [Brz99, Proposition 3.13] (which in turn generalizes [DT89, Theorem 2.11]):

Proposition 2.2. *Assume A/B to be $\mathcal{C}$-Galois.*

1. *If ${}_BA$ is flat, then $\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem.*
2. *Assume there exists $\widehat{q} \in Q$, such that $\widehat{q}(\mathbf{x}) = 1_A$. If ${}_BA$ is flat, or for all $b \in B$ and $c \in \mathcal{C}$ we have $\widehat{q}(cb) = q(c)b$, then $\mathcal{M}^{\mathcal{C}}$ satisfies the strong structure theorem.*

Proof.

1. The proof is the first part of the proof of [Brz02, Theorem 5.6].
2. By assumption and Corollary 1.8, Φ_N is an isomorphism for all $N \in \mathcal{M}_B$. If ${}_BA$ is flat, then $\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem by (1). On the other hand, if for all $b \in B$ and $c \in \mathcal{C}$ we have $\widehat{q}(cb) = q(c)b$, then an analog argument to that in the proof of [Brz99, Proposition 3.13] shows that $\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem.■

Theorem 2.3. *Assume ${}_A\mathcal{C}$ to be locally projective. Then the following are equivalent:*

1. *$\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem;*
2. *${}_BA$ is flat and A/B is $\mathcal{C}$-Galois;*
3. *${}_BA$ is flat and $\beta' := \Psi'_{\mathcal{C}}$ is an isomorphism;*
4. *${}_BA$ is flat and for every A-generated $M \in \mathcal{M}^{\mathcal{C}}$, Ψ'_M is bijective;*
5. *for every $M \in \mathcal{M}^{\mathcal{C}} = \sigma[\mathcal{C}_{^*\mathcal{C}}]$, the $\mathcal{C}$-colinear morphism Ψ'_M is bijective;*
6. $\sigma[\mathcal{C}_{^*\mathcal{C}}] = \mathrm{Gen}(A_{^*\mathcal{C}})$;
7. *${}_BA$ is flat, $\sigma[\mathcal{C}_{^*\mathcal{C}}] = \sigma[A_{^*\mathcal{C}}]$ and $\mathrm{Hom}_{-^*\mathcal{C}}(A,-) : \mathrm{Gen}(A_{^*\mathcal{C}}) \longrightarrow \mathcal{M}_B$ is full faithful;*
8. *$\mathrm{Hom}^{\mathcal{C}}(A,-) : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}_B$ is faithful;*
9. *A is a generator in $\mathcal{M}^{\mathcal{C}}$.*

Proof. (1) $\Longleftrightarrow$ (5) & (2) $\Longleftrightarrow$ (3) follow by Lemma 1.1. The equivalences (4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) follow by [MZ97, Theorem 2.3]. The equivalence (8) $\Longleftrightarrow$ (9) is evident for any category, and moreover (6) $\Longleftrightarrow$ (9) by the fact that $\mathrm{Gen}(A_{^*\mathcal{C}}) \subseteq \sigma[A_{^*\mathcal{C}}] \subseteq \sigma[\mathcal{C}_{^*\mathcal{C}}] = \mathcal{M}^{\mathcal{C}}$. By Lemma 2.1 we have (3) $\Rightarrow$ (4). Now assuming (1) we conclude that A/B is $\mathcal{C}$-Galois and that ${}_BA$ is flat (since (1) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (7)), hence (1) $\Rightarrow$ (2) follows and we are done.■

Definition 2.4. ([MZ97, Definition 2.4]) A left module P over a ring $\mathcal{S}$ is called a **weak generator**, if for any right $\mathcal{S}$-module Y, $Y \otimes_{\mathcal{S}} P = 0$ implies $Y = 0$. A right module P over a ring $\mathcal{R}$ is called **quasiprogenerator** (resp. **progenerator**), if $P_{\mathcal{R}}$ is f.g. quasiprojective and generates each of its submodules (resp. $P_{\mathcal{R}}$ is f.g., projective and a generator). $P_{\mathcal{R}}$ is called **faithful** (resp. **balanced**), if the canonical morphism $\mathcal{R} \longrightarrow \mathrm{End}({}_{\mathrm{End}(P_{\mathcal{R}})}P)^{op}$ is injective (resp. surjective).

Theorem 2.5. *Assume ${}_A\mathcal{C}$ to be flat. Then the following are equivalent:*

1. *$\mathcal{M}^{\mathcal{C}}$ satisfies the strong structure theorem;*
2. *${}_BA$ is faithfully flat and A/B is $\mathcal{C}$-Galois.*

   If moreover ${}_A\mathcal{C}$ is locally projective, then (1) & (2) *are moreover equivalent to:*
3. *${}_BA$ is faithfully flat and $\beta' := \Psi'_{\mathcal{C}}$ is bijective;*
4. *${}_BA$ is faithfully flat and for every $M \in \sigma[A_{*\mathcal{C}}]$, Ψ'_M is bijective;*
5. *$A_{*\mathcal{C}}$ is quasiprojective and generates each of its submodules, ${}_BA$ is a weak generator and $\sigma[\mathcal{C}_{*\mathcal{C}}] = \sigma[A_{*\mathcal{C}}]$;*
6. *$A_{*\mathcal{C}}$ is a quasiprogenerator and $\sigma[\mathcal{C}_{*\mathcal{C}}] = \sigma[A_{*\mathcal{C}}]$;*
7. *${}_BA$ is a weak generator, Ψ'_M is an isomorphism for every $M \in \mathrm{Gen}(A_{*\mathcal{C}})$ and $\sigma[A_{*\mathcal{C}}] = \sigma[\mathcal{C}_{*\mathcal{C}}]$;*
8. *$\mathrm{Hom}^{\mathcal{C}}(A, -) : \mathcal{M}^{\mathcal{C}} \longrightarrow \mathcal{M}_B$ is an equivalence;*
9. *A is a progenerator in $\mathcal{M}^{\mathcal{C}}$.*

Proof. (1) $\Longleftrightarrow$(2) is [Brz02, Theorem 5.6]. Assume ${}_A\mathcal{C}$ to be locally projective. Then (2) $\Longleftrightarrow$ (3) follows by Lemma 1.1 and we get (1) $\Longleftrightarrow$ (8) $\Longleftrightarrow$ (9) by characterizations of progenerators in categories of type $\sigma[M]$ (see [Wis88, 18.5, 46.2]). Moreover (4) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6) $\Longleftrightarrow$ (7) follow from [MZ97, Theorem 2.5]. Obviously (3) $\Rightarrow$ (4) (note that (3) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (1)). Assume now (4). Then ${}_BA$ is faithfully flat and moreover $\Psi'_{\mathcal{C}}$ is bijective, since $\mathcal{C} \in \sigma[A_{*\mathcal{C}}]$ by (6), i.e. (4) $\Rightarrow$ (3) and the proof is complete.■

Remark 2.6. Assume ${}_A\mathcal{C}$ to be locally projective. Then $\mathrm{Im}(F) \subseteq {}^{\Box}\mathcal{C}$. In fact we have for all $q \in Q$, $a \in A$, $g \in {}^*\mathcal{C}$ and $c \in C$:

$$((qa) \cdot g)(c) = \sum g(c_1 q(c_2)a) = g(q(c)\mathbf{x}a) = q(c)g(\mathbf{x}a) = (qg(\mathbf{x}a)(c),$$

hence $qa \in {}^{\Box}\mathcal{C}$, with $\varrho(qa) = q \otimes_A \mathbf{x}a$.

Proposition 2.7. *Assume ${}_A\mathcal{C}$ to be locally projective and that there exists $\widehat{q} \in Q$ with $\widehat{q}(\mathbf{x}) = 1_A$ (equivalently $G : A \otimes_{*\mathcal{C}} Q \longrightarrow B$ is surjective). Then $\mathcal{M}^{\mathcal{C}}$ satisfies the strong equivalence theorem, iff $\mathrm{Im}(F) = {}^{\Box}\mathcal{C}$ and the following map is surjective for every $M \in \mathcal{M}^{\mathcal{C}}$*

$$\varpi_M : M \otimes_{*\mathcal{C}} {}^{\Box}\mathcal{C} \longrightarrow M, \; m \otimes_{*\mathcal{C}} f \mapsto mf.$$

In this case $Q \otimes_B A \overset{F}{\simeq} {}^{\Box}\mathcal{C}$ and $M \otimes_{*\mathcal{C}} {}^{\Box}\mathcal{C} \overset{\varpi_M}{\simeq} M$ for every $M \in \mathcal{M}^{\mathcal{C}}$.

Proof. Consider for every $M \in \mathcal{M}^{\mathcal{C}}$ the commutative diagram

$$\begin{tikzcd}
M \otimes_{^*\mathcal{C}} Q \otimes_B A \arrow[r, "\xi_M \otimes id_A"] \arrow[d, "id_M \otimes F"'] & M^{co\mathcal{C}} \otimes_B A \arrow[d, "\Psi_M"] \\
M \otimes_{^*\mathcal{C}} {}^{\Box}\mathcal{C} \arrow[r, "\varpi_M"'] & M
\end{tikzcd}$$

Assume $\mathrm{Im}(F) = {}^{\Box}\mathcal{C}$ and ϖ_M to be surjective for every $M \in \mathcal{M}^{\mathcal{C}}$. Then Ψ_M is obviously surjective. Let $K = \mathrm{Ke}(\Psi_M)$. Since Ψ_M is a morphism in $\mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{^*\mathcal{C}}]$ we have $K \in \mathcal{M}^{\mathcal{C}}$, hence $\Psi_K : K^{co\mathcal{C}} \otimes_B A \longrightarrow K$ is surjective. By Theorem 1.7 we have $K \otimes_{^*\mathcal{C}} Q \overset{\xi_K}{\simeq} K^{co\mathcal{C}}$ and $A \otimes_{^*\mathcal{C}} Q \overset{\xi_A}{\simeq} B$, hence

$$K^{co\mathcal{C}} \simeq K \otimes_{^*\mathcal{C}} Q = \mathrm{Ke}(\Psi_M) \otimes_{^*\mathcal{C}} Q = \mathrm{Ke}(\Psi_M \otimes_{^*\mathcal{C}} id_Q) = \mathrm{Ke}(id_{M^{co\mathcal{C}}}) = 0,$$

i.e. Ψ_M is bijective. By corollary 1.8 Φ_N is bijective for every $N \in \mathcal{M}_B$. Consequently $\mathcal{M}^{\mathcal{C}}$ satisfies the strong structure theorem.

On the other hand, assume that $\mathcal{M}^{\mathcal{C}}$ satisfies the strong structure theorem. Note that F is the adjunction of $\Psi_{^{\Box}\mathcal{C}}$, hence $Q \otimes_B A \overset{F}{\simeq} {}^{\Box}\mathcal{C}$ and consequently ϖ_M is also bijective for every $M \in \mathcal{M}^{\mathcal{C}}$ by the commutativity of the above diagram.■

Remarks 2.8. Assume ${}_A\mathcal{C}$ to be locally projective.

1. $\varpi_A : A \otimes_{^*\mathcal{C}} {}^{\Box}\mathcal{C} \longrightarrow A$ is surjective iff there exists $\widehat{g} \in {}^{\Box}\mathcal{C}$ with $\widehat{g}(\mathbf{x}) = 1_A$. To prove this assume first that ϖ_A is surjective. Then there exist $\{(a_i, g_i)\}_{i=1}^{k} \subset A \times {}^{\Box}\mathcal{C}$, such that $\sum_{i=1}^{k} a_i \leftharpoonup g_i = 1_A$. Set $\widehat{g} := \sum_{i=1}^{k} a_i g_i \in {}^{\Box}\mathcal{C}$. Then $\widehat{g}(\mathbf{x}) = (\sum_{i=1}^{k} a_i g_i)(\mathbf{x}) = \sum_{i=1}^{k} g_i(\mathbf{x}a_i) = \sum_{i=1}^{k} a_i \leftharpoonup g_i = 1_A$. On the other hand, assume there exists $\widehat{g} \in {}^{\Box}\mathcal{C}$ with $\widehat{g}(\mathbf{x}) = 1_A$. Then for every $a \in A$ we have $1_A \leftharpoonup (\widehat{g}a) = (\widehat{g}a)(\mathbf{x}) = \widehat{g}(\mathbf{x})a = a$, i.e. ϖ_A is surjective.

2. Assume ϖ_A to be surjective. If Ψ_M is surjective for $M \in \mathcal{M}^{\mathcal{C}}$, then ϖ_M is surjective, since
$$\varpi_M \circ (\Psi_M \otimes_{^*\mathcal{C}} id_{^{\Box}\mathcal{C}}) = \Psi_M \circ (id_{M^{co\mathcal{C}}} \otimes_B \varpi_A).$$

Theorem 2.9. *Assume ${}_A\mathcal{C}$ to be f.g. and projective. Then the following are equivalent:*

1. *$\mathcal{M}^{\mathcal{C}}$ satisfies the weak structure theorem;*

2. *${}_BA$ is flat and A/B is $\mathcal{C}$-Galois;*

3. *${}_BA$ is flat and $\beta' := \Psi'_{\mathcal{C}}$ is an isomorphism;*

4. *${}_BA$ is flat and for every A-generated $M \in \mathcal{M}^{\mathcal{C}} = \mathcal{M}_{^*\mathcal{C}}$, the $\mathcal{C}$-colinear morphism Ψ'_M is bijective;*

5. *for every* $M \in \mathcal{M}^{\mathcal{C}}$, *the* $\mathcal{C}$*-colinear morphism* Ψ'_M *is bijective;*

6. ${}_BA$ *is flat,* $\mathcal{M}_{^*\mathcal{C}} = \sigma[A_{^*\mathcal{C}}]$ *and* $\mathrm{Hom}_{-^*\mathcal{C}}(A,-) : \mathrm{Gen}(A_{^*\mathcal{C}}) \longrightarrow \mathcal{M}_B$ *is full faithful;*

7. $\mathrm{Hom}_{-^*\mathcal{C}}(A,-) : \mathcal{M}_{^*\mathcal{C}} \longrightarrow \mathcal{M}_B$ *is faithful;*

8. $A_{^*\mathcal{C}}$ *is a generator;*

9. $F : Q \otimes_B A \longrightarrow {}^*\mathcal{C}$ *is surjective (bijective);*

10. *(a)* Q_B *is f.g. and projective;*
    (b) $\Omega : A \longrightarrow \mathrm{Hom}_{-B}(Q,B)$, $a \mapsto [q \mapsto a \leftharpoonup q]$ *is a bimodule isomorphism;*
    (c) ${}_{^*\mathcal{C}}Q$ *is faithful;*

11. *(a)* ${}_BA$ *is f.g. and projective;*
    (b) $\Lambda : {}^*\mathcal{C} \longrightarrow \mathrm{End}({}_BA)^{op}$, $g \mapsto [a \mapsto a \leftharpoonup g]$ *is a ring isomorphism.*

Proof. The result follows by Theorems 1.11, 2.3 and the fact that in case ${}_A\mathcal{C}$ is f.g. and projective $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_{^*\mathcal{C}} = \sigma[\mathcal{C}_{^*\mathcal{C}}]$.$\blacksquare$

Theorem 2.10. (Morita, e.g. [Fai81, 4.1.3, 4.3], [MZ97, 2.6]). *Let* $\mathcal{R}$ *be a ring,* P *a right* $\mathcal{R}$*-module,* $\mathcal{S} := \mathrm{End}(P_{\mathcal{R}})$ *and* $P^* := \mathrm{Hom}_{\mathcal{R}}(P,\mathcal{R})$.

1. *The following are equivalent:*
    (a) $P_{\mathcal{R}}$ *is a generator;*
    (b) ${}_{\mathcal{S}}P$ *is f.g. projective and* $\mathcal{R} \simeq \mathrm{End}({}_{\mathcal{S}}P)^{op}$ *canonically.*

2. *The following are equivalent:*
    (a) $P_{\mathcal{R}}$ *is a faithful quasiprogenerator and* ${}_{\mathcal{S}}P$ *is finitely generated;*
    (b) $P_{\mathcal{R}}$ *is a progenerator;*
    (c) ${}_{\mathcal{S}}P$ *is a progenerator and* $P_{\mathcal{R}}$ *is faithfully balanced;*
    (d) $P_{\mathcal{R}}$ *and* ${}_{\mathcal{S}}P$ *are generators;*
    (e) $P_{\mathcal{R}}$ *and* ${}_{\mathcal{S}}P$ *are f.g. and projective;*
    (f) $\mathrm{Hom}_{-\mathcal{R}}(P,-) : \mathcal{M}_{\mathcal{R}} \longrightarrow \mathcal{M}_{\mathcal{S}}$ *is an equivalence with inverse* $\mathrm{Hom}_{-\mathcal{S}}(P^*,-)$;
    (g) $- \otimes_{\mathcal{R}} P^* : \mathcal{M}_{\mathcal{R}} \longrightarrow \mathcal{M}_{\mathcal{S}}$ *is an equivalence with inverse* $- \otimes_{\mathcal{S}} P$.

As a consequence of Theorems 2.5 and 2.10 we get

Theorem 2.11. *Assume* ${}_A\mathcal{C}$ *to be f.g. and projective. Then the following are equivalent:*

1. $\mathcal{M}^{\mathcal{C}}$ *satisfies the strong structure theorem;*

2. ${}_BA$ *is faithfully flat and* A/B *is* $\mathcal{C}$*-Galois;*

3. *${}_BA$ is faithfully flat and $\beta' := \Psi'_{\mathcal{C}}$ is bijective;*

4. *${}_BA$ is faithfully flat and for every $M \in \sigma[A_{*\mathcal{C}}]$, the map Ψ'_M is bijective;*

5. *$A_{*\mathcal{C}}$ is quasiprojective and generates each of its submodules, ${}_BA$ is a weak generator and $\mathcal{M}_{*\mathcal{C}} = \sigma[A_{*\mathcal{C}}]$;*

6. *$A_{*\mathcal{C}}$ is a quasiprogenerator and $\mathcal{M}_{*\mathcal{C}} = \sigma[A_{*\mathcal{C}}]$;*

7. *${}_BA$ is a weak generator, Ψ'_M is an isomorphism for every $M \in \mathrm{Gen}(A_{*\mathcal{C}})$ and $\mathcal{M}_{*\mathcal{C}} = \sigma[A_{*\mathcal{C}}]$;*

8. *$A_{*\mathcal{C}}$ is a faithful quasiprogenerator and ${}_BA$ is finitely generated;*

9. *${}_BA$ is a progenerator and $A_{*\mathcal{C}}$ is faithfully balanced;*

10. *$\mathrm{Hom}_{-*\mathcal{C}}(A,-) : \mathcal{M}_{*\mathcal{C}} \longrightarrow \mathcal{M}_B$ is an equivalence with inverse $\mathrm{Hom}_{-B}(Q,-)$;*

11. *$- \otimes_{*\mathcal{C}} Q : \mathcal{M}_{*\mathcal{C}} \longrightarrow \mathcal{M}_B$ is an equivalence with inverse $- \otimes_B A$;*

12. *$A_{*\mathcal{C}}$ and ${}_BA$ are generators;*

13. *$A_{*\mathcal{C}}$ and ${}_BA$ are f.g. and projective;*

14. *$A_{*\mathcal{C}}$ is a progenerator.*

3 Cleft C-Galois Extensions

In what follows R is a *commutative* ring with $1_R \neq 0_R$ and $\mathcal{M}_R$ is the category of R-(bi)modules. For an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ and an R-algebra (A, μ_A, η_A) we consider $(\mathrm{Hom}_R(C,A), \star) := \mathrm{Hom}_R(C,A)$ as an R-algebra with the *usual convolution product* $(f \star g)(c) := \sum f(c_1)g(c_2)$ and unity $\eta_A \circ \varepsilon_C$. The unadorned $- \otimes -$ means $- \otimes_R -$.

3.1. Entwined modules. A right-right **entwining structure** (A, C, ψ) over R consists of an R-algebra (A, μ_A, η_A), an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ and an R-linear map

$$\psi : C \otimes_R A \longrightarrow A \otimes_R C,\ c \otimes a \mapsto \sum a_\psi \otimes c^\psi,$$

such that

$$\begin{array}{rclrcl} \sum (a\widetilde{a})_\psi \otimes c^\psi & = & \sum a_\psi \widetilde{a}_\Psi \otimes c^{\psi\Psi}, & \sum (1_A)_\psi \otimes c^\psi & = & 1_A \otimes c, \\ \sum a_\psi \otimes \Delta_C(c^\psi) & = & \sum a_{\psi\Psi} \otimes c_1^\Psi \otimes c_2^\psi, & \sum a_\psi \varepsilon_C(c^\psi) & = & \varepsilon_C(c)a. \end{array}$$

3.2. Let (A, C, ψ) be a right-right entwining structure. An **entwined module** corresponding to (A, C, ψ) is a right A-module, which is also a right C-comodule through ϱ_M, such that

$$\varrho_M(ma) = \sum m_{<0>} a_\psi \otimes m_{<1>}^\psi \text{ for all } m \in M \text{ and } a \in A.$$

The category of right-right entwined modules and A-linear C-colinear morphisms is denoted by $\mathcal{M}_A^C(\psi)$. For $M, N \in \mathcal{M}_A^C(\psi)$ we denote by $\mathrm{Hom}_A^C(M,N)$ the set of A-linear C-colinear morphisms from M to N. With $\#_{\psi}^{op}(C,A) := \mathrm{Hom}_R(C,A)$, we denote the A-ring with $(af)(c) = \sum a_{\psi} f(c^{\psi})$, $(fa)(c) = f(c)a$, multiplication $(f \cdot g)(c) = \sum f(c_2)_{\psi} g(c_1^{\psi})$ and unity $\eta_A \circ \varepsilon_C$ (see [Abu, Lemma 3.3]).

Entwined modules were introduced by T. Brzeziński and S. Majid [BM98] as a generalization of the Doi-Koppinen modules presented in [Doi92] and [Kop95]. By a remark of M. Takeuchi (e.g. [Brz02, Proposition 2.2]), we have an A-coring structure on $\mathcal{C} := A \otimes_R C$, where $\mathcal{C}$ is an A-bimodule through $a(\widetilde{a} \otimes c) := a\widetilde{a} \otimes c$, $(\widetilde{a} \otimes c)a := \sum \widetilde{a}a_{\psi} \otimes c^{\psi}$ and has comultiplication

$$\Delta_{\mathcal{C}} : A \otimes_R C \longrightarrow (A \otimes_R C) \otimes_A (A \otimes_R C),\ a \otimes c \mapsto \sum (a \otimes c_1) \otimes_A (1_A \otimes c_2)$$

and counity $\varepsilon_{\mathcal{C}} := id_A \otimes \varepsilon_C$. Moreover $\mathcal{M}_A^C(\psi) \simeq \mathcal{M}^{\mathcal{C}}$, $\#_{\psi}^{op}(C,A) \simeq {}^*\mathcal{C}$ as A-rings and ${}_A\mathcal{C}$ is flat (resp. f.g., projective), if ${}_RC$ is so (e.g. [Abu]).

Inspired by [Doi94, 3.1] we make the following definition:

3.3. Let (A, C, ψ) be a right-right entwining structure over R and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$. We say that (A, C, ψ) satisfies the **left α-condition**, if for every right A-module M the following map is injective

$$\alpha_M^{\psi} : M \otimes_R C \longrightarrow \mathrm{Hom}_R(\#_{\psi}^{op}(C,A), M),\ m \otimes c \mapsto [f \mapsto mf(c)]$$

(equivalently, if ${}_A\mathcal{C}$ is locally projective).

Let M be a right $\#_{\psi}^{op}(C,A)$-module M and consider the canonical map $\rho_M : M \longrightarrow \mathrm{Hom}_R(\#_{\psi}^{op}(C,A), M)$. Set $\mathrm{Rat}^C(M_{\#_{\psi}^{op}(C,A)}) := (\rho_M^{\psi})^{-1}(M \otimes_R C)$. We call M **#-rational**, if $\mathrm{Rat}^C(M_{\#_{\psi}^{op}(C,A)}) = M$ and set $\varrho_M := (\alpha_M^{\psi})^{-1} \circ \rho_M$. The category of #-rational right $\#_{\psi}^{op}(C,A)$-modules will be denoted by $\mathrm{Rat}^C(\mathcal{M}_{\#_{\psi}^{op}(C,A)})$.

Theorem 3.4. ([Abu, Theorem 3.10]) *Let (A, C, ψ) be a right-right entwining structure and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

1. *If ${}_RC$ is flat, then $\mathcal{M}_A^C(\psi)$ is a Grothendieck category with enough injective objects.*
2. *If ${}_RC$ is locally projective (resp. f.g. and projective), then*

$$\mathcal{M}_A^C(\psi) \simeq \mathrm{Rat}^C(\mathcal{M}_{\#_{\psi}^{op}(C,A)}) \simeq \sigma[(A \otimes_R C)_{\#_{\psi}^{op}(C,A)}]\ (\textit{resp. } \mathcal{M}_A^C(\psi) \simeq \mathcal{M}_{\#_{\psi}^{op}(C,A)}). \quad (5)$$

In what follows we fix a right-right entwining structure (A, C, ψ) with $\mathcal{C} := A \otimes_R C$ the corresponding A-coring and assume that $A \in \mathcal{M}_A^C(\psi) \simeq \mathcal{M}^{\mathcal{C}}$ with

$$\varrho_A : A \longrightarrow A \otimes_R C,\ a \mapsto \sum a_{<0>} \otimes a_{<1>} = \sum 1_{<0>} a_{\psi} \otimes 1_{<1>}^{\psi}.$$

Then $\sum 1_{<0>} \otimes 1_{<1>} \in \mathcal{C}$ is a group-like element and

$$Q \simeq \{q \in \mathrm{Hom}_R(C,A) \mid \sum q(c_2)_{\psi} \otimes c_1^{\psi} = \sum q(c) 1_{<0>} \otimes 1_{<1>} \text{ for all } c \in C\}.$$

For every $M \in \mathcal{M}_A^C(\psi)$, we set

$$M^{co\mathcal{C}} := \{m \in M \mid \sum m_{<0>} \otimes m_{<1>} = \sum m 1_{<0>} \otimes 1_{<1>}\}.$$

Moreover we set $B := A^{co\mathcal{C}}$.

Remark 3.5. Let $x \in C$ be a group-like element. For every right C-comodule M we put $M^{coC} := \{m \in M \mid \varrho_M(m) = m \otimes x\}$. If $\varrho_A(1_A) = 1_A \otimes x$, then we have $M^{coC} = M^{co\mathcal{C}}$ for every $M \in \mathcal{M}_A^C(\psi)$.

By [Brz99, Corollaries 3.4, 3.7] $- \otimes_R^c A : \mathcal{M}^C \longrightarrow \mathcal{M}_A^C(\psi)$ is a functor, which is left adjoint to the forgetful functor. Here, for every $N \in \mathcal{M}^C$, we consider the canonical right A-module $N \otimes_R^c A := N \otimes_R A$ with the C-coaction $n \otimes a \mapsto \sum n_{<0>} \otimes a_\psi \otimes n_{<1>}^\psi$.

Proposition 3.6. *Let R be a QF ring and assume C be right semiperfect. Let ${}_RC$ to be locally projective (projective) and put $C^\Box := \mathrm{Rat}({}_{^*C}C^*)$.*

1. *The following are equivalent:*
   - *(a) A is a generator in $\mathcal{M}_A^C(\psi)$;*
   - *(b) A generates $C^\Box \otimes_R^c A$ in $\mathcal{M}_A^C(\psi)$;*
   - *(c) the map $\Psi'_{C^\Box \otimes_R^c A} : \mathrm{Hom}_A^C(A, C^\Box \otimes_R^c A) \otimes_B A \longrightarrow C^\Box \otimes_R^c A$ is surjective (bijective).*
2. *The following are equivalent:*
   - *(a) A is a progenerator in $\mathcal{M}_A^C(\psi)$;*
   - *(b) $\Psi'_{C^\Box \otimes_R^c A}$ is surjective (bijective) and ${}_BA$ is a weak generator.*

Proof. By [MTW01, 2.6] $C^\Box$ is a generator in $\mathcal{M}^C$, hence $C^\Box \otimes_R^c A$ is a generator in $\mathcal{M}_A^C(\psi)$ by the functorial isomorphism $\mathrm{Hom}_A^C(C^\Box \otimes_R^c A, M) \simeq \mathrm{Hom}^C(C^\Box, M)$ for every $M \in \mathcal{M}_A^C(\psi)$.

1. The assertions follow form the note above and Theorem 2.3.
2. (a) $\Rightarrow$ (b) follows by Theorem 2.5.

   (b) $\Rightarrow$ (a). By the note above $C^\Box \otimes_R^c A$ is a generator in $\mathcal{M}_A^C(\psi)$, and the surjectivity of $\Psi'_{C^\Box \otimes_R^c A}$ makes A a generator in $\mathcal{M}_A^C(\psi)$. So ${}_BA$ is flat by Theorem 2.3. The weak generator property makes ${}_BA$ faithfully flat and we are done by Theorem 2.5.■

Definition 3.7. A **(total) integral** for C is a C-colinear morphism $\lambda : C \longrightarrow A$ (with $\sum 1_{<0>}\lambda(1_{<1>}) = 1_A$). We call the ring extension A/B **cleft,** if there exists a **$\star$-invertible integral**. We say A has the **right normal basis property**, if there exists a left B-linear right C-colinear isomorphism $A \simeq B \otimes_R C$.

Lemma 3.8. *Let $\lambda \in \mathrm{Hom}_R(C, A)$ be $\star$-invertible with inverse $\overline{\lambda}$. Then:*

1. *$\lambda \in \mathrm{Hom}^C(C, A)$ iff $\overline{\lambda} \in Q$.*
2. *If $\varrho(a) = \sum a_\psi \otimes x^\psi$ for some group-like element $x \in C$, then there exists $\widehat{\lambda} \in Q$, such that $\sum 1_{<0>}\widehat{\lambda}(1_{<1>}) = \widehat{\lambda}(x) = 1_A$ (in this case C admits a total integral, namely the $\star$-inverse of $\widehat{\lambda}$).*

Proof. Let $\lambda \in \mathrm{Hom}_R(C,A)$ be $\star$-invertible with inverse $\overline{\lambda}$.

1. If $\overline{\lambda} \in Q$, then we have for all $c \in C$:

$$\begin{array}{rcl} \sum \lambda(c_1) \otimes c_2 & = & \sum \lambda(c_1) 1_\psi \otimes c_2^\psi \\ & = & \sum \lambda(c_1)\varepsilon(c_3) 1_\psi \otimes c_2^\psi \\ & = & \sum \lambda(c_1)(\overline{\lambda}(c_3)\lambda(c_4))_\psi \otimes c_2^\psi \\ & = & \sum \lambda(c_1)\overline{\lambda}(c_3)_\psi \lambda(c_4)_\Psi \otimes c_2^{\psi\Psi} \\ & = & \sum \lambda(c_1)\overline{\lambda}(c_{22})_\psi \lambda(c_3)_\Psi \otimes c_{21}^{\psi\Psi} \\ & = & \sum \lambda(c_1)\overline{\lambda}(c_2) 1_{<0>} \lambda(c_3)_\Psi \otimes 1_{<1>}^\Psi \\ & = & \sum 1_{<0>} \lambda(c)_\Psi \otimes 1_{<1>}^\Psi \\ & = & \sum \lambda(c)_{<0>} \otimes \lambda(c)_{<1>} \end{array}$$

   i.e. $\lambda \in \mathrm{Hom}^C(C,A)$. On the other hand, if $\lambda \in \mathrm{Hom}^C(C,A)$, then we have for all $c \in C$:

$$\begin{array}{rcl} \sum \overline{\lambda}(c_2)_\psi \otimes c_1^\psi & = & \sum \overline{\lambda}(c_1)\lambda(c_2)\overline{\lambda}(c_4)_\psi \otimes c_3^\psi \\ & = & \sum \overline{\lambda}(c_1)\lambda(c_2)_{<0>}\overline{\lambda}(c_3)_\psi \otimes \lambda(c_2)_{<1>}^\psi \\ & = & \sum \overline{\lambda}(c_1) 1_{<0>} \lambda(c_2)_\psi \overline{\lambda}(c_3)_\Psi \otimes 1_{<1>}^{\psi\Psi} \\ & = & \sum \overline{\lambda}(c_1) 1_{<0>} (\lambda(c_2)\overline{\lambda}(c_3))_\psi \otimes 1_{<1>}^\psi \\ & = & \sum \overline{\lambda}(c) 1_{<0>} 1_\psi \otimes 1_{<1>}^\psi \\ & = & \sum \overline{\lambda}(c) 1_{<0>} \otimes 1_{<1>}, \end{array}$$

   i.e. $\overline{\lambda} \in Q$.

2. Assume $\varrho(a) = \sum a_\psi \otimes x^\psi$ for some group-like element $x \in C$. Let $\lambda \in \mathrm{Hom}^C(C,A)$ with $\overline{\lambda} \in Q$ (see (1)). Then $\widehat{\lambda} := \overline{\lambda}\lambda(x) \in Q$, since $\lambda(x) \in B$, and moreover $\sum 1_{<0>}\widehat{\lambda}(1_{<1>}) = \widehat{\lambda}(x) = \overline{\lambda}(x)\lambda(x) = (\overline{\lambda} \star \lambda)(x) = \varepsilon_C(x) 1_A = 1_A$.■

Proposition 3.9. *Assume A/B to be cleft.*

1. *$\mathcal{M}_A^C(\psi)$ satisfies the weak structure theorem (in particular A/B is $\mathcal{C}$-Galois).*

2. *For every $M \in \mathcal{M}_A^C(\psi)$, the C-colinear morphism*

$$\gamma_M : M \longrightarrow M^{coC} \otimes_R C, \; m \mapsto \sum m_{<0>}\overline{\lambda} \otimes m_{<1>}$$

   is an isomorphism.

3. *A has the right normal basis property.*

4. *If ${}_RC$ is faithfully flat, then $\mathcal{M}_A^C(\psi)$ satisfies the strong structure theorem.*

Proof. Assume there exists a $\star$-invertible $\lambda \in \mathrm{Hom}^C(C,A)$ with inverse $\overline{\lambda} \in Q$ (see Lemma 3.8 (1)).

1. Let $M \in \mathcal{M}_A^C(\psi)$ and consider

$$\widetilde{\Psi}_M : M \longrightarrow M^{co\mathcal{C}} \otimes_B A,\ m \mapsto \sum m_{<0>}\overline{\lambda} \otimes \lambda(m_{<1>}).$$

   Then we have for all $n \in M^{co\mathcal{C}}$, $m \in M$ and $a \in A$:

$$\begin{aligned}
(\widetilde{\Psi}_M \circ \Psi_M)(n \otimes a) &= \widetilde{\Psi}_M(na) \\
&= \textstyle\sum (na_{<0>})\overline{\lambda} \otimes_B \lambda(a_{<1>}) \\
&= \textstyle\sum na_{<0><0>}\overline{\lambda}(a_{<0><1>}) \otimes_B \lambda(a_{<1>}) \\
&= \textstyle\sum n \otimes_B a_{<0><0>}\overline{\lambda}(a_{<0><1>})\lambda(a_{<1>}) \\
&= \textstyle\sum n \otimes_B a_{<0>}\overline{\lambda}(a_{<1>1})\lambda(a_{<1>2}) \\
&= n \otimes_B a
\end{aligned}$$

   and

$$\begin{aligned}
(\Psi_M \circ \widetilde{\Psi}_M)(m) &= \textstyle\sum (m_{<0>}\overline{\lambda})\lambda(m_{<1>}) \\
&= \textstyle\sum m_{<0><0>}\overline{\lambda}(m_{<0><1>})\lambda(m_{<1>}) \\
&= \textstyle\sum m_{<0>}\overline{\lambda}(m_{<1>1})\lambda(m_{<1>2}) \\
&= \textstyle\sum m_{<0>}\varepsilon_C(m_{<1>})1_A \\
&= m.
\end{aligned}$$

2. For every $M \in \mathcal{M}_A^C(\psi)$, γ_M is bijective with inverse

$$\widetilde{\gamma}_M : M^{co\mathcal{C}} \otimes_R C \longrightarrow M,\ n \otimes c \mapsto n\lambda(c).$$

   In fact we have for all $m \in M$, $n \in M^{co\mathcal{C}}$ and $c \in C$:

$$\begin{aligned}
(\widetilde{\gamma}_M \circ \gamma_M)(m) &= \textstyle\sum (m_{<0>}\overline{\lambda})\lambda(m_{<1>}) \\
&= \textstyle\sum m_{<0><0>}\overline{\lambda}(m_{<0><1>})\lambda(m_{<1>}) \\
&= \textstyle\sum m_{<0>}\overline{\lambda}(m_{<1>1})\lambda(m_{<1>2}) \\
&= \textstyle\sum m_{<0>}\varepsilon_C(m_{<1>}) \\
&= m
\end{aligned}$$

   and

$$\begin{aligned}
(\gamma_M \circ \widetilde{\gamma}_M)(n \otimes_B c) &= \textstyle\sum (n\lambda(c))_{<0>}\overline{\lambda} \otimes (n\lambda(c))_{<0>} \\
&= \textstyle\sum (n\lambda(c)_{<0>})\overline{\lambda} \otimes \lambda(c)_{<1>} \\
&= \textstyle\sum (n\lambda(c_1))\overline{\lambda} \otimes c_2 \\
&= \textstyle\sum n\lambda(c_1)_{<0>}\overline{\lambda}(\lambda(c_1)_{<1>}) \otimes c_2 \\
&= \textstyle\sum n\lambda(c_{11})\overline{\lambda}(c_{12}) \otimes c_2 \\
&= n \otimes c.
\end{aligned}$$

3. By (2) the left B-linear right C-colinear map

$$\gamma_A : A \longrightarrow B \otimes_R C, a \mapsto \sum a_{<0>} \leftharpoonup \overline{\lambda} \otimes a_{<1>}$$

   is an isomorphism with inverse $b \otimes c \mapsto b\lambda(c)$.

4. Assume ${}_RC$ to be faithfully flat. By (3) $A \simeq B \otimes_R C$ as left B-modules, hence ${}_BA$ is faithfully flat. By (1) A/B is $\mathcal{C}$-Glaois and we are done by Theorem 2.5.■

Theorem 3.10. *The following statements are equivalent:*

1. *A/B is cleft;*
2. *$\mathcal{M}_A^C(\psi)$ satisfies the weak structure theorem and A has the right normal basis property;*
3. *A/B is $\mathcal{C}$-Galois and A has the right normal basis property;*
4. *$\Lambda : \#_{\psi}^{op}(C, A) \simeq \mathrm{End}({}_BA)^{op}$, $g \mapsto [a \mapsto a \leftharpoonup g]$ is a ring isomorphism and A has the right normal basis property.*

   If moreover ${}_RC$ is faithfully flat, then (1)-(4) *are equivalent to*

5. *$\mathcal{M}_A^C(\psi)$ satisfies the strong structure theorem and A has the right normal basis property.*

Proof. (1) ⇒ (2). This follows by Proposition 3.9.

(2) ⇒ (3). By assumption $\beta := \Psi_{A\otimes_R C}$ is an isomorphism.

(3) ⇒ (4). By assumption $A \otimes_B A \simeq A \otimes_R C$ as left A-modules, hence we have the canonical isomorphisms

$$\begin{aligned} \#_{\psi}^{op}(C, A) &\simeq \mathrm{Hom}_{A-}(A \otimes_R C, A) &&\simeq \mathrm{Hom}_{A-}(A \otimes_B A, A) \\ &\simeq \mathrm{Hom}_{B-}(A, \mathrm{End}({}_AA)) &&\simeq \mathrm{End}({}_BA). \end{aligned}$$

(4) ⇒ (1). Assume $\theta : B \otimes_R C \longrightarrow A$ to be a left B-linear right C-colinear isomorphism and consider the right C-colinear morphism $\lambda : C \longrightarrow A$, $c \mapsto \theta(1_A \otimes c)$ and the left B-linear morphism $\delta := (id \otimes \varepsilon_C) \circ \theta^{-1} : A \longrightarrow B$. Define $\overline{\lambda} := \Lambda^{-1}(\delta) \in \#_{\psi}^{op}(C, A)$. Then we have for all $c \in C$:

$$\begin{aligned} \sum \lambda(c_1)\overline{\lambda}(c_2) &= \sum \lambda(c)_{<0>}\overline{\lambda}(\lambda(c)_{<1>}) &&= \lambda(c) \leftharpoonup \overline{\lambda} \\ &= \delta(\lambda(c)) &&= ((id \otimes \varepsilon_C) \circ \theta^{-1})(\lambda(c)) \\ &= ((id \otimes \varepsilon_C) \circ \theta^{-1})(\theta(1_A \otimes c)) &&= \varepsilon_C(c)1_A. \end{aligned}$$

On the other hand we have for all $a \in A$:

$$\begin{aligned} \Lambda(\overline{\lambda} \star \lambda)(a) &= a \leftharpoonup (\overline{\lambda} \star \lambda) &&= \sum a_{<0>}(\overline{\lambda} \star \lambda)(a_{<1>}) \\ &= \sum a_{<0>}\overline{\lambda}(a_{<1>1})\lambda(a_{<1>2}) &&= \sum a_{<0><0>}\overline{\lambda}(a_{<0><1>})\lambda(a_{<1>}) \\ &= \sum (a_{<0>} \leftharpoonup \overline{\lambda})\lambda(a_{<1>}) &&= \sum (a_{<0>} \leftharpoonup \Lambda^{-1}(\delta))\lambda(a_{<1>}) \\ &= \sum \delta(a_{<0>})\lambda(a_{<1>}) &&= \sum \delta(a_{<0>})\theta(1_A \otimes a_{<1>}) \\ &= \sum \theta(\delta(a_{<0>}) \otimes a_{<1>}) &&= \theta(\theta^{-1}(a)) = a, \end{aligned}$$

hence $\overline{\lambda} \star \lambda = \eta_A \circ \varepsilon_C$.

Now assume ${}_RC$ to be faithfully flat. Then (1) ⇒ (5) follows by Proposition 3.9 (4) and we are done.■

The following result deals with the special case $\varrho(a) = \sum a_{\psi} \otimes x^{\psi}$, for some group-like element $x \in C$. In this case we obtain the equivalent statements (1)-(5) in Theorem 3.10 without any assumptions on C.

Theorem 3.11. *Assume that* $\varrho(a) = \sum a_\psi \otimes x^\psi$ *for some group-like element* $x \in C$. *The following statements are equivalent:*

1. *A/B is cleft;*
2. *$\mathcal{M}_A^C(\psi)$ satisfies the strong structure theorem and A has the right normal basis property;*
3. *$\mathcal{M}_A^C(\psi)$ satisfies the weak structure theorem and A has the right normal basis property;*
4. *A/B is $\mathcal{C}$-Galois and A has the right normal basis property;*
5. *$\Lambda : \#_\psi^{op}(C, A) \simeq \mathrm{End}({}_BA)^{op}$, $g \mapsto [a \mapsto a \leftharpoonup g]$ is a ring isomorphism and A has the right normal basis property.*

Proof. By Theorem 3.10 it remains to prove that Φ_N is an isomorphism for every $N \in \mathcal{M}_B$, if A/B is cleft. But in our special case there exists by Lemma 3.8 some $\widehat{\lambda} \in Q$ with $\sum 1_{<0>}\widehat{\lambda}(1_{<1>}) = 1_A$ and we are done by Corollary 1.8 (2).■

Remark 3.12. Let (H, A, C) be a right-right resp. a left-right Doi-Koppinen structure. Then (A, C, ψ) is a right-right entwining structure with

$$\psi : C \otimes_R A \longrightarrow A \otimes_R C,\ c \otimes a \mapsto \sum a_{<0>} \otimes ca_{<1>}$$

resp. a left-right entwining structure with

$$\psi : A \otimes_R C \longrightarrow A \otimes_R C,\ a \otimes c \mapsto \sum a_{<0>} \otimes a_{<1>}c.$$

If x is a group-like element of C, then $A \in \mathcal{M}(H)_A^C$ with $\varrho(a) := \sum a_{<0>} \otimes xa_{<1>}$ (resp. $\varrho(a) = \sum a_{<0>} \otimes a_{<1>}x$) and we get [DM92, Theorem 1.5] (resp. [Doi94, Theorem 2.5]) as special cases of Theorem 3.11.

References

[Abu] J.Y. Abuhlail, *Rational modules for corings*, to appear in Commun. Algebra.

[BDR97] M. Beattie, D. Dăscălescu and Ş. Raianu, *Galois extensions for co-Frobenius Hopf algebras,* J. Algebra **198**, 164-183 (1997).

[BM98] T. Brzeziński and S. Majid, *Coalgebra bundles*, Comm. Math. Phys. **191**, 467-492 (1998).

[Brz02] T. Brzeziński, *The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois-type properties*, Algebr. Represent. Theory **5,** 389-410 (2002).

[Brz99] T. Brzeziński, *On modules associated to coalgebra Galois extensions*, J. Algebra **215**, 290-317 (1999).

[CFM90] M. Cohen, D. Fischman and S. Montgomery, *Hopf Galois extensions, smash products and Morita equivalence*, J. Algebra **133**, 351-372 (1990).

[CMZ02] S. Caenepeel, G. Militaru, and S. Zhu, *Frobenius and Separable Functors for Generalized Module Categories and Nonlinear Equations*, Lect. Not. Math. **1787**, Springer-Verlag, Berlin (2002).

[CS69] S. Chase and M. Sweedler, *Hopf algebras and Galois theory*, Lec. Not. Math. **97** Springer-Verlag Berlin (1969).

[CVW] S. Caenepeel, J. Vercruysse and S. Wang, *Morita Theory for corings and cleft entwining structures*, preprint, math.RA/0206198 (2002).

[DM92] Y. Doi and A. Masuoka, *Generalization of cleft comodule algebras*, Comm. Algebra **20,** 3703-3721 (1992).

[Doi94] Y. Doi, *Generalized smash products and Morita contexts for arbitrary Hopf algebras*, Bergen-Montgomery (ed.), Advances in Hopf algebras, Marcle Dekker, Lec. Notes Pure Applied Maths **158**, 39-53 (1994).

[Doi92] Y. Doi, *Unifying Hopf modules*, J. Algebra **153**, 373-385 (1992).

[Doi83] Y. Doi, *On the structure of relative Hopf modules*, Comm. Algebra **11(3)**, 243-255 (1983).

[DT89] Y. Doi and M. Takeuchi, *Hopf-Galois extensions of algebras, the Miyashita-Ulbrich action, and the Azumaya algebras*, J. Algebra, **121**, 488-516 (1989).

[DT86] Y. Doi and M. Takeuchi, *Cleft comodule algebras for a bialgebra*, Comm. Algebra **14**, 801-817 (1986).

[Fai81] C. Faith, *Algebra I*, *Rings, Modules and Categories*, Springer-Verlag (1981).

[Kop95] M. Koppinen, *Variations on the smash product with applications to group-graded rings*, J. Pure Appl. Algebra **104**, 61-80 (1995).

[KT81] H. Kreimer and M. Takeuchi, *Hopf Algebras and Galois extensions of an algebra,* Indiana Univ. Math. J. **30**, 675-692 (1981).

[MSTW01] C. Menini, A. Seidel, B. Torrecillas and R. Wisbauer, *A-H-bimodules and equivalences*, Comm. Algebra **29(10)**, 4619-4640 (2001).

[MTW01] C. Menini, B. Torrecillas and R. Wisbauer, *Strongly rational comodules and semiperfect Hopf algebras over QF Rings*, J. Pure Appl. Algebra **155**, 237-255 (2001).

[MZ97] C. Menini and M. Zuccoli, *Equivalence theorems and Hopf-Galois extensions*, J. Algebra **194**, 245-274 (1997).

[Sch90] H-J. Schneider, *Principal homogenous spaces for arbitrary Hopf algebras*, Israel J. Math. **72(1-2)** 167-195 (1990).

[Swe75] M. Sweedler, *The Predual theorem to the Jacobson-Bourbaki theorem*, Trans. Amer. Math. Soc. **213**, 391-406 (1975).

[Wis02] R. Wisbauer, *On the category of comodules for corings*, Proc. 3rd. Int. Pal. Conf.: Math. & Math. Edu., Bethlehem (Palestine), S. Elaydi et al. (ed.), World Scientific, New Jersy, ISBN 981-02-4720-6, 325-336 (2002).

[Wis88] R. Wisbauer, *Grundlagen der Modul- und Ringtheorie*, München: Verlag Reinhard Fischer (1988); *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991).

[Z-H76] B. Zimmermann-Huignes, *Pure submodules of direct products of free modules*, Math. Ann. **224**, 233-245 (1976).