# On Linear Difference Equations over Rings and Modules *

**Jawad Y. Abuhlail**
Mathematics Department
Birzeit University
P.O.Box 14, Birzeit - Palestine
jabuhlail@birzeit.edu

**Abstract**

In this note we develop a coalgebraic approach to the study of solutions of linear difference equations over modules and rings. Some known results about linearly recursive sequences over base fields are generalized to linearly (bi)recursive (bi)sequences of modules over arbitrary commutative ground rings.

## Introduction

Although the theory of linear difference equations over base fields is well understood, the theory over arbitrary ground rings and modules is still under development. It is becoming more interesting and is gaining increasingly special importance mainly because of recent applications in coding theory and cryptography (e.g. [HN99], [KKMMN99]).

In a series of papers E. Taft et al. (e.g. [PT80], [LT90], [Taf95]) developed a coalgebraic aspect to the study of linearly recursive sequences over fields. Moreover L. Grünenfelder et al. studied in ([GO93], [GK97]) the linearly recursive sequences over *finite dimensional* vector spaces. Linearly recursive (bi)sequences over arbitrary rings and modules were studied intensively by A. Nechaev et al. (e.g. [Nec96], [KKMN95], [Nec93]), however the coalgebraic approach in their work was limited to the field case. Generalization to the case of arbitrary commutative ground rings was studied by several authors including V. Kurakin ([Kur94], [Kur00] & [Kur02]) and eventually Abuhlail, Gomez-Torrecillas and Wisbauer [AG-TW00].

In this note we develop a coalgebraic aspect to the study of solutions of linear difference equations over *arbitrary* rings and modules. For some of our results we assume that the

---

*2000 MSC: Primary 16W30; Secondary 39A99.
Key words and phrases: difference equations, linearly (bi)recursive (bi)sequences, reversible sequences, Hopf algebras, dual coalgebra, rings, modules.

ground ring is artinian. Our results generalize also previous results of us in [AG-TW00] and [Abu01, Kapitel 4]. A standard reference for the theory of linearly recursive sequences over rings and modules is the comprehensive work of A. Mikhalev et al. [KKMN95]. For the theory of Hopf algebras the reader may refer to any of the classical references (e.g. [Swe69], [Abe80] and [Mon93]).

With $R$ we denote a commutative ring with $1_R \neq 0_R$ and with $U(R) = \{r \in R|\ r$ is invertible$\}$ the group of *units* of $R$. The category of $R$-(bi)modules will be denoted by $\mathcal{M}_R$. For an $R$-module $M$, we call an $R$-submodule $K \subset M$ *pure* (in the sense of Cohn), if for every $R$-module $N$ the induced map $\iota_k \otimes \mathrm{id}_N : K \otimes_R N \to M \otimes_R N$ is injective.

For an $R$-algebra $A$ and an $A$-module $M$, we call an $A$-submodule $K \subset M$ $R$-cofinite, if $M/K$ is f.g. in $\mathcal{M}_R$. For an $R$-algebra $A$ we denote by $\mathcal{K}_A$ the class of $R$-cofinite ideals. If $A$ is an $R$-algebra with $\mathcal{K}_A$ a filter, then we define for every left $A$-module $M$ the *finite dual* right $A$-module

$$M^{\circ} := \{f \in M^* \mid \ \mathrm{Ke}(f) \supset IM \text{ for some } A\text{-ideal } I \text{ with } A/I \text{ f.g.}\}. \tag{1}$$

With $\mathbb{N}$ resp. $\mathbb{Z}$ we denote the set of natural numbers resp. the ring of integers. Moreover we set $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$. For an $n \times n$ matrix $M$ over $R$ we denote the characteristic polynomial with $\chi(M)$. The identity matrix of order $n$ over $R$ is denoted by $E_n$. For an $m \times n$ matrix $A$ and a $k \times l$ matrix $B$, the *Kronecker product* (*tensor product*) of $A$ and $B$ is the $mk \times nl$ matrix

$$A \otimes B := \begin{bmatrix} a_{11} \cdot B & a_{12} \cdot B & ... & ... & a_{1n} \cdot B \\ a_{21} \cdot B & a_{22} \cdot B & ... & ... & a_{2n} \cdot B \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ a_{m1} \cdot B & a_{m2} \cdot B & ... & ... & a_{mn} \cdot B \end{bmatrix}$$

# 1 Preliminaries

Let $M$ be an $R$-module and

$$M[\mathbf{x}] := M[x_1, ..., x_k],\ M[\mathbf{x}, \mathbf{x}^{-1}] := M[x_1, x_1^{-1}, ..., x_k, x_k^{-1}]. \tag{2}$$

We consider the *polynomial ring* $R[\mathbf{x}]$ and the *ring of Laurent polynomials* $R[\mathbf{x}, \mathbf{x}^{-1}]$ as commutative $R$-algebras with the usual multiplication and the usual unity. For every $R$-module $M$, $M[\mathbf{x}]$ (resp. $M[\mathbf{x}, \mathbf{x}^{-1}]$) is an $R[\mathbf{x}]$-module (resp. an $R[\mathbf{x}, \mathbf{x}^{-1}]$-module) with action induced from the $R$-module structure on $M$ and we have moreover canonical $R$-module isomorphisms

$$M[\mathbf{x}] \simeq M \otimes_R R[\mathbf{x}] \simeq M^{(\mathbb{N}_0^k)} \text{ and } M[\mathbf{x}, \mathbf{x}^{-1}] \simeq M \otimes_R R[\mathbf{x}, \mathbf{x}^{-1}] \simeq M^{(\mathbb{Z}^k)}.$$

For $\mathbf{n} = (n_1, ..., n_k) \in \mathbb{N}_0^k$ resp. $\mathbf{z} = (z_1, ..., z_k) \in \mathbb{Z}^k$ we set $\mathbf{x}^{\mathbf{n}} := x_1^{n_1} \cdot ... \cdot x_k^{n_k}$ resp. $\mathbf{x}^{\mathbf{z}} := x_1^{z_1} \cdot ... \cdot x_k^{z_k}$.

**1.1.** Let $M$ be an $R$-module, $\mathbf{l} = (l_1, ..., l_k) \in \mathbb{N}_0^k$ and consider the *system of linear difference equations* (ab. SLDE)

$$\begin{array}{lcllcl} x_{\mathbf{n}+(l_1,0,...,0)} & + & \sum\limits_{i=1}^{l_1} p_{(1,l_1-i)}(\mathbf{n}) x_{\mathbf{n}+(l_1-i,0,...,0)} & = & g_1(\mathbf{n}), \\ x_{\mathbf{n}+(0,l_2,0,...,0)} & + & \sum\limits_{i=1}^{l_2} p_{(2,l_2-i)}(\mathbf{n}) x_{\mathbf{n}+(0,l_2-i,0,...,0)} & = & g_2(\mathbf{n}), \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ x_{\mathbf{n}+(0,...,0,l_k)} & + & \sum\limits_{i=1}^{l_k} p_{(k,l_k-i)}(\mathbf{n}) x_{\mathbf{n}+(0,...,0,l_k-i)} & = & g_k(\mathbf{n}), \end{array} \tag{3}$$

where the $p_{jl}$'s are $R$-valued functions and the $g_j$'s are $M$-valued functions defined for all $\mathbf{n} \in \mathbb{N}_0^k$. If the $g_j$'s are identically zero, then (3) is said to be a *homogenous* SLDE. If the $p_{jl}$'s are constants, then (3) is said to be a SLDE *with constant coefficients.*

**1.2.** For an $R$-module $M$ and $k \geq 1$ let

$$\mathcal{S}_M^{<k>} := \{u : \mathbb{N}_0^k \to M\} \simeq M^{\mathbb{N}_0^k}$$

be the $R$-module of *$k$-sequences* over $M$. If $M$ (resp. $k$) is not mentioned, then we mean $M = R$ (resp. $k = 1$). For $f(\mathbf{x}) = \sum\limits_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in R[\mathbf{x}]$ and $w \in \mathcal{S}_M^{<k>}$ define

$$f(\mathbf{x}) \rightharpoonup w = u \in \mathcal{S}_M^{<k>}, \text{ where } u(\mathbf{n}) := \sum_{\mathbf{i}} a_{\mathbf{i}} w(\mathbf{n} + \mathbf{i}) \text{ for all } \mathbf{n} \in \mathbb{N}_0^k. \tag{4}$$

With this action $\mathcal{S}_M^{<k>}$ is an $R[\mathbf{x}]$-module. For subsets $I \subset R[\mathbf{x}]$ and $L \subset \mathcal{S}_M^{<k>}$ consider the annihilator submodules

$$\begin{array}{lcl} \mathrm{An}_{\mathcal{S}_M^{<k>}}(I) & = & \{w \in \mathcal{S}_M^{<k>} \mid f \rightharpoonup w = 0 \text{ for every } f \in I\}, \\ \mathrm{An}_{R[\mathbf{x}]}(L) & = & \{h \in R[\mathbf{x}] | \, h \rightharpoonup u = 0 \text{ for every } u \in L\}. \end{array}$$

Note that $\mathrm{An}_{\mathcal{S}_M}^{<k>}(I) \subset \mathcal{S}_M^{<k>}$ is an $R[\mathbf{x}]$-submodule and $\mathrm{An}_{R[\mathbf{x}]}(L) \lhd R[\mathbf{x}]$ is an ideal.

**1.3.** A polynomial $f(x) \in R[x]$ is called *monic*, if its leading coefficient is $1_R$. For every monic polynomial $f(x) = x^l + a_{l-1}x^{l-1} + ... + a_1 x + a_0 \in R[x]$, the *companion matrix* of $f$ is defined to be the $l \times l$ matrix

$$S_f := \begin{bmatrix} 0_R & 0_R & ... & 0_R & -a_0 \\ 1_R & 0_R & ... & 0_R & -a_1 \\ 0_R & 1_R & ... & 0_R & -a_2 \\ ... & ... & ... & ... & ... \\ 0_R & 0_R & ... & 1_R & -a_{l-1} \end{bmatrix} \tag{5}$$

$S_f$ is a matrix that has $f(x)$ as its characteristic polynomial as well as its minimum polynomial ([Jon73, Theorem 4.18]).

**Definition 1.4.** An ideal $I \lhd R[\mathbf{x}]$ will be called *monic*, if it contains a non-empty subset of monic polynomials

$$\{f_j(x_j) = x_j^{l_j} + a_{l_j-1}^{(j)} x_j^{l_j-1} + ... + a_1^{(j)} x_j + a_0^{(j)} | \; j = 1, ..., k\}. \tag{6}$$

In this case the polynomials (6) are called *elementary polynomials* and $(f_1(x_1), ..., f_k(x_k)) \lhd R[\mathbf{x}]$ an *elementary ideal.* A monic polynomial $q(x) \in R[x]$ is called *reversible*, if $q(0) \in U(R)$. An ideal $I \lhd R[\mathbf{x}, \mathbf{x}^{-1}]$ will be called *reversible*, if it contains a subset of reversible polynomials $\{q_1(x_1), ..., q_k(x_k)\}$.

**1.5.** Let $M$ be an $R$-module. We call $u \in \mathcal{S}_M^{<k>}$ a *linearly recursive $k$-sequence* (resp. a *linearly birecursive $k$-sequence*), if $\mathrm{An}_{R[\mathbf{x}]}(u)$ is a monic ideal (resp. a reversible ideal). Note that a $k$-sequence $u \in \mathcal{S}_M^{<k>}$ is linearly recursive, iff it's a solution of a homogenous SLDE with constants coefficients of the form (3). If $\mathrm{An}_{R[\mathbf{x}]}(u)$ contains a set of monic polynomials $\{f_1(x_1), ..., f_k(x_k)\}$, where $f_j(x_j)$ is of order $m_j$, $j = 1, ..., k$, then these are called *elementary characteristic polynomials* of $u$ and $u$ is said to have *order* $\mathbf{m} := (m_1, ..., m_k)$. Characteristic polynomials of $u$ of least degree $n_j$, $j = 1, ..., k$ are called *minimal polynomials* of $u$ and $\mathbf{n} := (n_1, ..., n_k)$ is called the *rank* of $u$. The subsets $\mathcal{L}_M^{<k>} \subseteq \mathcal{S}_M^{<k>}$ of linearly recursive $k$-sequences and $\mathcal{B}_M^{<k>} \subseteq \mathcal{S}_M^{<k>}$ of linearly birecursive $k$-sequences are obviously $R[\mathbf{x}]$-submodules.

**1.6.** ([MN96, Page 170]) The *lexicographical linear order* ($\preceq$) on $\mathbb{N}_0^k$ is defined as follows: for $\mathbf{i} = (i_1, ..., i_k)$ and $\mathbf{n} = (n_1, ..., n_k) \in \mathbb{N}_0^k$ we say $\mathbf{i} \preceq \mathbf{n}$, if the first number in the sequence of integers

$$(n_1 + ... + n_k) - (i_1 + ... + i_k), \; n_1 - i_1, ..., n_k - i_k$$

that is different from zero is positive.

Let $M$ be an $R$-module, $\mathbf{F} := \{f_1(x_1), ..., f_k(x_k)\} \subset R[\mathbf{x}]$ a subset of monic polynomials with $\deg(f_j(x_j)) = l_j$ for $j = 1, ..., k$, $\mathbf{l} := (l_1, ..., l_k)$, $\mathbf{1} : = (1, ..., 1)$, and $I_{\mathbf{F}} := (f_1, ..., f_k) \lhd R[\mathbf{x}]$. Note that the natural order "$\leq$ " on $\mathbb{N}_0$ induces on $\mathbb{N}_0^k$ a *partial order* and we define the *polyhedron* $\Pi_{\mathbf{F}} = \Pi(\mathbf{l}) := \{\mathbf{i} \in \mathbb{N}_0^k | \; \mathbf{i} \leq \mathbf{l} - \mathbf{1}\}$. The *initial polyhedron of values* of $\omega \in \mathcal{S}_M^{<k>}$ is defined as $\omega(\Pi_{\mathbf{F}}) := \{\omega(\mathbf{i}) | \; \mathbf{i} \in \Pi_{\mathbf{F}}\}$. For $l = l_1 \cdot ... \cdot l_k$ the points of the polyhedron $\Pi_{\mathbf{F}}$ build a chain $\mathbf{0} = \mathbf{i}_0 \preceq \mathbf{i}_1 \preceq ... \preceq \mathbf{i}_{l-1}$ and we can write $\omega(\Pi_{\mathbf{F}})$ as an *initial vector of values* $(\omega(\mathbf{0}), \omega(\mathbf{i}_1), ..., \omega(\mathbf{i}_{l-1})) \in M^l$.

Let $\omega \in \mathrm{An}_{\mathcal{S}_M^{<k>}}(f_1(x_1), ..., f_k(x_k))$, where $f_j(x_j)$ is monic for $j = 1, ..., n$ and write for every $\mathbf{n} = (n_1, ..., n_k) \in \mathbb{N}_0^k$ :

$$x_j^{n_j} = h_j(x_j) f_j(x_j) + r_j(x_j), \text{ where } \deg(r_j(x_j)) < l_j.$$

If we set

$$g^{(\mathbf{n})}(\mathbf{x}) := \prod_{j=1}^{k} r_j(x_j) = \sum_{\mathbf{i} \in \Pi_{\mathbf{F}}} a_{\mathbf{i}}^{(\mathbf{n})} \mathbf{x}^{\mathbf{i}} \text{ and } v := \mathbf{x}^{\mathbf{n}} \rightharpoonup \omega = g^{(\mathbf{n})}(\mathbf{x}) \rightharpoonup \omega,$$

then

$$\omega(\mathbf{n}) = v(\mathbf{0}) = \sum_{\mathbf{i} \in \Pi_{\mathbf{F}}} a_{\mathbf{i}}^{(\mathbf{n})} \omega(\mathbf{i}) \text{ for every } \mathbf{n} \in \mathbb{N}_0^k.$$

Consequently $\omega$ is completely determined by the initial polyhedron of values $\omega(\Pi_{\mathbf{F}})$. For $\mathbf{t} \in \Pi_{\mathbf{F}}$ define the sequence $e_{\mathbf{t}}^{\mathbf{F}} \in \mathrm{An}_{\mathcal{S}_R^{<k>}}(I_{\mathbf{F}})$ with initial polyhedron of values $e_{\mathbf{t}}^{\mathbf{F}}(\mathbf{i}) = \delta_{\mathbf{i},\mathbf{t}}$ for all $\mathbf{i} \in \Pi_{\mathbf{F}}$. The sequence $e_{\mathbf{1}-\mathbf{1}}^{\mathbf{F}}$ is called the *impulse sequence* of $\mathrm{An}_{\mathcal{S}_R^{<k>}}(I_{\mathbf{F}})$.

# Examples

We give now some examples of linearly recursive sequences. For more examples the reader may refer to [KKMN95].

**Example 1.7.** (*Geometric progression*). Let $M$ be an $R$-module, $m \in M$, $r \in R$ and consider $w \in \mathcal{S}_M$ given by

$$w(n) := r^n m \text{ for every } n \in \mathbb{N}_0.$$

Then $w \in \mathcal{L}_M$ with initial condition $w(0) = m$ and elementary characteristic polynomial $f(x) = x - r$. Moreover $\mathrm{An}_{R[x]}(w) = R[x](x - r) + R[x]\mathrm{An}_R(r)$.

**Example 1.8.** (*Arithmetic progression*). Let $M$ be an $R$-module, $\{p, q\} \subset M$ and consider $w \in \mathcal{S}_M$ given by

$$w(n) := p + nq \text{ for every } n \in \mathbb{N}_0.$$

Then $w \in \mathcal{L}_M$ with initial vector $(p, p + q)$ and elementary characteristic polynomial $f(x) = (x - 1)^2$. If $\mathrm{An}_R(q) = 0$, then $f(x)$ is a unique minimal polynomial of $w$. If $r \in \mathrm{An}_R(q)$, then $f_r(x) = (x - 1)^2 + r(x - 1)$ is another minimal polynomial of $w$.

**Remark 1.9.** An example of a *non* linearly recursive sequences over $\mathbb{Z}$ is the sequence of prime positive numbers $\{2, 3, 5, 7, ...\}$.

**Example 1.10.** Let $E = \{f_1(x), ..., f_k(x)\} \subset R[x]$ be a subset of monic polynomials.

1. Let $M$ be an $R$-module, $u_i \in \mathrm{An}_{\mathcal{S}_M}(f_i)$ for $i = 1, ..., k$ and consider $u := u_1 \dot{+} ... \dot{+} u_k \in \mathcal{S}_M^{<k>}$ defined by $u(\mathbf{n}) = u_1(n_1) + ... + u_k(n_k)$. Then $u \in \mathrm{An}_{\mathcal{S}_M^{<k>}}(g_1(x_1), ..., g_k(x_k))$, where for $i = 1, ..., k$ :
$$g_i(x_i) = \begin{cases} f_i(x_i), & f_i(1_R) = 0_R \\ f_i(x_i)(x_i - 1_R), & \text{otherwise.} \end{cases} \tag{7}$$

2. Let $M_1, ..., M_k$ be $R$-modules, $u_i \in \mathrm{An}_{\mathcal{S}_{M_i}}(f_i)$ for $i = 1, ..., k$, $M := M_1 \oplus ... \oplus M_k$ and consider $u \in \mathcal{S}_M^{<k>}$ defined by $u(\mathbf{n}) := (u_1(n_1), ..., u_k(n_k))$. Then $u \in \mathrm{An}_{\mathcal{S}_M^{<k>}}(g_1(x_1), ..., g_k(x_k))$, where the $g_i$'s are defined as in (7).

3. Let $u_i \in \mathrm{An}_{\mathcal{S}_R}(f_i)$ for $i = 1, ..., k$ and consider $u \in \mathcal{S}_R^{<k>}$ defined by $u(\mathbf{n}) := u_1(n_1) \cdot ... \cdot u_k(n_k)$. Then $u \in \mathrm{An}_{\mathcal{S}_R^{<k>}}(f_1(x_1), ..., f_k(x_k))$ and
$$\mathrm{An}_{\mathcal{S}_R^{<k>}}(f_1(x_1), ..., f_k(x_k)) \simeq \mathrm{An}_{\mathcal{S}_R}(f_1) \otimes_R ... \otimes_R \mathrm{An}_{\mathcal{S}_R}(f_k).$$

4. Let $M_1, ..., M_k$ be $R$-modules, $u_i \in \mathrm{An}_{\mathcal{S}_{M_i}}(f_i)$ for $i = 1, ..., k$, $M := M_1 \otimes_R ... \otimes_R M_k$ and consider $u \in \mathcal{S}_M^{<k>}$ defined by $u(\mathbf{n}) := u_1(n_1) \otimes ... \otimes u_k(n_k)$. Then $u \in \mathrm{An}_{\mathcal{S}_M^{<k>}}(f_1(x_1), ..., f_k(x_k))$ and
$$\mathrm{An}_{\mathcal{S}_M^{<k>}}(f_1(x_1), ..., f_k(x_k)) \simeq \mathrm{An}_{\mathcal{S}_{M_1}}(f_1) \otimes_R ... \otimes_R \mathrm{An}_{\mathcal{S}_{M_k}}(f_k).$$

# Admissible $R$-bialgebras and Hopf $R$-algebras

For every $R$-coalgebra $(C, \Delta_C, \varepsilon_C)$ there is a *dual $R$-algebra* $C^* := \mathrm{Hom}_R(C, R)$ with multiplication the so called *convolution product*

$$(f \star g)(c) := \sum f(c_1)g(c_2) \text{ for all } f, g \in C^*, \; c \in C$$

and unity $\varepsilon_C$. Although every algebra $A$ has a *dual coalgebra,* if the ground ring is hereditary noetherian (e.g. a field), the existence of dual coalgebras of algebras over an arbitrary commutative ground rings is not guaranteed!! One way to handle this problem is to restrict the class of $R$-algebras, for which the dual $R$-coalgebras are defined.

**Definition 1.11.** Let $A$ be an $R$-algebra (resp. an $R$-bialgebra, a Hopf $R$-algebra). Then we call $A$ :

1. an *$\alpha$-algebra* (resp. an *$\alpha$-bialgebra,* a *Hopf $\alpha$-algebra*), if $\mathcal{K}_A$ is a filter and $A^\circ \subset R^A$ is pure.
2. *cofinitary,* if $\mathcal{K}_A$ is a *filter* and for every $I \in \mathcal{K}_A$ there exists an $A$-ideal $\overline{I} \subseteq I$ with $A/\overline{I}$ f.g. and projective.

**1.12.** Let $H$ be an $R$-bialgebra and consider the class of $R$-cofinite $H$-ideals $\mathcal{K}_H$. We call $H$ an *admissible $R$-bialgebra*, if $H$ is cofinitary and $\mathcal{K}_H$ satisfies the following axioms:

$$\text{(A1)} \quad \forall\, I, J \in \mathcal{K}_H \text{ there exists } L \in \mathcal{K}_H, \text{ s.t. } \Delta_H(L) \subseteq I \otimes_R H + H \otimes_R J \tag{8}$$

and

$$\text{(A2)} \quad \exists\, I \in \mathcal{K}_H, \text{ s.t. } \mathrm{Ke}(\varepsilon_H) \supset I. \tag{9}$$

We call a Hopf $R$-algebra $H$ an *admissible Hopf $R$-algebra*, if $H$ is cofinitary, $\mathcal{K}_H$ satisfies (A1), (A2) and

$$\text{(A3)} \quad \text{for every } I \in \mathcal{K}_H \text{ there exists } J \in \mathcal{K}_H, \text{ s.t. } S_H(J) \subseteq I. \tag{10}$$

**Remark 1.13.** It follows from the proof of [AG-TL01, Proposition 4.2.], that every cofinitary $R$-algebra (resp. $R$-bialgebra, Hopf $R$-algebra) is an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra). By ([Abu01, Lemma 2.5.6.]) every cofinitary bialgebra (Hopf algebra) over a *noetherian* ground ring is admissible.

**Proposition 1.14.** ([Abu01, Proposition 2.4.13, Proposition 2..5.7])

1. *If $A$ is a cofinitary $R$-algebra, then $A^\circ$ is an $R$-coalgebra. If $H$ is an admissible $R$-bialgebra (resp. an admissible Hopf $R$-algebra), then $H^\circ$ is an $R$-bialgebra (resp. a Hopf $R$-algebra).*
2. *Let $R$ be noetherian. If $A$ is an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra), then $A^\circ$ is an $R$-coalgebra (resp. an $R$-bialgebra, a Hopf $R$-algebra).*

**Proposition 1.15.** *Let $A$ be an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra), $B$ a cofinitary $R$-algebra (resp. $R$-bialgebra, Hopf $R$-algebra) and consider the canonical map $\sigma : A^\circ \otimes_R B^\circ \to (A \otimes_R B)^\circ$. Then:*

1. *$\sigma$ is injective.*
2. *If $R$ is noetherian, then $\sigma$ is an isomorphism of $R$-coalgebras (resp. $R$-bialgebras, Hopf $R$-algebras).*

**Proof.**

1. The proof is along the lines of the proof of [Kur02, Proposition 5].
2. The proof is along the lines of the proof of [AG-TL01, Theorem 4.10].

The proof of [AG-TL01, Lemma 4.12] can be generalized to get

**Lemma 1.16.** *For any set of reversible polynomials $\{q_1(x_1), ..., q_k(x_k)\} \subseteq R[\mathbf{x}]$ we have an isomorphism of $R$-algebras*

$$R[\mathbf{x}]/(q_1(x_1), ..., q_k(x_k)) \simeq R[\mathbf{x}, \mathbf{x}^{-1}]/(q_1(x_1), ..., q_k(x_k)).$$

**Lemma 1.17.** ([Kur02, Proposition 1]) *Let $R$ be an* arbitrary *commutative ring.*

1. *An ideal $I \lhd R[\mathbf{x}]$ is $R$-cofinite, iff it's monic. Consequently every $R$-cofinite $R[\mathbf{x}]$-ideal contains an ideal $\overline{I} \lhd R[\mathbf{x}]$, such that $R[\mathbf{x}]/\overline{I}$ is free of finite rank. In particular $R[\mathbf{x}]$ is cofinitary.*
2. *An ideal $I \lhd R[\mathbf{x}, \mathbf{x}^{-1}]$ is $R$-cofinite, iff it's reversible. Consequently every $R$-cofinite $R[\mathbf{x}, \mathbf{x}^{-1}]$-ideal contains an ideal $\overline{I} \lhd R[\mathbf{x}, \mathbf{x}^{-1}]$, such that $R[\mathbf{x}, \mathbf{x}^{-1}]/\overline{I}$ is free of finite rank. In particular $R[\mathbf{x}, \mathbf{x}^{-1}]$ is cofinitary.*

# 2 Linearly (bi)recursive sequences

In this section we study the *linearly (bi)recursive $k$-sequences* over $R$-modules, where $R$ is an arbitrary commutative ground ring.

**2.1.** Let $(G, \mu_G, e_G)$ be a (commutative) monoid. Considering the elements of the basis $G$ as *group-like elements,* the monoid algebra $RG$ becomes a (commutative) cocommutative $R$-bialgebra $(RG, \mu, \eta, \Delta_g, \varepsilon_g)$, where

$$\Delta_g(x) = x \otimes x \text{ and } \varepsilon_g(x) = 1_R \text{ for every } x \in G.$$

If $G$ is a group, then $RG$ is a Hopf $R$-algebra with antipode

$$S_g : RG \to RG,\ x \mapsto x^{-1} \text{ for every } x \in G.$$

**2.2. Bialgebra structures on $R[\mathbf{x}]$.** Consider the commutative monoid $G$ generated by $\{x_j \mid j = 1, ..., k\}$. Then $R[\mathbf{x}] = RG$ has the structure of a *commutative cocommutative* $R$-bialgebra $R[\mathbf{x}; g] = (R[\mathbf{x}], \mu, \eta, \Delta_g, \varepsilon_g)$, where $\mu$ is the usual multiplication, $\eta$ is the usual unity and

$$\begin{array}{lllllll} \Delta_g : & R[\mathbf{x}] & \to & R[\mathbf{x}] \otimes_R R[\mathbf{x}], & x_j^n & \mapsto & x_j^n \otimes x_j^n, \quad \forall\, n \geq 0,\, j = 1, ..., k, \\ \varepsilon_g : & R[\mathbf{x}] & \to & R, & x_j^n & \mapsto & 1_R, \quad \forall\, n \geq 0,\, j = 1, ..., k. \end{array}$$

On the other hand $R[\mathbf{x}; p] = (R[\mathbf{x}], \mu, \eta, \Delta_g, \varepsilon_g)$ is a *commutative cocommutative* Hopf $R$-algebra, where $\mu$ is the usual multiplication, $\eta$ is the usual unity and

$$\begin{array}{lllllll} \Delta_p & : R[\mathbf{x}] & \to & R[\mathbf{x}] \otimes_R R[\mathbf{x}], & x_j^n & \mapsto & \sum\limits_{t=0}^{n} \binom{n}{t}\, x_j^t \otimes x_j^{n-t}, \quad \forall\, n \geq 0,\, j = 1, ..., k, \\ \varepsilon_p & : R[\mathbf{x}] & \to & R, & x_j^n & \mapsto & \delta_{n,0}, \quad \forall\, n \geq 0,\, j = 1, ..., k, \\ S_p & : R[\mathbf{x}] & \to & R[\mathbf{x}], & x_j^n & \mapsto & (-1)^n x_j^n, \quad \forall\, n \geq 0,\, j = 1, ..., k. \end{array}$$

**Remarks 2.3.**

1. Let $R$ be an integral domain, then it follows by [Grü69, Theorem 1.3.6.] that for every set $G$, the class of group-like elements of the $R$-coalgebra $RG$ is $G$ itself. Then one can show as in the field case [CG93], that $R[\mathbf{x}; g]$ and $R[\mathbf{x}; p]$ are the only possible $R$-bialgebra structures on $R[\mathbf{x}]$ with the usual multiplication and the usual unity.

2. The $R$-bialgebra $R[\mathbf{x}; g]$ has no antipode, because the group-like elements in a Hopf $R$-algebra should be invertible.

The proof of the following result depends mainly on arguments of [Kur02, Theorem 2]:

**Proposition 2.4.** *Let $R$ be an* arbitrary *commutative ring. Then $R[\mathbf{x}; g]$ is an admissible $R$-bialgebra and $R[\mathbf{x}; p]$ is an admissible Hopf $R$-algebra. Hence $R[\mathbf{x}; g]^\circ$ is an $R$-bialgebra and $R[\mathbf{x}; p]^\circ$ is a Hopf $R$-algebra.*

**Proof.** Denote with $(R[\mathbf{x}], \Delta, \varepsilon)$ either of the cofinitary $R$-bialgebras $R[\mathbf{x}; g]$ and $R[\mathbf{x}; p]$. Let $I, J \lhd R[\mathbf{x}]$ be $R$-cofinite ideals and assume w.l.o.g. that $R[\mathbf{x}]/I$ and $R[\mathbf{x}]/J$ are free of finite rank (see Lemma 1.17). Let $\beta$ be a basis of the free $R$-module $B := R[\mathbf{x}]/I \otimes_R R[\mathbf{x}]/J$ and consider the $R$-algebra morphism $\overline{\Delta} := (\pi_I \otimes \pi_J) \circ \Delta : R[\mathbf{x}] \to R[\mathbf{x}]/I \otimes_R R[\mathbf{x}]/J$. For $j = 1, ..., k$ let $M_j$ be the matrix of the $R$-linear map

$$T_j : B \to B,\ b \mapsto \overline{\Delta}(x_j)b$$

w.r.t. $\beta$ and $\chi_j(\lambda)$ its characteristic polynomial. Then $\chi_j(\overline{\Delta}(x_j)) = 0$ for $j = 1, ..., k$. Since $\overline{\Delta}$ is an $R$-algebra morphism, it follows that $\chi_j(x_j) \in \mathrm{Ke}(\overline{\Delta}) = \Delta^{-1}(I \otimes_R R[\mathbf{x}] + R[\mathbf{x}] \otimes_R J)$ for $j = 1, ..., k$. If we set $L := (\chi_1(x_1), ..., \chi_k(x_k)) \lhd R[\mathbf{x}]$, then $\Delta(L) \subseteq I \otimes_R R[\mathbf{x}] + R[\mathbf{x}] \otimes_R J$, i.e. $\mathcal{K}_{R[\mathbf{x}]}$ satisfies axiom (8). Note that $R[\mathbf{x}]/\mathrm{Ke}(\varepsilon) \simeq R$, hence $\mathcal{K}_{R[\mathbf{x}]}$ satisfies axiom (9). Consequently $R[\mathbf{x}; g]$ and $R[\mathbf{x}; p]$ are admissible $R$-bialgebras. Consider now the Hopf $R$-algebra $R[\mathbf{x}; p]$ with the *bijective* antipode $S_p$. For every ideal $I \lhd R[\mathbf{x}]$, $S_p^{-1}(I) \lhd R[\mathbf{x}; p]$ is an ideal and we have an isomorphism of $R$-modules $R[\mathbf{x}]/S_p^{-1}(I) \simeq R[\mathbf{x}]/I$, hence $\mathcal{K}_{R[\mathbf{x};p]}$ satisfies axiom (10). Consequently $R[\mathbf{x}; p]$ is an admissible Hopf $R$-algebra. The last statement follows now by Proposition 1.14.■

If $M$ is an arbitrary $R$-module, then we have obviously an isomorphism of $R[\mathbf{x}]$-modules

$$\Phi_M : M[\mathbf{x}]^* \to \mathcal{S}_{M^*}^{<k>},\ \varkappa \mapsto [\mathbf{n} \mapsto [m \mapsto \varkappa(m\mathbf{x}^{\mathbf{n}})]] \tag{11}$$

with inverse $u \mapsto [m\mathbf{x}^{\mathbf{n}} \mapsto u(\mathbf{n})(m)]$.

**Proposition 2.5.** *Let $M$ be an $R$-module. Then* (11) *induces an isomorphism of $R[\mathbf{x}]$-modules*

$$M[\mathbf{x}]^{\circ} \simeq \mathcal{L}_{M^*}^{<k>}. \tag{12}$$

**Proof.** Consider the $R[\mathbf{x}]$-module isomorphism $M[\mathbf{x}]^* \overset{\Phi_M}{\simeq} \mathcal{S}_{M^*}^{<k>}$ (11). Let $\varkappa \in M[\mathbf{x}]^{\circ}$. Then there exists an $R$-cofinite $R[\mathbf{x}]$-ideal $I$, such that $I \rightharpoonup \varkappa = 0$. So $I \rightharpoonup \Phi(\varkappa) = \Phi(I \rightharpoonup \varkappa) = 0$, i.e. $I \subset \mathrm{An}_{R[\mathbf{x}]}(\Phi(\varkappa))$. By Lemma 1.17 (1) $I$ is monic, i.e. $\Phi(\varkappa) \in \mathcal{L}_{M^*}^{<k>}$.

On the other hand, let $u \in \mathcal{L}_{M^*}^{<k>}$. By definition $J := \mathrm{An}_{R[\mathbf{x}]}(u)$ is a monic ideal and it follows by Lemma 1.17 (1) that $J \lhd R[\mathbf{x}]$ is $R$-cofinite. For $\varkappa := \Phi^{-1}(u)$ we have $J \rightharpoonup \varkappa = J \rightharpoonup \Phi^{-1}(u) = \Phi^{-1}(J \rightharpoonup u) = 0$, i.e. $\varkappa \in M[\mathbf{x}]^{\circ}$.■

### 2.6. The coalgebra structure on $\mathcal{L}^{<k>}$.

By Lemma 1.17 (1) $(R[\mathbf{x}], \mu, \eta)$ is a cofinitary $R$-algebra, where $\mu$ is the usual multiplication and $\eta$ is the usual unity. Hence $(R[\mathbf{x}]^{\circ}, \mu^{\circ}, \eta^{\circ})$ is (by Proposition 1.14) an $R$-coalgebra, where

$$\begin{array}{llllll} \mu^{\circ}: & R[\mathbf{x}]^{\circ} \to & R[\mathbf{x}]^{\circ} \otimes_R R[\mathbf{x}]^{\circ}, & f & \mapsto & [x_i^s \otimes x_j^t \mapsto f(x_i^s x_j^t),\ s, t \geq 0, i, j = 1, ..., k], \\ \eta^{\circ}: & R[\mathbf{x}]^{\circ} \to & R, & f & \mapsto & f(1_R). \end{array}$$

So $\mathcal{L}^{<k>} \simeq R[\mathbf{x}]^{\circ}$ has the structure of an $R$-coalgebra with counity

$$\varepsilon_{\mathcal{L}^{<k>}} : \mathcal{L}^{<k>} \to R,\ u \mapsto u(\mathbf{0}). \tag{13}$$

and comultiplication described as follows (see [KKMN95, Proposition 14.16]):

Let $u \in \mathcal{L}^{<k>}$, $\{f_1(x_1), ..., f_k(x_k)\} \subseteq \mathrm{An}_{R[\mathbf{x}]}(u)$ a subset of elementary characteristic polynomials with $\deg(f_j(x_j)) = l_j$ and $\mathbf{l} := (l_1, ..., l_k)$. So we have for all $\mathbf{n}, \mathbf{i} \in \mathbb{N}_0^k$ :

$$u(\mathbf{n} + \mathbf{i}) = (\mathbf{x}^{\mathbf{i}} \rightharpoonup u)(\mathbf{n}) = (\sum_{\mathbf{t} \leq \mathbf{l}-\mathbf{1}} (\mathbf{x}^{\mathbf{i}} \rightharpoonup u)(\mathbf{t}) \cdot e_{\mathbf{t}}^{\mathbf{F}})(\mathbf{n}) = \sum_{\mathbf{t} \leq \mathbf{l}-\mathbf{1}} (\mathbf{x}^{\mathbf{t}} \rightharpoonup u)(\mathbf{i}) \cdot e_{\mathbf{t}}^{\mathbf{F}}(\mathbf{n}).$$

The comultiplication of $\mathcal{L}^{<k>}$ is given then by

$$\Delta_{\mathcal{L}^{<k>}} : \mathcal{L}^{<k>} \to \mathcal{L}^{<k>} \otimes_R \mathcal{L}^{<k>},\ u \mapsto \sum_{\mathbf{t} \leq \mathbf{l}-\mathbf{1}} (\mathbf{x}^{\mathbf{t}} \rightharpoonup u) \otimes e_{\mathbf{t}}^{\mathbf{F}}. \tag{14}$$

**Example 2.7.** Consider the *Fibonacci sequence* $\mathcal{F} = (0, 1, 1, 2, 3, 5, ...)$. Clearly $\mathcal{F}$ is given by

$$\mathcal{F}(0) = 0,\ \mathcal{F}(1) = 1,\ \mathcal{F}(n+2) = \mathcal{F}(n+1) + \mathcal{F}(n) \text{ for all } n \geq 0,$$

i.e. $\mathcal{F} \in \mathcal{L}_{\mathbb{Z}}$ with initial vector $(0, 1)$ and elementary characteristic polynomial $f(x) = x^2 - x - 1 \in \mathbb{Z}[x]$. By (14) one can easily calculate

$$\Delta_{\mathcal{L}_{\mathbb{Z}}}(\mathcal{F}) = \mathcal{F} \otimes_{\mathbb{Z}} (x \rightharpoonup \mathcal{F}) + (x \rightharpoonup \mathcal{F}) \otimes_{\mathbb{Z}} \mathcal{F} - \mathcal{F} \otimes_{\mathbb{Z}} \mathcal{F}.$$

**2.8. The $R$-bialgebra $(\mathcal{L}_R^{<k>}; g)$.** Consider the $R$-bialgebra $R[\mathbf{x}; g]$. Then $\mathcal{S}^{<k>} \simeq R^{\mathbb{N}_0^k} \simeq R[\mathbf{x}; g]^*$ is an $R$-algebra with multiplication given by the *Hadamard product*

$$*_g : \mathcal{S}^{<k>} \otimes_R \mathcal{S}^{<k>} \longrightarrow \mathcal{S}^{<k>},\ u \otimes v \mapsto [\mathbf{n} \mapsto u(\mathbf{n})v(\mathbf{n})] \tag{15}$$

and the unity

$$\eta_g : R \longrightarrow \mathcal{S}^{<k>},\ 1_R \mapsto [\mathbf{n} \mapsto 1_R] \text{ for every } \mathbf{n} \in \mathbb{N}_0^k. \tag{16}$$

By Propositions 2.4 and 2.5 $(\mathcal{L}_R^{<k>}; g) \simeq R[\mathbf{x}; g]^\circ$ has the structure of an $R$-bialgebra with the coalgebra structure described in 2.6, the Hadamard product (15) and the unity (16).

**2.9. The Hopf $R$-algebra $(\mathcal{L}_R^{<k>}; p)$.** Consider the Hopf $R$-algebra $R[\mathbf{x}; p]$. Then $\mathcal{S}^{<k>} \simeq R^{\mathbb{N}_0^k} \simeq R[\mathbf{x}; p]^*$ is an $R$-algebra with multiplication given by the *Hurwitz product*

$$*_p : \mathcal{S}^{<k>} \otimes_R \mathcal{S}^{<k>} \longrightarrow \mathcal{S}^{<k>},\ u \otimes v \mapsto [\mathbf{n} \mapsto \sum_{\mathbf{t} \leq \mathbf{n}} \binom{\mathbf{n}}{\mathbf{t}} u(\mathbf{t})v(\mathbf{n} - \mathbf{t})] \tag{17}$$

and the unity

$$\eta_p : R \longrightarrow \mathcal{S}^{<k>},\ 1_R \mapsto [\mathbf{n} \mapsto \delta_{\mathbf{n},\mathbf{0}}] \text{ for every } \mathbf{n} \in \mathbb{N}_0^k. \tag{18}$$

By Propositions 2.4 and 2.5 $(\mathcal{L}_R^{<k>}; p) \simeq R[\mathbf{x}; p]^\circ$ has the structure of a Hopf $R$-algebra with the coalgebra structure described in 2.6, the Hurwitz product (17), the unity (18) and the antipode

$$S_{\mathcal{L}^{<k>}} : \mathcal{L}^{<k>} \longrightarrow \mathcal{L}^{<k>},\ u \mapsto [\mathbf{i} \mapsto (-1)^{\mathbf{i}} u(\mathbf{i})].$$

**Proposition 2.10.** ([Kur02, Theorem 3]) *Let $u$ and $v$ be linearly recursive sequences over $R$ of orders $m$, $n$ and with characteristic polynomials $f(x)$, $g(x)$ respectively. Then*

1. *$u \star_g v$ is a linearly recursive sequence over $R$ of order $m \cdot n$ and characteristic polynomial $\chi(S_f \otimes S_g)$;*

2. *$u \star_p v$ is a linearly recursive sequence over $R$ of order $m \cdot n$ and characteristic polynomial $\chi(S_f \otimes E_n + E_m \otimes S_g)$.*

**Example 2.11.** Let $R$ be any ring and $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \in \mathcal{S}_R$ be solutions of the difference equations

$$\begin{array}{rcl} x_{n+3} - x_{n+2} + x_{n-1} - x_n & = & 0; \quad x_0 = 0, \quad x_1 = 1, \quad x_2 = 2; \\ y_{n+2} - y_{n+1} + y_n & = & 0; \quad y_0 = 1, \quad y_1 = 0. \end{array}$$

Then $\{x_n\}_{n=0}^{\infty}$ is a linearly recursive sequence over $R$ with characteristic polynomial $f(x) = x^3 - x^2 + x - 1$ and $\{y_n\}_{n=0}^{\infty}$ is a linearly recursive sequence over $R$ with characteristic polynomial $g(x) = x^2 - x + 1$.

Notice that

$$\begin{aligned} S_f \otimes S_g &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

Hence $\{z_n\}_{n=0}^{\infty} := \{x_n\}_{n=0}^{\infty} \star_g \{y_n\}_{n=0}^{\infty}$ is by Proposition 2.10 a linearly recursive sequence over $R$ with characteristic polynomial

$$\chi(S_f \otimes S_g) = x^6 - x^5 + x^3 - x + 1,$$

i.e. $\{z_n\}_{n=0}^{\infty}$ is a solution of the difference equation

$$z_{n+6} - z_{n+5} + z_{n+3} - z_{n+1} + z_n = 0 \text{ with initial vector } (0, 0, -2, -1, 0, 1).$$

The following table gives the first 11 terms of the sequences $\{z_n\}_{n=0}^{\infty}$ :

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 0 | 1 | 2 |
| $y_n$ | 1 | 0 | $-1$ | $-1$ | 0 | 1 | 1 | 0 | $-1$ | $-1$ | 0 |
| $z_n$ | 0 | 0 | $-2$ | $-1$ | 0 | 1 | 2 | 0 | 0 | $-1$ | 0 |

**Example 2.12.** Consider the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ of the previous example. Then

$$S_f \otimes E_2 + E_3 \otimes S_g = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

By Proposition 2.10 $\{z_n\}_{n=0}^{\infty} = \{x_n\}_{n=0}^{\infty} \star_p \{y_n\}_{n=0}^{\infty} := \{\sum_{j=0}^{n} \binom{n}{j} x_j \cdot y_{n-j}\}_{n=0}^{\infty}$ is a linearly recursive sequence over $R$ with characteristic polynomial

$$\chi(S_f \otimes E_2 + E_3 \otimes S_g) = x^6 - 5x^5 + 14x^4 - 25x^3 + 28x^2 - 15x + 3.$$

Hence $\{z_n\}_{n=0}^{\infty}$ is a solution of the difference equation

$$z_{n+6} - 5z_{n+5} + 14z_{n+4} - 25z_{n+3} + 28z_{n+2} - 15z_{n+1} + 3z_n = 0$$

with initial vector $(0, 1, 2, -2, -16, -29)$.

The following table gives the first 9 terms of the sequences $\{z_n\}_{n=0}^{\infty}$ :

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 0 |
| $y_n$ | 1 | 0 | $-1$ | $-1$ | 0 | 1 | 1 | 0 | $-1$ |
| $z_n$ | 0 | 1 | 2 | $-2$ | $-16$ | $-29$ | $-12$ | 29 | 0 |

**2.13. Cofree comodules.** Let $C$ be an $R$-coalgebra. A right $C$-comodule $(M, \varrho_M)$ is called *cofree*, if there exists an $R$-module $K$, such that $(M, \varrho_M) \simeq (K \otimes_R C, id_K \otimes \Delta_C)$ as right $C$-comodules. Note that if $K \simeq R^{(\Lambda)}$, a free $R$-module, then $M \simeq R^{(\Lambda)} \otimes_R C \simeq C^{(\Lambda)}$ as right $C$-comodules (this is one reason of the terminology *cofree*).

As a direct consequence of Lemma 1.17 we get

**Corollary 2.14.** *Let $M$ be an $R[\mathbf{x}]$-module. Then we have an isomorphism of $R[\mathbf{x}]^{\circ}$-comodules*

$$\mathcal{L}_{M^*}^{<k>} \simeq M[\mathbf{x}]^{\circ} \simeq M^* \otimes_R R[\mathbf{x}]^{\circ} \simeq M^* \otimes_R \mathcal{L}_R^{<k>}.$$

*In particular $M[\mathbf{x}]^{\circ}$ ($\mathcal{L}_{M^*}^{<k>}$) is a cofree $R[\mathbf{x}]^{\circ}$-comodule ($\mathcal{L}_R^{<k>}$-comodule).*

# 3 Linearly (bi)recursive bisequences

In this section we consider the *linearly (bi)recursive $k$-bisequences* and the *reversible $k$-sequences* over $R$-modules, where $R$ is an arbitrary commutative ground ring. We generalize results of [LT90] and [KKMN95] concerning the bialgebra structure of the linearly recursive sequences over a base field to the case of arbitrary *artinian* ground rings.

**3.1.** Let $M$ be an $R$-module, $\mathbf{l} = (l_1, ..., l_k) \in \mathbb{N}_0^k$ and consider the *system of linear bidifference equations* (ab. SLBE)

$$\begin{array}{lcll} x_{\mathbf{z}+(l_1,0,...,0)} & + & \sum_{i=1}^{l_1} p_{(1,l_1-i)}(\mathbf{z}) x_{\mathbf{z}+(l_1-i,0,...,0)} & = & g_1(\mathbf{z}), \\ x_{\mathbf{z}+(0,l_2,0,...,0)} & + & \sum_{i=1}^{l_2} p_{(2,l_2-i)}(\mathbf{z}) x_{\mathbf{z}+(0,l_2-i,0,...,0)} & = & g_2(\mathbf{z}), \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ x_{\mathbf{z}+(0,...,0,l_k)} & + & \sum_{i=1}^{l_k} p_{(k,l_k-i)}(\mathbf{z}) x_{\mathbf{z}+(0,...,0,l_k-i)} & = & g_k(\mathbf{z}), \end{array} \tag{19}$$

where the $p_{jl}$'s are $R$-valued functions and the $g_j$'s are $M$-valued functions defined for all $\mathbf{z} \in \mathbb{Z}^{<k>}$. If the $g_j$'s are identically zero, then (19) is said to be a *homogenous* SLBE. If the $p_{jl}$'s are constants, then (19) is said to be a SLBE *with constant coefficients.*

**3.2. Bisequences.** For an $R$-module $M$ and $k \geq 0$ let

$$\widetilde{\mathcal{S}}_M^{<k>} := \{\widetilde{\nu} : \mathbb{Z}^k \to M\} \simeq M^{\mathbb{Z}^k}$$

be the $R$-module of *$k$-bisequences* over $M$. If $M$ (resp. $k$) is not mentioned, then we mean $M = R$ (resp. $k = 1$). For $\widetilde{w} \in \widetilde{\mathcal{S}}_M^{<k>}$ and $f(\mathbf{x}) = \sum\limits_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in R[\mathbf{x}, \mathbf{x}^{-1}]$ define

$$f(\mathbf{x}) \rightharpoonup \widetilde{w} = \widetilde{\nu} \in \widetilde{\mathcal{S}}_M^{<k>}, \text{ where } \widetilde{\nu}(\mathbf{z}) := \sum_{\mathbf{i}} a_{\mathbf{i}} \widetilde{w}(\mathbf{z} + \mathbf{i}) \text{ for all } \mathbf{z} \in \mathbb{Z}^k.$$

With this action $\widetilde{\mathcal{S}}_M^{<k>}$ becomes an $R[\mathbf{x}, \mathbf{x}^{-1}]$-module. For subsets $I \subset R[\mathbf{x}, \mathbf{x}^{-1}]$ and $Y \subset \widetilde{\mathcal{S}}_M^{<k>}$ consider

$$\begin{array}{lcl} \mathrm{An}_{\widetilde{\mathcal{S}}_M^{<k>}}(I) & = & \{\widetilde{w} \in \widetilde{\mathcal{S}}_M^{<k>} \mid g \rightharpoonup \widetilde{w} = 0 \text{ for every } g \in I\}, \\ \mathrm{An}_{R[\mathbf{x}, \mathbf{x}^{-1}]}(Y) & = & \{h \in R[\mathbf{x}, \mathbf{x}^{-1}] \mid h \rightharpoonup \widetilde{\nu} = 0 \text{ for every } \widetilde{\nu} \in Y\}. \end{array}$$

Obviously $\mathrm{An}_{\mathcal{S}_M}^{<k>}(I) \subset \mathcal{S}_M^{<k>}$ is an $R[\mathbf{x}, \mathbf{x}^{-1}]$-submodule and $\mathrm{An}_{R[\mathbf{x}, \mathbf{x}^{-1}]}(Y) \lhd R[\mathbf{x}, \mathbf{x}^{-1}]$ is an ideal.

**Definition 3.3.** Let $M$ be an $R$-module. We call $\widetilde{w} \in \widetilde{\mathcal{S}}_M^{<k>}$ a *linearly recursive $k$-bisequence* (resp. a *linearly birecursive $k$-bisequence*), if $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{w})$ is a monic ideal (resp. a reversible ideal). Note that a $k$-bisequence $\widetilde{u} \in \widetilde{\mathcal{S}}_M^{<k>}$ is linearly recursive, iff it's a solution of a homogenous SLBE with constants coefficients of the form (19). The subsets $\widetilde{\mathcal{L}}_M^{<k>} \subseteq \widetilde{\mathcal{S}}_M^{<k>}$ of linearly recursive $k$-bisequences and $\widetilde{\mathcal{B}}_M^{<k>} \subseteq \widetilde{\mathcal{S}}_M^{<k>}$ of linearly birecursive $k$-bisequences over $M$ are obviously $R[\mathbf{x}, \mathbf{x}^{-1}]$-submodules.

# Reversible sequences over modules

**3.4.** Let $M$ be an $R$-module. A $k$-bisequence $\widetilde{u}$ is said to be a *reverse* of $u \in \mathcal{S}_M^{<k>}$, if $\widetilde{u}|_{\mathbb{N}_0^k} = u$ and $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{u}) = \mathrm{An}_{R[\mathbf{x}]}(u)$. A linearly recursive $k$-sequence $u$ will be called *reversible*, if $u$ has a reverse $\widetilde{u} \in \widetilde{\mathcal{L}}_M^{<k>}$. With $\mathcal{R}_M^{<k>} \subset \mathcal{L}_M^{<k>}$ we denote the $R[\mathbf{x}]$-submodule of reversible $k$-sequences over $M$.

**Lemma 3.5.** (*Compare* [KKMN95, Proposition 14.11]) *Let $R$ be* artinian.

1. *Every monic ideal $I \lhd R[\mathbf{x}]$ contains a subset of monic polynomials*

$$\{x_j^{d_j} q_j(x_j) \mid \ q_j(x_j) \text{ is reversible for } j = 1, ..., k\}. \tag{20}$$

2. *Let $M$ be an $R$-module. Then every linearly recursive $k$-bisequence over $M$ is linearly birecursive (i.e. $\widetilde{\mathcal{B}}_M^{<k>} = \widetilde{\mathcal{L}}_M^{<k>}$).*

**Proof.** 1. By [AM69, 8.7] every commutative artinian ring is (up to isomorphism) a direct sum of local artinian rings. W.l.o.g. let $R$ be a local artinian ring. The Jacobson radical of $R$

$$J(R) = \{r \in R|\ r \text{ is not invertible in } R\}$$

is nilpotent, hence there exists a positive integer $n$, such that $J(R)^n = 0$. Let $I$ be a monic ideal with a subset of monic polynomials $\{g_1(x_1), ..., g_k(x_k)\} \subset I$. If $g_j(x_j) \equiv f_j(x_j) \pmod{J(R)[x_j]}$ for $j = 1, ..., k$, then $g_j(x_j)|f_j(x_j)^n$, where $n$ is the index of nilpotency of the ideal $J(R)$. Hence $f_j(x_j)^n \in I$. If we write $f_j(x_j)^n = x_j^{d_j} q_j(x_j)$ with $(x_j, q_j(x_j)) = 1$, then $q_j(0) \in U(R)$, i.e. $q_j(x_j)$ is a reversible polynomial for $j = 1, ..., k$.

2. Let $\widetilde{u}$ be a linearly recursive $k$-bisequence over $M$. If $R$ is artinian, then $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{u})$ contains by (1) a subset of monic polynomials $\{x_j^{d_j} q_j(x_j)|\ q_j(x_j)$ is reversible for $j = 1, ..., k\}$. Then for every $\mathbf{z} \in \mathbb{Z}^k$ we have $(q_j(x_j) \rightharpoonup \widetilde{u})(z_1, ..., z_j, ..., z_k) = (x_j^{d_j} q_j(x_j) \rightharpoonup \widetilde{u})(z_1, ..., z_j - d_j, ..., z_k) = 0$. Hence $\{q_j(x_j)|\ i = 1, ..., k\} \subset \mathrm{An}_{R[\mathbf{x}]}(\widetilde{u})$, i.e. $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{u})$ is a reversible ideal.■

**3.6. Backsolving**. Let $M$ be an $R$-module. Let $u$ be a linearly recursive sequence over $M$ and assume that $\mathrm{An}_{R[x]}(u)$ contains some monic polynomial of the form $x^d q(x) = x^d(a_0 + a_1 x + ... + a_{l-1}x^{l-1} + x^l)$, $a_0 \in U(R)$. Then

$$a_0 u(j+d) + a_1 u(j+d+1) + ... + a_{l-1} u(j+d+l-1) + u(j+d+l) = 0 \text{ for all } j \geq 0$$

and we get by *Backsolving* a *unique* linearly birecursive bisequence $\widetilde{u} \in \mathrm{An}_{\widetilde{\mathcal{S}}_M}(q(x))$ with $\widetilde{u}(n) = u(n)$ for all $n \geq d$. The bisequence $\widetilde{u} \equiv 0$ in case $l = 0$ and is given for $l \neq 0$ by

$$\widetilde{u}(z) := \begin{cases} u(z), & z \geq d \\ -a_0^{-1}(a_1\widetilde{u}(z+1) + ... + a_{l-1}\widetilde{u}(z+l-1) + \widetilde{u}(z+l)), & z < d. \end{cases}$$

If there are two bisequences $\widetilde{v}$, $\widetilde{w} \in \mathrm{An}_{\widetilde{\mathcal{S}}_M}(q(x))$ with $\widetilde{v}(n) = u(n) = \widetilde{w}(n)$ for all $n \geq d$, then one can easily show by backsolving using $q(x)$ that $\widetilde{v} = \widetilde{w}$. Moreover we claim that $\mathrm{An}_{R[x]}(\widetilde{u}) = \mathrm{An}_{R[x]}(u)$. It's obvious that $\mathrm{An}_{R[x]}(\widetilde{u}) \subseteq \mathrm{An}_{R[x]}(u)$. On the other hand assume $g(x) = \sum\limits_{j=0}^{m} b_j x^j \in \mathrm{An}_{R[x]}(u)$. We prove by induction that $(g \rightharpoonup \widetilde{u})(z) = 0$ for all $z \in \mathbb{Z}$. First of all, note that for all $z \geq d$ we have $(g \rightharpoonup \widetilde{u})(z) = (g \rightharpoonup u)(z) = 0$. Now let $z_0 < d$ and assume that $(g \rightharpoonup \widetilde{u})(z) = 0$ for $z \in \{z_0, z_0 + 1, ..., z_0 + l - 1\} \subseteq \mathbb{Z}$. Then we have for

$z = z_0 - 1:$

$$\begin{aligned}(g \rightharpoonup \widetilde{u})(z_0-1) &= \sum_{j=0}^{m} b_j \widetilde{u}(j+z_0-1) \\ &= \sum_{j=0}^{m} b_j (\sum_{i=1}^{l} -a_0^{-1} a_i \widetilde{u}(j+z_0-1+i)) \\ &= -\sum_{i=1}^{l} a_0^{-1} a_i \sum_{j=0}^{m} b_j \widetilde{u}(j+z_0-1+i) \\ &= -\sum_{i=1}^{l} a_0^{-1} a_i (g \rightharpoonup \widetilde{u})(z_0-1+i) \\ &= 0.\end{aligned}$$

If $u$ is a linearly recursive $k$-sequence over $M$ with $k > 1$ and $\mathrm{An}_{R[\mathbf{x}]}(u)$ contains a set of monic polynomials $\{x_j^{d_j} q_j(x_j) \mid q_j$ is reversible for $j = 1, ..., k\}$, then we get by *backsolving* through $q_j(x_j)$ along the $j$-th row for $j = 1, ..., k$ a unique linearly birecursive $k$-bisequence $\widetilde{u} \in \mathrm{An}_{\widetilde{\mathcal{S}}_M^{<k>}}(q_1(x_1), ..., q_k(x_k))$ with $\widetilde{u}(\mathbf{n}) = u(\mathbf{n})$ for all $\mathbf{n} \geq \mathbf{d}$ and it follows moreover that $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{u}) = \mathrm{An}_{R[\mathbf{x}]}(u)$.

**Lemma 3.7.** *Let $M$ be an $R$-module.*

1. *Every birecursive $k$-sequence over $M$ is reversible with unique reverse (which we denote by* $\mathrm{Rev}(u)$*). Moreover $\mathcal{B}_M^{<k>}$ becomes a structure of an $R[\mathbf{x}, \mathbf{x}^{-1}]$-module through $f \rightharpoonup u := (f \rightharpoonup \mathrm{Rev}(u))_{|_{\mathbb{N}_0^k}}$.*

2. *If $R$ is artinian, then every reversible $k$-sequence over $M$ is birecursive as well (i.e. $\mathcal{B}_M^{<k>} = \mathcal{R}_M^{<k>}$).*

**Proof.** Let $M$ be an $R$-module.

1. If $u \in \mathcal{B}_M^{<k>}$, then $\mathrm{An}_{R[\mathbf{x}]}(u)$ contains a set of reversible polynomials $\{q_j(x_j) \mid j = 1, ..., k\}$ and we get by backsolving (see 3.6) a *unique* linearly birecursive $k$-bisequence $\widetilde{u} \in \mathrm{An}_{\widetilde{\mathcal{S}}_M^{<k>}}(q_1(x_1), ..., q_k(x_k))$ with $\widetilde{u}(\mathbf{n}) = u(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}_0^k$. For the bisequence $\widetilde{u}$ we have as shown above $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{u}) = \mathrm{An}_{R[\mathbf{x}]}(u)$, i.e. $\widetilde{u}$ is a reverse of $u$. The last statement is obvious.

2. By (1) $\mathcal{B}_M^{<k>} \subseteq \mathcal{R}_M^{<k>}$. If $R$ is artinian and $u \in \mathcal{R}_M^{<k>}$ with reverse $\widetilde{u}$, then $\mathrm{An}_{R[\mathbf{x}]}(u) = \mathrm{An}_{R[\mathbf{x}]}(\widetilde{u})$ is by Lemma 3.5 (2) reversible, i.e. $u \in \mathcal{B}_M^{<k>}$.■

**Example 3.8.** The Fibonacci sequence $\mathcal{F} = (0, 1, 1, 2, 3, 5, ...)$ has elementary characteristic polynomial $f(x) = x^2 - x - 1$. Since $f(0) = -1$ is invertible in $\mathbb{Z}$, we conclude that $\mathcal{F}$ is reversible with reverse

$$\mathrm{Rev}(\mathcal{F})(z) = \begin{cases} \mathcal{F}(z) & z \geq 0 \\ \mathrm{Rev}(\mathcal{F})(z+2) - \mathrm{Rev}(\mathcal{F})(z+1) & z < 0. \end{cases}$$

The following tables lists some of the terms of the bisequence $\mathrm{Rev}(\mathcal{F}) \in \mathrm{An}_{\mathcal{S}_{\mathbb{Z}}}(x^2 - x - 1)$ :

| $z$ | ... | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\mathrm{Rev}(\mathcal{F})(z)$ | ... | $-3$ | 2 | $-1$ | 1 | 0 | 1 | 1 | 2 | 3 |

**Lemma 3.9.** *We have an isomorphism of* $R[\mathbf{x}, \mathbf{x}^{-1}]$*-modules*

$$\widetilde{\mathcal{B}}_M^{<k>} \simeq \mathcal{B}_M^{<k>}. \tag{21}$$

**Proof.** By Lemma 3.7 we have the well defined $R[\mathbf{x}, \mathbf{x}^{-1}]$-linear map

$$\mathrm{Rev}(-) : \mathcal{B}_M^{<k>} \rightarrow \widetilde{\mathcal{B}}_M^{<k>},\ u \mapsto \mathrm{Rev}(u).$$

It's easy to see that $\mathrm{Rev}(-)$ is bijective with inverse $\widetilde{u} \mapsto \widetilde{u}_{|_{\mathbb{N}_0^k}}$.■

**3.10.** Let $M$ be an $R$-module. We call a $k$-sequence $u \in \mathcal{S}_M^{<k>}$ *periodic* (resp. *degenerating*), if $\mathbf{x}^{\mathbf{d}}(\mathbf{x}^{\mathbf{t}} \rightharpoonup u) = 0$ for some $\mathbf{d} \in \mathbb{N}_0^k$ and $\mathbf{t} \in \mathbb{N}^k$ (resp. $\mathbf{x}^{\mathbf{d}} \rightharpoonup u = 0$ for some $\mathbf{d} \in \mathbb{N}_0^k$). It's clear that the subsets $\mathcal{P}_M^{<k>} \subseteq \mathcal{L}_M^{<k>}$ of periodic $k$-sequences and $\mathcal{D}_M^{<k>} \subseteq \mathcal{L}_M^{<k>}$ of degenerating $k$-sequences are $R[\mathbf{x}]$-submodules.

**Remark 3.11.** ([KKMN95, Proposition 5.2]) If $M$ is a *finite* $R$-module, then every linearly recursive sequence over $M$ is periodic (i.e. $\mathcal{P}_M^{<1>} = \mathcal{L}_M^{<1>}$).

**Proposition 3.12.** ([KKMN95, Proposition 5.27]) *Let $R$ be an* arbitrary *commutative ring, $M$ an $R$-module and denote with $\mathcal{RP}_M^{<k>}$ the set of reversible periodic $k$-sequences over $M$. Then we have an isomorphism of* $R[\mathbf{x}]$*-modules*

$$\mathcal{P}_M^{<k>} \simeq \mathcal{D}_M^{<k>} \oplus \mathcal{RP}_M^{<k>}. \tag{22}$$

The following result generalizes Proposition 3.12 and describes the $R[\mathbf{x}]$-module structure of arbitrary linearly recursive $k$-sequences of $R$-modules, where $R$ is an *artinian* commutative ground ring:

**Proposition 3.13.** *Let $M$ be an $R$-module. If $R$ is artinian, then we have isomorphisms of* $R[\mathbf{x}]$*-modules*

$$\mathcal{L}_M^{<k>} \simeq \mathcal{D}_M^{<k>} \oplus \widetilde{\mathcal{L}}_M^{<k>} = \mathcal{D}_M^{<k>} \oplus \widetilde{\mathcal{B}}_M^{<k>} \simeq \mathcal{D}_M^{<k>} \oplus \mathcal{B}_M^{<k>} = \mathcal{D}_M^{<k>} \oplus \mathcal{R}_M^{<k>}. \tag{23}$$

**Proof.** Let $R$ be artinian and $M$ an $R$-module. If $u$ is a linearly recursive sequence over $M$, then $\mathrm{An}_{R[\mathbf{x}]}(u)$ contains by Lemma 3.5 (1) a set of monic polynomials $\{x^{d_j} q_j(x_j) \mid q_j(x_j)$ is reversible for $j = 1, ..., k\}$. By *backsolving* (see 3.6) we have a well defined morphism of $R[\mathbf{x}]$-modules

$$\gamma : \mathcal{L}_M^{<k>} \rightarrow \widetilde{\mathcal{L}}_M^{<k>},\ u \mapsto \widetilde{u}, \tag{24}$$

where $\widetilde{u}$ is the *unique* linearly birecursive bisequence $\widetilde{u} \in \mathrm{An}_{\widetilde{\mathcal{S}}_M}(q_1, ..., q_k)$ with $\widetilde{u}(\mathbf{n}) = u(\mathbf{n})$ for all $\mathbf{n} \geq \mathbf{d}$. It's clear that $\mathrm{Ke}(\gamma) = \mathcal{D}_M^{<k>}$. On the other hand, there is a morphism of $R[\mathbf{x}]$-modules

$$\beta = \widetilde{\mathcal{L}}_M^{<k>} \rightarrow \mathcal{L}_M^{<k>},\ \widetilde{w} \mapsto \widetilde{w}_{|_{\mathbb{N}_0^k}}. \tag{25}$$

It's obvious that $\gamma \circ \beta = id_{\widetilde{\mathcal{L}}_M^{<k>}}$, hence the following exact sequence of $R[\mathbf{x}]$-modules splits

$$0 \to \mathcal{D}_M^{<k>} \to \mathcal{L}_M^{<k>} \xrightarrow{\gamma} \widetilde{\mathcal{L}}_M^{<k>} \to 0,$$

i.e. $\mathcal{L}_M^{<k>} \simeq \mathcal{D}_M^{<k>} \oplus \widetilde{\mathcal{L}}_M^{<k>}$. Since $R$ is artinian, we have by Lemmata 3.5 (2) and 3.7 (2) $\widetilde{\mathcal{L}}_M^{<k>} = \widetilde{\mathcal{B}}_M^{<k>}$ and $\mathcal{B}_M^{<k>} = \mathcal{R}_M^{<k>}$. We are done now by the isomorphism of $R[\mathbf{x}]$-modules $\mathcal{B}_M^{<k>} \simeq \widetilde{\mathcal{B}}_M^{<k>}$ (Lemma 3.9).■

**3.14. The Hopf $R$-algebra $R[\mathbf{x}, \mathbf{x}^{-1}]$.** Consider the commutative group $G$ generated by $\{x_j \mid j = 1, ..., k\}$. Then the ring of Laurent polynomials $R[\mathbf{x}, \mathbf{x}^{-1}] = RG$ has the structure of a *commutative cocommutative* Hopf $R$-algebra $(R[\mathbf{x}, \mathbf{x}^{-1}], \mu, \eta, \Delta, \varepsilon, S)$, where $\mu$ resp. $\eta$ are the usual multiplication resp. the usual unity and

$$\begin{array}{llllll} \Delta: & R[\mathbf{x},\mathbf{x}^{-1}] & \to & R[\mathbf{x},\mathbf{x}^{-1}] \otimes_R R[\mathbf{x},\mathbf{x}^{-1}], & x_j^z \mapsto & x_j^z \otimes x_j^z, & \forall\, z \in \mathbb{Z},\ j = 1, ..., k, \\ \varepsilon: & R[\mathbf{x},\mathbf{x}^{-1}] & \to & R, & x_j^z \mapsto & 1_R, & \forall\, z \in \mathbb{Z},\ j = 1, ..., k, \\ S: & R[\mathbf{x},\mathbf{x}^{-1}] & \to & R[\mathbf{x},\mathbf{x}^{-1}], & x_j^z \mapsto & x_j^{-z}, & \forall\, z \in \mathbb{Z},\ j = 1, ..., k. \end{array} \tag{26}$$

**Proposition 3.15.** *Let $R$ be an* arbitrary *commutative ring. Then $R[\mathbf{x}, \mathbf{x}^{-1}]$ is an admissible Hopf $R$-algebra and $R[\mathbf{x}, \mathbf{x}^{-1}]^\circ$ is a Hopf $R$-algebra.*

**Proof.** Notice that $R[\mathbf{x}, \mathbf{x}^{-1}]$ is a cofinitary Hopf $R$-algebra by Lemma 1.17 (2). Consider the proof of Proposition 2.4 and replace $R[\mathbf{x}]$ with $R[\mathbf{x}, \mathbf{x}^{-1}]$. Then the map

$$T_j : B \to B,\ b \mapsto \overline{\Delta}(x_j)b$$

is invertible with inverse

$$\overline{T}_j : B \to B,\ b \mapsto \overline{\Delta}(x_j^{-1})b.$$

Then the matrix $M_j$ of $T_j$ is invertible and $\chi_j(0) \in U(R)$ for $j = 1, ..., k$. Consequently $\mathcal{K}_{R[\mathbf{x},\mathbf{x}^{-1}]}$ satisfies axiom (8). Since $R[\mathbf{x}, \mathbf{x}^{-1}]/\mathrm{Ke}(\varepsilon) \simeq R$, $\mathcal{K}_{R[\mathbf{x};p]}$ satisfies axiom (9). Consider the *bijective* antipode $S$ of $R[\mathbf{x}; p]$. For every ideal $I \lhd R[\mathbf{x}, \mathbf{x}^{-1}]$, $S^{-1}(I) \lhd R[\mathbf{x}, \mathbf{x}^{-1}]$ is an ideal and we have an isomorphism of $R$-modules $R[\mathbf{x}, \mathbf{x}^{-1}]/S^{-1}(I) \simeq R[\mathbf{x}, \mathbf{x}^{-1}]/I$. Hence $\mathcal{K}_{R[\mathbf{x},\mathbf{x}^{-1}]}$ satisfies axiom (10). Consequently $R[\mathbf{x}, \mathbf{x}^{-1}]$ is an admissible Hopf $R$-algebra. The last statement follows now by Proposition 1.14.■

For every $R$-module $M$ we have an isomorphism of $R[\mathbf{x}, \mathbf{x}^{-1}]$-modules

$$\Psi_M : M[\mathbf{x}, \mathbf{x}^{-1}]^* \to \widetilde{\mathcal{S}}_{M^*}^{<k>},\ \widetilde{\varphi} \mapsto [\mathbf{z} \mapsto [m \mapsto \widetilde{\varphi}(m\mathbf{x}^{\mathbf{z}})]] \tag{27}$$

with inverse $\widetilde{u} \mapsto [m\mathbf{x}^{\mathbf{z}} \mapsto \widetilde{u}(\mathbf{z})(m)]$.

As in the proof of Proposition 2.5 we get

**Proposition 3.16.** *Let $R$ be an arbitrary ring. Then (27) induces an isomorphism of $R[\mathbf{x}, \mathbf{x}^{-1}]$-modules*

$$M[\mathbf{x}, \mathbf{x}^{-1}]^\circ \simeq \widetilde{\mathcal{B}}_{M^*}^{<k>}. \tag{28}$$

**Proof.** Consider the isomorphism of $R[\mathbf{x},\mathbf{x}^{-1}]$-modules $M[\mathbf{x},\mathbf{x}^{-1}]^* \overset{\Psi_M}{\simeq} \widetilde{\mathcal{S}}^{<k>}_{M^*}$ (27). Let $\varkappa \in M[\mathbf{x},\mathbf{x}^{-1}]^\circ$. Then $I \rightharpoonup \varkappa = 0$ for some $R$-cofinite $R[\mathbf{x},\mathbf{x}^{-1}]$-ideal $I \lhd R[\mathbf{x},\mathbf{x}^{-1}]$ and so $I \rightharpoonup \Psi(\varkappa) = \Psi(I \rightharpoonup \varkappa) = 0$. By Lemma 1.17 (2) $I$ is a reversible ideal and so $\mathrm{An}_{R[\mathbf{x}]}(u) \supset I \cap R[\mathbf{x}]$ is a reversible ideal, i.e. $\Psi(\varkappa)$ is linearly birecursive.

On the other hand, let $\widetilde{u} \in \widetilde{\mathcal{B}}^{<k>}_{M^*}$. Then $\mathrm{An}_{R[\mathbf{x}]}(\widetilde{u})$ is by definition a reversible ideal, i.e. it contains a subset of reversible polynomials $\{q_j(x_j),\ j = 1,...,k\}$. Note that for arbitrary $g \in R[\mathbf{x},\mathbf{x}^{-1}]$ we have $gq_j \rightharpoonup \Psi^{-1}(\widetilde{u}) = \Psi^{-1}(gq_j \rightharpoonup \widetilde{u}) = \Psi^{-1}(g \rightharpoonup (q_j \rightharpoonup \widetilde{u})) = 0$ for $j = 1,...,k$. By Lemma 1.17 (2) the reversible ideal $(q_1(x_1),...,q_k(x_k)) \lhd R[\mathbf{x},\mathbf{x}^{-1}]$ is $R$-cofinite, i.e. $\Psi^{-1}(\widetilde{u}) \in M[\mathbf{x},\mathbf{x}^{-1}]^\circ$.■

**3.17. The Hopf $R$-algebra structures on $\widetilde{\mathcal{B}}^{<k>}$** and $\mathcal{B}^{<k>}$. Let $R$ be an arbitrary ring and consider the Hopf $R$-algebra $R[\mathbf{x},\mathbf{x}^{-1}]$. Then $\widetilde{\mathcal{S}}^{<k>} \simeq R^{\mathbb{Z}^k} \simeq R[\mathbf{x},\mathbf{x}^{-1}]^*$ is an $R$-algebra with the *Hadamard product*

$$\star : \widetilde{\mathcal{S}}^{<k>} \otimes_R \widetilde{\mathcal{S}}^{<k>} \longrightarrow \widetilde{\mathcal{S}}^{<k>},\ \widetilde{u} \otimes \widetilde{v} \mapsto [\mathbf{z} \mapsto \widetilde{u}(\mathbf{z})\widetilde{v}(\mathbf{z})] \tag{29}$$

and the unity

$$\eta : R \longrightarrow \widetilde{\mathcal{S}}^{<k>},\ 1_R \mapsto [\mathbf{z} \mapsto 1_R] \text{ for every } \mathbf{z} \in \mathbb{Z}^k. \tag{30}$$

By Proposition 3.15 $R[\mathbf{x},\mathbf{x}^{-1}]^\circ$ is a Hopf $R$-algebra. So $\mathcal{B}^{<k>} \simeq R[\mathbf{x},\mathbf{x}^{-1}]^\circ$ inherits the structure of a Hopf $R$-algebra $(\mathcal{B}^{<k>}, \star_g, \eta_g, \Delta_{\mathcal{B}^{<k>}}, \varepsilon_{\mathcal{B}^{<k>}}, S_{\mathcal{B}^{<k>}})$, where $\star_g$ is the Hadamard product (15), $\eta_g$ is the unity (16) and

$$\begin{array}{lclclcll}
\Delta_{\mathcal{B}^{<k>}} & : & \mathcal{B}^{<k>} & \longrightarrow & \mathcal{B}^{<k>} \otimes_R \mathcal{B}^{<k>}, & u & \longmapsto & \sum\limits_{\mathbf{t} \leq \mathbf{l}-\mathbf{1}} (\mathbf{x}^{\mathbf{t}} \rightharpoonup u) \otimes e^{\mathbf{F}}_{\mathbf{t}}, \\
\varepsilon_{\mathcal{B}^{<k>}} & : & \mathcal{B}^{<k>} & \longrightarrow & R, & u & \longmapsto & u(\mathbf{0}), \\
S_{\mathcal{B}^{<k>}} & : & \mathcal{B}^{<k>} & \longrightarrow & \mathcal{B}^{<k>}, & u & \longmapsto & [\mathbf{n} \mapsto \mathrm{Rev}(u)(-\mathbf{n})].
\end{array}$$

Moreover $\widetilde{\mathcal{B}}^{<k>} \simeq R[\mathbf{x},\mathbf{x}^{-1}]^\circ$ becomes a Hopf $R$-algebra $(\widetilde{\mathcal{B}}^{<k>}, \star_g, \eta_g, \Delta_{\widetilde{\mathcal{B}}^{<k>}}, \varepsilon_{\widetilde{\mathcal{B}}^{<k>}}, S_{\widetilde{\mathcal{B}}^{<k>}})$, where $\star$ is the Hadamard product (29), $\eta$ is the unity (30) and

$$\begin{array}{lclclcll}
\Delta_{\widetilde{\mathcal{B}}^{<k>}} & : & \widetilde{\mathcal{B}}^{<k>} & \longrightarrow & \widetilde{\mathcal{B}}^{<k>} \otimes_R \widetilde{\mathcal{B}}^{<k>}, & \widetilde{u} & \longmapsto & \sum\limits_{\mathbf{t} \leq \mathbf{l}-\mathbf{1}} \mathrm{Rev}(\mathbf{x}^{\mathbf{t}} \rightharpoonup \widetilde{u}_{|_{\mathbb{N}_0^k}})) \otimes \mathrm{Rev}(e^{\mathbf{F}}_{\mathbf{t}}), \\
\varepsilon_{\widetilde{\mathcal{B}}^{<k>}} & : & \widetilde{\mathcal{B}}^{<k>} & \longrightarrow & R, & \widetilde{u} & \longmapsto & \widetilde{u}(\mathbf{0}), \\
S_{\widetilde{\mathcal{B}}^{<k>}} & : & \widetilde{\mathcal{B}}^{<k>} & \longrightarrow & \widetilde{\mathcal{B}}^{<k>}, & \widetilde{u} & \longmapsto & [\mathbf{z} \mapsto \widetilde{u}(-\mathbf{z})].
\end{array}$$

Note that with these structures the isomorphism $\mathcal{B}^{<k>} \simeq \widetilde{\mathcal{B}}^{<k>}$ of Lemma 3.9 turns to be an isomorphism of Hopf $R$-algebras.

The following theorem extends the corresponding result from the case of a base field [LT90, Page 124] (see also [KKMN95, 14.15]) to the case of arbitrary *artinian* ground rings:

**Theorem 3.18.** *If $R$ is artinian, then there are isomorphisms of $R$-bialgebras*

$$\mathcal{L}^{<k>} \simeq \mathcal{D}^{<k>} \oplus \widetilde{\mathcal{L}}^{<k>} = \mathcal{D}^{<k>} \oplus \widetilde{\mathcal{B}}^{<k>} \simeq \mathcal{D}^{<k>} \oplus \mathcal{B}^{<k>} = \mathcal{D}^{<k>} \oplus \mathcal{R}^{<k>}. \tag{31}$$

**Proof.** Consider the isomorphism $\mathcal{L}^{<k>} \simeq \mathcal{D}^{<k>} \oplus \widetilde{\mathcal{L}}^{<k>}$ (23). With the help of Lemmata 1.16 and 3.5 one can show as in [LT90, Seite 123], that $\gamma : \mathcal{L}^{<k>} \to \widetilde{\mathcal{L}}^{<k>}$ (24) and $\beta : \widetilde{\mathcal{L}}^{<k>} \to \mathcal{L}^{<k>}$ (25) are in fact bialgebra morphisms. Obviously $\mathrm{Ke}(\gamma) = \mathcal{D}^{<k>} \subset \mathcal{L}^{<k>}$ is an $\mathcal{L}^{<k>}$-subbialgebra and we are done.■

As an analog to Corollary (2.14) we get

**Corollary 3.19.** *Let $M$ be an $R[\mathbf{x}, \mathbf{x}^{-1}]$-module. Then we have an isomorphism of $R[\mathbf{x}, \mathbf{x}^{-1}]^\circ$-comodules*

$$\widetilde{\mathcal{L}}_{M^*}^{<k>} \simeq M[\mathbf{x}, \mathbf{x}^{-1}]^\circ \simeq M^* \otimes_R R[\mathbf{x}, \mathbf{x}^{-1}]^\circ \simeq M^* \otimes_R \widetilde{\mathcal{L}}_R^{<k>}.$$

*In particular $M[\mathbf{x}, \mathbf{x}^{-1}]^\circ$ $(\widetilde{\mathcal{L}}_{M^*}^{<k>})$ is a cofree $R[\mathbf{x}, \mathbf{x}^{-1}]^\circ$-comodule $(\widetilde{\mathcal{L}}_R^{<k>}$-comodule$)$.*

As a consequence of [Abu01, Satz 2.4.7] and [Abu01, Folgerung 2.5.10] we get

**Corollary 3.20.** *Let $R$ be noetherian and consider the $R$-bialgebra $R[\mathbf{x}; g]^\circ$ (resp. the Hopf $R$-algebra $R[\mathbf{x}; p]^\circ$, the Hopf $R$-algebra $R[\mathbf{x}, \mathbf{x}^{-1}]^\circ$). If $A$ is an $\alpha$-algebra (resp. an $\alpha$-bialgebra, a Hopf $\alpha$-algebra), then we have isomorphism of $R$-coalgebras (resp. $R$-bialgebras, Hopf $R$-algebras)*

$$A[\mathbf{x}; g]^\circ \simeq A^\circ \otimes_R R[\mathbf{x}; g]^\circ,\ A[\mathbf{x}; p]^\circ \simeq A^\circ \otimes_R R[\mathbf{x}; p]^\circ \text{ and } A[\mathbf{x}, \mathbf{x}^{-1}]^\circ \simeq A^\circ \otimes_R R[\mathbf{x}, \mathbf{x}^{-1}]^\circ. \tag{32}$$

**3.21. Representative functions.** Let $G$ be a monoid (a group) and consider the $R$-algebra $B = R^G$ with pointwise multiplication. Then $B$ is an $RG$-bimodule under the left and right actions

$$(yf)(x) = f(xy) \text{ and } (fy)(x) = f(yx) \text{ for all } x, y \in G.$$

We call $f \in R^G$ an *$R$-valued representative function* on the monoid $G$, if $(RG)f(RG)$ is finitely generated as an $R$-module. If $R$ is noetherian, then the subset $\mathcal{R}(G) \subset R^G$ of all representative functions on $G$ is an $RG$-subbimodule. Moreover we deduce from [AG-TW00, Theorem 2.13, Corollary 2.15] that in case $(RG)^\circ \subset R^G$ is pure, we have an isomorphism of $R$-bialgebras (Hopf $R$-algebras) $\mathcal{R}(G) \simeq (RG)^\circ$.

**Corollary 3.22.** *Let $R$ be noetherian.*

1. *Considering the monoid $(\mathbb{N}_0^k, +)$ we have isomorphisms of $R$-bialgebras*

$$\mathcal{R}(\mathbb{N}_0^k) \simeq R[\mathbf{x}; p]^\circ \simeq \mathcal{L}_R^{<k>}.$$

2. *Considering the monoid $(\mathbb{Z}^k, +)$ we have isomorphisms of Hopf $R$-algebras*

$$\mathcal{R}(\mathbb{Z}^k) \simeq R[\mathbf{x}, \mathbf{x}^{-1}]^\circ \simeq \widetilde{\mathcal{B}}_R^{<k>} \simeq \mathcal{B}_R^{<k>}.$$

**Acknowledgment.** Besides the new results, this note extends results in Kapitel 4 of my doctoral thesis at the Heinrich-Heine Universität - Düsseldorf (Germany). I am so grateful to my advisor Prof. Robert Wisbauer for his wonderful supervision and the continuous encouragement and support.

The author is also grateful to the referees for helpful suggestions and for drawing his attention to the articles [Kur00] and [Kur94] by V. Kurakin.

# References

[Abe80] E. Abe, *Hopf algebras*, Cambridge Tracts in Mathematics **74**, Cambridge University Press (1980).

[Abu01] J.Y. Abuhlail, *Dualitätstheoreme für Hopf-Algebren über Ringen*, **Ph.D. Dissertation,** Heinrich-Heine Universität, Düsseldorf-Germany (2001). http://www.ulb.uni-duesseldorf.de/diss/mathnat/2001/abuhlail.html

[AG-TL01] J.Y. Abuhlail, J. Gómez-Torrecillas and F. Lobillo, *Duality and rational modules in Hopf algebras over commutative rings*, J. Algebra **240**, 165-184 (2001).

[AG-TW00] J.Y. Abuhlail, J. Gómez-Torrecillas and R. Wisbauer, *Dual coalgebras of algebras over commutative rings*, J. Pure Appl. Algebra **153**, 107-120 (2000).

[AM69] M. Atiya and I. McDonald, *Introduction to commutative algebra*, Addison-Wesley, Reading Mass (1969).

[CG93] W. Chin and J. Goldman, *Bialgebras of linearly recursive sequences*, Commun. Algebra **21**, 3935-3952 (1993).

[CP91] L. Cerlienco and F. Piras, *On the continuous dual of a polynomial bialgebra*, Commun. Algebra **19**, 2707-2727 (1991).

[CV95] U. Cerruti and F. Vaccarino, *R-Algebras of linear recurrent sequences*, J. Algebra **175,** 332-338 (1995).

[GK97] L. Grünenfelder and T. Košir, *Koszul cohomology for finite families of comodule maps and applications*, Commun. Algebra, **25**, 459-479 (1997).

[GO93] L. Grünenfelder and M. Omladič, *Linearly recursive sequences and operator polynomials*, Linear Algebra Appl. **182**, 127-145 (1993).

[Grü69] L. Grünenfelder, *Über die Struktur von Hopf-Algebren*, **Ph.D. Dissertation**, Eidgenössische Technische Hochschule - Zürich (1969).

[HN99] T. Honold and A. Nechaev, *Weighted modules and representations of codes*, Probl. Inf. Transm. **35**, 205-223 (1999).

[Jon73] B. Jones, *Linear Algebra,* Holden-Day, Inc., San Francisco (1973).

[KKMMN99] V. Kurakin, A. Kuzamin, V. Markov, A. Mikhalev, and A. Nechaev, *Linear codes and polylinear recurrences over finite rings and modules* (*a survey*), Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, M. Fossorier et al. (ed.), Lect. Notes Comput. Sci. **1719**, Berlin: Springer-Verlag, 365-391 (1999).

[KKMN95] V. Kurakin, A. Kuzmin, A. Mikhalev and A. Nechaev, *Linear recurring sequences over rings and modules*, J. Math. Sci., New York **76**, 2793-2915 (1995).

[Kur02] V. Kurakin, *Hopf algebra dual to a polynomial algebra over a commutative ring*, Math. Zametki **71**, 677-685 (2002).

[Kur00] V. Kurakin, *Hopf algebras of linear recurring sequences over rings and modules* (2000) (unpublished).

[Kur94] V. Kurakin, *Hopf algebras of linear recurring sequences over commutative rings.* Proceedings of the Second Mathematical Workshop of MSSU, Moscow 67-69 (1994) (in Russian).

[LT90] R. Larson and E. Taft, *The algebraic structure of linearly recursive sequences under Hadamard product*, Isr. J. Math. **72**, 118-132 (1990).

[MN96] A. Mikhalev and A. Nechaev, *Linear recurring sequences over modules*, Acta Appl. Math. **42**, 161-202 (1996).

[Mon93] S. Montogomery, *Hopf Algebras and their Actions on Rings*, Reg. Conf. Series in Math. **82**, AMS (1993).

[Nec96] A. Nechaev, *Polylinear recurring sequences over modules and quasi-Frobenius modules*, First international Tainan-Moscow algebra workshop, Proceedings of the international conference, Y. Fong et al. (ed.), Berlin: de Gruyter, 283-298 (1996).

[Nec93] A. Nechaev, *Linear recurrent sequences over quasi-Frobenius modules*, Russ. Math. Surv. **48**, 209-210 (1993).

[Nec92] A. Nechaev, *Linear recurring sequences over commutative rings,* Discrete Math. Appl. **2**, 659-683 (1992).

[PT80] B. Peterson and E. Taft, *The Hopf algebra of linearly recursive sequences*, Aequationes Math. **20**, 1-17 (1980).

[Swe69] M. Sweedler, *Hopf algebras*, New York: Benjamin, (1969).

[Taf95] E. Taft, *Linearly recursive tableaux,* Congr. Numerantium **107**, 33-36 (1995).

[Taf94] E. Taft, *Algebraic aspects of linearly recursive sequences*, J. Bergen et al. (ed.), Advances in Hopf algebras, Marcel Dekker. Lect. Notes Pure Appl. Math. **158**, 299-317 (1994).