

DUALITY AND RATIONAL MODULES IN HOPF ALGEBRAS OVER COMMUTATIVE RINGS.

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ABSTRACT. Let A be an algebra over a commutative ring R . If R is noetherian and A° is pure in R^A , then the categories of rational left A -modules and right A° -comodules are isomorphic. In the Hopf algebra case, we can also strengthen the Blattner-Montgomery duality theorem. Finally, we give sufficient conditions to get the purity of A° is R^A .

INTRODUCTION

It is well known that the theory of Hopf algebras over a field cannot be trivially passed to Hopf algebras over a commutative ring. For instance let us consider $\mathbb{Z}[x]$ as Hopf algebra and let \mathfrak{a} be the Hopf ideal generated by $\langle 4, 2x \rangle$. Let H be the Hopf \mathbb{Z} -algebra $H = \mathbb{Z}[x]/\mathfrak{a}$. The finite dual is zero in this situation. However $H \cong \mathbb{Z}_4[x]/\langle 2x \rangle$, so we can view H as a Hopf \mathbb{Z}_4 -algebra. If I is a \mathbb{Z}_4 -cofinite ideal of H then every element nonzero in H/I has order 2, which implies every element in H° has order 2. In this situation H° is not pure in $\mathbb{Z}_4^H = \text{Map}(H, \mathbb{Z}_4)$ as a \mathbb{Z}_4 -module (since H° is not free) and a canonical \mathbb{Z}_4 -coalgebra structure on H° cannot be expected.

A basic result from the theory of coalgebras over fields is that the comodules are essentially rational modules. Thus, given a (left) non-singular pairing of a coalgebra C and an algebra A (see [8]), the categories of rational left A -modules and right C -comodules are isomorphic. This applies in particular for the canonical pairings (C, C^*) and (A°, A) derived from a coalgebra C and an algebra A , respectively. An attempt to develop systematically the theory of rational modules associated to a pairing (C, A) , where C is a coalgebra and A is an algebra over an arbitrary commutative ring R , is [4]. A corollary of the theory there developed is that if C is projective as an R -module, then the category of right C -comodules is isomorphic to the category of rational left modules over the convolution dual R -algebra $C^* = \text{Hom}_R(C, R)$ (this result was independently obtained in [13] by means of a different approach). However, the results of [4] are proved in a framework which does not allow to apply them directly to the pairing (A°, A) , for a given R -algebra A . In fact, the first problem is to endow the finite dual A° with a comultiplication, which entails some serious technical difficulties at the very beginning due to the lack of exactness of the tensor product bifunctor $- \otimes_R -$. Nevertheless, it has been recently proved in [1, Theorem 2.8]

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that if R is noetherian and A° is pure in the R -module R^A of all maps from A to R , then A° is a coalgebra. We have observed that the notion of rational pairing introduced in [4] can be restated in order that the methods developed there can be applied to the pairing (A°, A) to prove that the category of right A° -comodules is isomorphic to the category of rational left A -modules. This applies, in particular, for any algebra A over a hereditary noetherian commutative ring. We explain our general theory of rational modules and comodules in Sections 2 and 3.

We apply our methods to strengthen the Blattner-Montgomery duality theorem for Hopf algebras over commutative rings. Let H be a Hopf algebra over the commutative ring R . When R is a field, the Blattner-Montgomery duality theorem says that if U is a Hopf subalgebra of H° , A is an H -module algebra such that the H -action is locally finite in a sense appropriate to the choice of U , H and U have bijective antipodes and there is a certain right-left symmetry in the action of $H\#U$ on H , then

$$(A\#H)\#U \cong A \otimes (H\#U)$$

There are two proofs of this theorem in the literature. The first one appeared in [2, Theorem 2.1], and a new one, due to Blattner, appears in [5, Theorem 9.4.9]. Since, in this situation, the U -comodules are just the U -locally finite H -modules, it is easy to see that the two theorems are equivalent. In the case of a general commutative ring R , there is a similar theorem due to Van den Bergh (see [11]) when H is finitely generated and projective over R . A generalization of [2] for Hopf algebras over a Dedekind domain R was proved by Chen and Nichols, under the technical condition that U is R -closed in H° (see [3, Theorem 5]). This condition guarantees that every U -locally finite is rational. However, it is not evident that [3, Theorem 5] generalizes [5, Theorem 9.4.9].

We show that the ideas used in the proof of [5, Theorem 9.4.9], together with our results on rational modules and comodules, can be combined to get a duality theorem for Hopf algebras over a noetherian commutative ring R which generalizes both [5, Theorem 9.4.9] and [3, Theorem 5] and, hence, [2, Theorem 2.1].

In Section 4 we introduce a class of R -algebras $P_\ell\text{Alg}_R$ (in case R is noetherian) which satisfy the property that $A^\circ \subset R^A$ is an R -pure submodule for each $A \in P_\ell\text{Alg}_R$ and, hence, the canonical pairing $(A^\circ, A, \langle -, - \rangle)$ is a rational pairing. We give several examples of such algebras, among them the polynomial algebra $R[x_1, \dots, x_n]$ and the algebra of Laurent polynomials $R[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$.

1. PRELIMINARIES AND BASIC NOTIONS

In this paper R is a commutative ring with unit. Let A be an associative R -algebra with unit. The category of all left A -modules is denoted by ${}_A\mathcal{M}$. As usual, the notation $X \in \mathcal{C}$, where \mathcal{C} is a category, means X is an object of \mathcal{C} . By \otimes we denote the tensor product \otimes_R unless otherwise explicitly stated. Moreover, if $\pi \in S_n$ (the symmetric group on n symbols) then τ_π is the canonical isomorphism

$$\tau_\pi : M_1 \otimes \dots \otimes M_n \longrightarrow M_{\pi(1)} \otimes \dots \otimes M_{\pi(n)}$$

Let M, X be R -modules. If N is an R -submodule of M then N is called X -pure if $N \otimes X \subseteq M \otimes X$. The inclusion $N \subseteq M$ is called pure if N is X -pure for all R -modules X . Unless otherwise stated, pure, projective and flat mean pure, projective and flat in ${}_R\mathcal{M}$.

Let \mathcal{A} be a Grothendieck category. A *preradical* for \mathcal{A} is a subfunctor of the identity endofunctor $id_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$. We follow [9] for categorical basic notions.

Definition 1.1. An R -coalgebra is an R -module C together with two homomorphisms of R -modules

$$\Delta : C \rightarrow C \otimes C \text{ (comultiplication) and } \epsilon : C \rightarrow R \text{ (counit)}$$

such that

$$\begin{aligned} (id_C \otimes \Delta) \circ \Delta &= (\Delta \otimes id_C) \circ \Delta \quad \text{and} \\ (id_C \otimes \epsilon) \circ \Delta &= (\epsilon \otimes id_C) \circ \Delta = id_C \end{aligned}$$

Definition 1.2. A right C -comodule is an R -module together with an R -homomorphism

$$\rho_M : M \rightarrow M \otimes C$$

such that

$$\begin{aligned} (id_M \otimes \Delta) \circ \rho_M &= (\rho_M \otimes id_C) \circ \rho_M \quad \text{and} \\ (id_M \otimes \epsilon) \circ \rho_M &= id_M. \end{aligned}$$

Let M, N be right C -comodules. A homomorphism of R -modules $f : M \rightarrow N$ is said to be a comodule morphism (or C -colinear) if $\rho_N \circ f = (f \otimes id) \circ \rho_M$. By $\text{Hom}^C(M, N)$ we denote the R -module of all colinear maps from M to N . The right C -comodules with the C -colinear maps between them constitute an additive category denoted by \mathcal{M}^C . In case that C is a flat R -module, \mathcal{M}^C is a Grothendieck category (see [13, Corollary 3.15])

For basic notions on coalgebras and comodules over commutative rings we refer to [4, 13].

2. RATIONAL MODULES

In [4], the theory of rational modules is developed under the assumption of the existence of the there-called rational pairing. The main example exhibited there was (C^*, C) with C an R -projective coalgebra. Our aim is to deal with the finite dual of an R -algebra, so in this section we provide a weaker definition of rational pairing (in order to cover the finite dual example) which also implies the results on rational modules developed in [4].

2.1. Rational Systems. Let A, P be R -modules and let

$$\langle -, - \rangle : P \times A \rightarrow R$$

be a bilinear form. For every R -module M , define the R -linear map

$$\begin{aligned} \alpha_M : M \otimes P &\longrightarrow \text{Hom}_R(A, M) \\ m \otimes p &\longmapsto [a \mapsto m \langle p, a \rangle] \end{aligned}$$

Proposition 2.1. *In the previous situation the following statements are equivalent:*

- (1) α_M is injective
(2) If $\sum m_i \otimes p_i \in M \otimes P$, then $\sum m_i \otimes p_i = 0$ if and only if for every $a \in A$, $\sum m_i \langle p_i, a \rangle = 0$.

Proof. Note that for each R -module M ,

$$\ker(\alpha_M) = \left\{ \sum m_i \otimes p_i \mid \sum \langle p_i, - \rangle m_i = 0 \right\}.$$

□

Definition 2.2. The three-tuple $(P, A, \langle -, - \rangle)$ is a *rational system* if α_M is injective for every R -module M .

Remark 2.3. By [4, Proposition 2.3] and Proposition 2.1, a rational system as defined in [4, Definition 2.1] is a rational system in the present setting.

Remark 2.4. Let $(P, A, \langle -, - \rangle)$ be a rational system. Let M be an R -module and let N be an R -submodule of M . Consider the following commutative diagram

$$\begin{array}{ccc} N \otimes P & \xrightarrow{\alpha_N} & \text{Hom}_R(A, N) \\ i_N \otimes id_P \downarrow & & \downarrow i \\ M \otimes P & \xrightarrow{\alpha_M} & \text{Hom}_R(A, M) \end{array}$$

Note that α_N is injective since the three-tuple $(P, A, \langle -, - \rangle)$ is a rational system, and $i : \text{Hom}_R(A, N) \rightarrow \text{Hom}_R(A, M)$, $g \mapsto i_N \circ g$ is injective. Hence $i_N \otimes id_P$ is injective. Since M was arbitrary in ${}_R\mathcal{M}$ we conclude that P should be flat as an R -module.

The following Proposition replaces [4, Proposition 2.2] in order to show that the canonical comodule structure over a rational module is pseudocoassociative

Proposition 2.5. *If $(P, A, \langle -, - \rangle)$ and $(Q, B, \langle -, - \rangle)$ are rational systems, then the induced pairing*

$$[-, -] : P \otimes Q \times A \otimes B \rightarrow R$$

defined by

$$\left[\sum p_i \otimes q_i, \sum a_j \otimes b_j \right] = \sum \langle p_i, a_j \rangle \langle q_i, b_j \rangle$$

is a rational system.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} M \otimes P \otimes Q & \xrightarrow{\beta_M} & \text{Hom}_R(A \otimes B, M) \\ \downarrow \alpha_{M \otimes P} & & \downarrow \eta \\ \text{Hom}_R(B, M \otimes P) & \xrightarrow{(\alpha_M)_*} & \text{Hom}_R(B, \text{Hom}_R(A, M)) \end{array}$$

where η is the adjunction isomorphism, β_M is the mapping analogous to α with respect to the pairing $[-, -]$ and $(\alpha_M)_*$ is the homomorphism induced by α_M . This last morphism is monic and $\alpha_{M \otimes P}$ is monic. Therefore, β_M is a monomorphism. □

For each set S , let R^S denote the R -module of all maps from S to R . If α_R is injective, then P is isomorphic to a submodule of $A^* = \text{Hom}_R(A, R) \subseteq R^A$. We identify the R -module P with its image in R^A , so every $p \in P$ is identified with the R -linear map $\langle p, - \rangle : A \rightarrow R$.

Definition 2.6. We say P is *mock-projective* (relative to A and $\langle -, - \rangle$) if α_R is injective and for every $p_1, \dots, p_n \in P$ there are $a_1, \dots, a_m \in A$ and $g_1, \dots, g_m \in R^A$ such that for every $i = 1, \dots, n$, $p_i = \sum \langle p_i, a_l \rangle g_l$.

Proposition 2.1 can be improved under the assumption that P is mock-projective:

Proposition 2.7. *Assume P is mock-projective. If $M \in {}_R\mathcal{M}$, then $P \subseteq R^A$ is M -pure if and only if α_M is injective. Therefore, $(P, A, \langle -, - \rangle)$ is a rational system if and only if $P \subseteq R^A$ is pure.*

Proof. Assume $P \subseteq R^A$ is M -pure and let $\sum m_i \otimes p_i \in M \otimes P$. Assume $\sum \langle p_i, a \rangle m_i = 0$ for all $a \in A$. By Definition 2.6 there are $a_1, \dots, a_m \in A$ and $g_1, \dots, g_m \in R^A$ such that

$$p_i = \sum \langle p_i, a_l \rangle g_l, \text{ for each } i = 1, \dots, n.$$

As $\sum m_i \otimes p_i \in M \otimes R^A$ we have

$$\begin{aligned} \sum m_i \otimes p_i &= \sum m_i \otimes \sum \langle p_i, a_l \rangle g_l = \sum m_i \langle p_i, a_l \rangle \otimes g_l \\ &= \sum \left(\sum m_i \langle p_i, a_l \rangle \right) \otimes g_l = \sum 0 \otimes g_l = 0, \end{aligned} \tag{1}$$

and by purity $\sum m_i \otimes p_i = 0$ as an element in $M \otimes P$. By Proposition 2.1 α_M is injective.

Assume now α_M is injective. The diagram

$$\begin{array}{ccc} M \otimes P & \xrightarrow{\alpha_M} & \text{Hom}_R(A, M) \\ \text{id} \otimes i \downarrow & & \downarrow i \\ M \otimes R^A & \xrightarrow{\beta_M} & M^A \end{array}$$

is commutative where i denotes inclusion and $\beta_M(m \otimes f)(a) = mf(a)$. Since $i \circ \alpha_M$ is injective, we get $\text{id} \otimes i$ is injective and $P \subseteq R^A$ is M -pure. \square

2.2. Rational Pairings. Let (C, Δ, ϵ) be an R -coalgebra and let (A, m, u) be an R -algebra.

Definition 2.8. A *rational pairing* is a rational system $(C, A, \langle -, - \rangle)$ where C is an R -coalgebra, A is an R -algebra and the map $\varphi : A \rightarrow C^*$ given by $\varphi(a)(c) = \langle c, a \rangle$ is a homomorphism of R -algebras. This is equivalent to require

$$\begin{aligned} \langle -, - \rangle \circ (\text{id}_C \otimes m) &= (\langle -, - \rangle \otimes \langle -, - \rangle) \circ \tau_{(23)} \circ (\Delta \otimes \text{id}_A^{\otimes 2}) \\ \text{and } \epsilon &= \langle -, - \rangle \circ (\text{id}_C \otimes u) = \langle -, 1 \rangle. \end{aligned} \tag{2}$$

Now we can parallel the definition and properties of rational modules in [4]. The proofs are formally the same as in [4]. We include some of them for convenience of the reader.

Definition 2.9. Let $T = (C, A, \langle -, - \rangle)$ be a rational pairing. An element m in a left A -module M is called *rational* (with respect to the pairing T) if there exist finite subsets $\{m_i\} \subseteq M$ and $\{c_i\} \subseteq C$ such that $am = \sum m_i \langle c_i, a \rangle$ for every $a \in A$. The subset $\{(m_i, c_i)\} \subseteq M \times C$ is called a *rational set of parameters for m* (with respect to the pairing T). The subset $\text{Rat}^T(M)$ of M consisting of all rational elements of M is clearly an R -submodule of M . A left A -module is called *rational* (with respect to the pairing T) if $M = \text{Rat}^T(M)$. The full subcategory of ${}_A\mathcal{M}$ whose objects are all the rational (with respect to the pairing T) left A -modules will be denoted by $\text{Rat}^T({}_A\mathcal{M})$. We use the notation Rat instead of Rat^T when the rational pairing T is clear from the context.

Remark 2.10. As a consequence of Proposition 2.1, if $\{(m_i, c_i)\} \subseteq M \times C$ is a rational set of parameters for m and $\sum m_i \otimes c_i = \sum n_j \otimes d_j$, then $\{(n_j, d_j)\} \subseteq M \times C$ is a rational set of parameters for m . In fact a rational set of parameters for m can be viewed as a representative of an element $\sum m_i \otimes c_i \in M \otimes C$.

Following [12], we will denote by $\sigma[{}_A C]$ the full subcategory of ${}_A\mathcal{M}$ consisting of all the left A -modules subgenerated by ${}_A C$. This means that a left A -module belongs to $\sigma[{}_A C]$ if and only if it is isomorphic to a submodule of a factor module of a direct sum of copies of ${}_A C$.

Theorem 2.11. *Let $T = (C, A, \langle -, - \rangle)$ be a rational pairing. Then*

- (1) $\text{Rat} = \text{Rat}^T : {}_A\mathcal{M} \rightarrow {}_A\mathcal{M}$ is a left exact preradical.
- (2) The categories $\text{Rat}({}_A\mathcal{M})$ and \mathcal{M}^C are isomorphic.
- (3) $\text{Rat}({}_A\mathcal{M}) = \sigma[{}_A C]$.

Proof. The proofs of these facts are formally the same that [4, Propositions 2.9 and 2.10, Theorems 3.12 and 3.13], using Propositions 2.1 and 2.5 instead of [4, Propositions 2.3 and 2.2] \square

The isomorphism given in Theorem 2.11 is defined in terms of sets of rational parameters: if $M \in \text{Rat}({}_A\mathcal{M})$ then the structure of right C comodule is $\omega_M(m) = \sum m_i \otimes c_i$ where $\{(m_i, c_i)\}$ is a set of rational parameters, if $(M, \delta_M) \in \mathcal{M}^C$ and $\delta_M(m) = \sum m_i \otimes c_i$, then $\{(m_i, c_i)\}$ is a set of rational parameters for m . See [4, Propositions 3.5 and 3.11] for details.

Sweedler's Σ -notation can be introduced in terms of sets of rational parameters. Let $T = (C, A, \langle -, - \rangle)$ be a rational pairing. We have the injective R -linear map $\alpha_R : C \rightarrow A^*$ defined by

$$\begin{aligned} \alpha_R : C &\longrightarrow A^* \\ c &\longmapsto [a \mapsto \langle c, a \rangle] \end{aligned}$$

Let us regard A^* as a left A -module via $(a\lambda)(b) = \lambda(ba)$ and let us identify C with $\alpha_R(C)$. As in [4, Proposition 3.2], $C = \text{Rat}({}_A A^*)$. Note that the set of rational parameters for $c \in C$ is given by $\Delta(c)$. If $\{(c_1, c_2)\}_{(c)} = \{(c_1, c_2)\} \subseteq C \times C$ represents a set of rational parameters of $c \in C$ then the comultiplication can be represented as

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = \sum c_1 \otimes c_2$$

Note that (2) means

$$\langle c, ab \rangle = \sum \langle c_1, a \rangle \langle c_2, b \rangle$$

in Σ -notation. Analogously, Let $(M, \delta_M) \in \mathcal{M}^C$. The set of rational parameters for $m \in M$ is given by $\delta_M(m)$. We are going to use Sweedler's Σ -notation on C -comodules, i.e., $\delta_M(m) = \sum_{(m)} m_0 \otimes m_1$ where $\{(m_0, m_1)\}_{(m)} \subseteq M \times C$ represents an arbitrary set of rational parameters for $m \in M$.

2.3. The finite dual coalgebra A° . *In this subsection, the commutative ring R is assumed to be noetherian.* Let A be an R -algebra. Recall that the canonical structure of an A -bimodule on A^* is given by

$$(af)(b) = f(ba) \text{ and } (fa)(b) = f(ab) \text{ for } f \in A^* \text{ and } a, b \in A. \quad (3)$$

Let $A^\circ = \{f \in A^* \mid Af \text{ is finitely generated as an } R\text{-module}\}$. Then by [1, Proposition 2.6],

$$\begin{aligned} A^\circ &= \{f \in A^* \mid Af \text{ is finitely generated as an } R\text{-module}\} \\ &= \{f \in A^* \mid fA \text{ is finitely generated as an } R\text{-module}\} \\ &= \{f \in A^* \mid \ker f \text{ contains an } R\text{-cofinite ideal of } A\} \\ &= \{f \in A^* \mid \ker f \text{ contains an } R\text{-cofinite left ideal of } A\} \\ &= \{f \in A^* \mid \ker f \text{ contains an } R\text{-cofinite right ideal of } A\}. \end{aligned} \quad (4)$$

By [1, 2.3], A° is an A -subbimodule of A^* .

Remark 2.12. If A is finitely generated and projective in ${}_R\mathcal{M}$ then $A^\circ = A^*$ is pure in R^A . In this case $(A \otimes A)^* \simeq A^* \otimes A^*$.

By [1, Theorem 2.8], A° is an R -coalgebra whenever A° is a pure submodule of R^A . In this section we will prove that $(A^\circ, A, \langle -, - \rangle)$ is a rational pairing. So we are going to describe the right A° -comodules as rational left A -modules. This applies in particular when R is hereditary.

Next lemma is used to prove the rationality of the three-tuple $(A^\circ, A, \langle -, - \rangle)$.

Lemma 2.13. *Let S be a set and let $f_1, \dots, f_n \in R^S$. Then there exist $s_1, \dots, s_m \in S$ and $g_1, \dots, g_m \in R^S$ such that*

$$f_i = \sum g_l f_l(s_l), \text{ for each } i = 1, \dots, n.$$

In particular, if A, P are R -modules and $\langle -, - \rangle : P \times A \rightarrow R$ a bilinear form such that α_R is injective, then P is mock-projective.

Proof. Define

$$\begin{aligned} \underline{f} : S &\longrightarrow R^n \\ s &\longmapsto (f_1(s), \dots, f_n(s)), \end{aligned}$$

and consider the R -submodule $M \subseteq R^n$ generated by the elements $\underline{f}(s)$, $s \in S$. Since R is noetherian, M is finitely generated, and there are $s_1, \dots, s_m \in S$ for which $M = \sum R\underline{f}(s_l)$. For any $s \in S$ we have the set

$$X(s) = \{(r_1, \dots, r_m) \in R^m \mid \underline{f}(s) = \sum r_l \underline{f}(s_l)\} \neq \emptyset.$$

For each $s \in S$, choose $r(s) \in X(s)$. This gives a map $r : S \rightarrow R^m$. Now, it is clear that there are maps $g_1, \dots, g_m \in R^S$ such that $r = (g_1, \dots, g_m)$. Finally, $\underline{f}(s) = \sum g_l(s) \underline{f}(s_l)$. But this is an equality in R^n , whence, for each $i = 1, \dots, n$ we obtain $f_i(s) = \sum g_l(s) f_i(s_l)$, and hence $f_i = \sum f_i(s_l) g_l$. \square

Remark 2.14. Assume α_R to be injective. Let M be an R -module. It follows directly from Lemma 2.13 and Proposition 2.7 that $P \subseteq R^A$ is M -pure if and only if α_M is injective.

Proposition 2.15. *Let A be an R -algebra and assume A° is pure in R^A . Then*

- (1) A° is an R -coalgebra. If in addition A is a bialgebra (resp. Hopf algebra) then A° is a bialgebra (resp. Hopf algebra).
- (2) Let B be an R -algebra such that B° is pure in R^B . For every morphism of R -algebras $\varphi : A \rightarrow B$ we have

$$\varphi^*(B^\circ) \subseteq A^\circ,$$

moreover $\varphi^\circ := \varphi^*_{|_{B^\circ}}$ is an R -coalgebra morphism.

- (3) Let $C \subseteq A^\circ$ be a subcoalgebra. Consider the bilinear form

$$\begin{aligned} \langle -, - \rangle : C \times A &\longrightarrow R \\ (f, a) &\longmapsto \langle f, a \rangle = f(a). \end{aligned}$$

Then $(C, A, \langle -, - \rangle)$ is a rational pairing.

- (4) If $(C, A, \langle -, - \rangle)$ is a rational pairing then C is an R -subcoalgebra of A° .

Proof. (1) By [1, Theorem 2.8] A° is a coalgebra. The rest of the first statement is similar to the argument in [5, 9.1.3] due to the fact that over noetherian rings, submodules of finitely generated modules are finitely generated.

(2) Let $f \in B^\circ$ and assume $I \subseteq B$ to be a cofinite left ideal contained in $\ker f$. Since R is noetherian, it is easy to check that $\varphi^{-1}(I) \subseteq A$ is a left cofinite ideal contained in $\ker(f \circ \varphi) = \ker(\varphi^*(f))$. A diagram chase shows that φ° is an R -coalgebra map and the second statement is proved.

(3) Equation (2) is clearly satisfied. By [13, 3.3] C is pure in A° , so Proposition 2.7 and Lemma 2.13 give injectivity of α_M for each R -module M .

(4) Since α_R is injective C can be viewed as an R -submodule of A^* . Let us see that $C \subseteq A^\circ$. If $c \in C$ and $\Delta(c) = \sum c_1 \otimes c_2$, then $ac = \sum c_1 \langle c_2, a \rangle$ by Eq. (2) and (3). It follows that Ac is finitely generated by $\{c_1\}_{(c)}$ as R -module for every $c \in C$. Moreover, Eq. (2) easily implies that the comultiplication and the counit on C are induced from A° . \square

Corollary 2.16. *Let A be an R -algebra and assume A° is pure in R^A . Consider the bilinear form*

$$\begin{aligned} [-, -] : A^\circ \times A^{\circ*} &\longrightarrow R \\ (f, \lambda) &\longmapsto [f, \lambda] = \lambda(f) \end{aligned}$$

for all $f \in A^\circ$ and $\lambda \in A^{\circ*}$. Then $(A^\circ, A^{\circ*}, [-, -])$ is a rational pairing.

Proof. Assume $\sum [f_i, \lambda] m_i = 0$ for all $\lambda \in A^{\circ*}$. Let $a \in A$ and consider

$$\begin{aligned} \langle -, a \rangle : A^\circ &\longrightarrow R \\ f &\longmapsto \langle f, a \rangle = f(a) \end{aligned}$$

for all $f \in A^\circ$. Then $\sum [f_i, \langle -, a \rangle] m_i = 0$ and so $\sum \langle f_i, a \rangle m_i = 0$ for all $a \in A$, which implies $\sum m_i \otimes f_i = 0$, since the pairing $(A^\circ, A, \langle -, - \rangle)$ is a rational pairing by Proposition 2.15. \square

As a consequence of Theorem 2.11, Propositions 2.15 and 2.1, and Corollary 2.16 we have:

Theorem 2.17. *Let A be an R -algebra such that A° is pure in R^A . Let $\varphi : A \rightarrow A^{\circ*}$ be the canonical morphism and let $\varphi_* : {}_{A^{\circ*}}\mathcal{M} \rightarrow {}_A\mathcal{M}$ be the restriction of scalars functor. Then*

- (1) *The functors $(-)^{A^\circ} : \text{Rat}({}_A\mathcal{M}) \rightarrow \mathcal{M}^{A^\circ}$ and $(-)^{A^\circ} : \text{Rat}({}_{A^{\circ*}}\mathcal{M}) \rightarrow \mathcal{M}^{A^\circ}$ are isomorphisms of categories.*
- (2) *$\text{Rat}({}_A\mathcal{M}) = \sigma[{}_A A^\circ]$ and $\text{Rat}({}_{A^{\circ*}}\mathcal{M}) = \sigma[{}_{A^{\circ*}} A^\circ]$*
- (3) *The following diagram of functors is commutative*

$$\begin{array}{ccc} {}_{A^{\circ*}}\mathcal{M} & \xrightarrow{\varphi_*} & {}_A\mathcal{M} \\ \text{Rat}^{T'} \downarrow & & \downarrow \text{Rat}^T \\ \text{Rat}^{T'}({}_{A^{\circ*}}\mathcal{M}) & & \text{Rat}^T({}_A\mathcal{M}) \\ & \searrow \cong & \swarrow \cong \\ & \mathcal{M}^{A^\circ} & \end{array}$$

where $T = (A^\circ, A, \langle -, - \rangle)$ and $T' = (A^\circ, A^{\circ*}, [-, -])$ are the canonical pairings.

3. AN APPLICATION: BLATTNER–MONTGOMERY DUALITY.

We are going to prove a Blattner–Montgomery like theorem (see [5, Theorem 9.4.9]) when R is any commutative noetherian ring and $(H, m, u, \Delta, \epsilon, S)$ is a Hopf algebra such that H° is pure in R^H (this condition holds if H is R -projective and H° is pure in H^*). When R is a Dedekind domain, we obtain as a corollary the version given in [3].

We are going to recall some definitions and notations. A left H -module algebra is an R -algebra (A, m_A, u_A) such that A is a left H -module and m_A, u_A are H -module maps.

This means in terms of Sweedler's notation:

$$h(ab) = \sum (h_1a)(h_2b) \quad \text{and} \quad h1_A = \epsilon(h)1_A$$

Analogously A is a right H -comodule algebra if A is a right H -comodule (via $\rho : A \rightarrow A \otimes H$) and m_A, u_A are H -comodule maps, i.e.,

$$\rho(ab) = \sum a_0b_0 \otimes a_1b_1 \quad \text{and} \quad \rho(1_A) = 1_A \otimes 1_H.$$

Let A be a left H -module algebra where the H -module action is denoted by $w_A : H \otimes A \rightarrow A$. The following composition of maps

$$\begin{array}{ccc} (A \otimes H) \otimes (A \otimes H) & \xrightarrow{id \otimes \Delta \otimes id^{\otimes 2}} & A \otimes H \otimes H \otimes A \otimes H \\ \downarrow m_{A\#H} & & \downarrow \tau_{(34)} \\ & & A \otimes H \otimes A \otimes H \otimes H \\ & & \downarrow id \otimes w_A \otimes id^{\otimes 2} \\ A \otimes H & \xleftarrow{m_A \otimes m} & A \otimes A \otimes H \otimes H \end{array}$$

provides a structure of an associative R -algebra on $A \otimes H$. This algebra is called the smash product of A and H , and it is denoted by $A\#H$. In Sweedler's notation, the multiplication can be viewed as follows:

$$(a\#h)(b\#k) = \sum a(h_1b)\#h_2k,$$

where $a\#h = a \otimes h$. Since H° is pure in R^H , by Proposition 2.15 we have $(H^\circ, \Delta^\circ, \epsilon^\circ, m^\circ, u^\circ, S^\circ)$ is a Hopf algebra. The left (and right) action of H on H° described in (3) makes H° a left (and right) H -module algebra (see [5, Example 4.1.10]). In order to make the notation consistent with the literature we denote the left (resp. right) action of H on H° by \rightharpoonup (resp. \leftharpoonup). Let U be a Hopf subalgebra of H° (by definition $U \subseteq H^\circ$ should be pure, see [13, 3.3]). Then U is also a left H -module algebra. The action can be described as

$$H \otimes U \xrightarrow{id \otimes m^\circ} H \otimes U \otimes U \xrightarrow{\tau_{(123)}} U \otimes H \otimes U \xrightarrow{\langle -, - \rangle \otimes id} U,$$

and in Sweedler's notation,

$$h \rightharpoonup f = \sum f_1 \langle f_2, h \rangle$$

which allows the construction of $U\#H$.

Analogously H is a left (resp. right) U -module algebra via

$$\begin{array}{c} U \otimes H \xrightarrow{id \otimes \Delta} U \otimes H \otimes H \xrightarrow{\tau_{(23)}} U \otimes H \otimes H \xrightarrow{\langle -, - \rangle \otimes id} H, \\ \left(\text{resp. } H \otimes U \xrightarrow{\Delta \otimes id} H \otimes H \otimes U \xrightarrow{\tau_{(123)}} U \otimes H \otimes H \xrightarrow{\langle -, - \rangle \otimes id} H, \right) \end{array}$$

This action is denoted by \rightharpoonup (resp. \leftharpoonup), and Sweedler's notation means

$$f \rightharpoonup h = \sum h_1 \langle f, h_2 \rangle \quad \left(\text{resp. } h \leftharpoonup f = \sum \langle f, h_1 \rangle h_2 \right)$$

and we can construct $H\#U$.

These actions and constructions are analogous to the ones over a field. See [5, 1.6.5, 1.6.6, 4.1.10] for details. Following [5, Definition 9.4.1] we have the following maps:

$$\begin{aligned}\lambda : H\#U &\longrightarrow \text{End}_R(H) \\ h\#f &\longmapsto [k \mapsto h(f \rightharpoonup k)] \\ \rho : U\#H &\longrightarrow \text{End}_R(H) \\ f\#h &\longmapsto [k \mapsto (k \leftarrow f)h]\end{aligned}$$

Lemma 3.1. *λ is an algebra morphism and ρ is an anti-algebra morphism. If also H has bijective antipode, then λ and ρ are injective.*

Proof. Following [5, Lemma 9.4.2], we consider λ , as the argument for ρ is similar. Straightforward computations show that λ is an algebra morphism. To see the injectivity we define $\lambda' : H\#U \rightarrow \text{End}_R(H)$ and $\psi : \text{End}_R(H) \rightarrow \text{End}_R(H)$ as follows:

$$\begin{aligned}\lambda'(h\#f)(k) &= \langle f, k \rangle h \\ \psi(\sigma) &= (\sigma \otimes \bar{S}) \circ \tau \circ \Delta\end{aligned}$$

where \bar{S} is the composition inverse of S . We can see that $\lambda' = \psi \circ \lambda$ as in [5, Lemma 9.4.2]. Moreover, $(U, H, \langle -, - \rangle)$ is a rational pairing by Proposition 2.15, so λ' is injective. \square

We say that U satisfies the RL-condition with respect to H if $\rho(U\#1) \subseteq \lambda(H\#U)$.

Let (A, ρ_A) be a right U -comodule algebra. Then A is a left H -module algebra with action

$$H \otimes A \xrightarrow{id \otimes \rho_A} H \otimes A \otimes U \xrightarrow{\tau_{(132)}} A \otimes U \otimes H \xrightarrow{id \otimes \langle -, - \rangle} A, \quad (5)$$

or in Sweedler's notation,

$$ha = \sum a_0 \langle a_1, h \rangle.$$

Theorem 3.2. *Let H be a Hopf algebra such that H° is pure in R^H , and let U be a Hopf subalgebra of H° . Assume that both H and U have bijective antipodes and U satisfies the RL-condition with respect to H . Let A be a right U -comodule algebra. Let U act on $A\#H$ by acting trivially on A and via \rightharpoonup on H . Then*

$$(A\#H)\#U \simeq A \otimes (H\#U)$$

Proof. The computations in [5, Theorem 9.4.9 and Lemma 9.4.10] remain valid here once we have proved Lemma 3.1. \square

Remark 3.3. Let R be a Dedekind domain and assume that A is an U -locally finite left H -module algebra and that U is R -closed in H° in the sense of [3]. By [3, Lemma 4], A is a rational left H -module which implies, by Theorem 2.17, that A is a right U -comodule algebra. Therefore, [3, Theorem 5] follows as a corollary of Theorem 3.2.

Remark 3.4. If H is cocommutative then U satisfies the RL-condition (see [5, 9.4.7 Example]), so examples in subsections 4.2 and 4.3 and Example 4.7 satisfy the RL-condition. So let G be a group such that $R[G]^\circ$ is pure in $R^{R[G]}$ (if G is either finite or R is hereditary, this condition is satisfied), and let A be an R -algebra such that G acts as automorphisms on A . Then we have

$$(A \# R[G]) \# R[G]^\circ \cong A \otimes (R[G] \# R[G]^\circ)$$

4. EXAMPLES

In this section R is assumed to be noetherian. We are going to consider a class of R -algebras for which A° is pure in R^A (and hence A° has a structure of an R -coalgebra). For every R -algebra A let \mathcal{L}_{cof} be the linear topology on A whose basic neighborhoods of 0 are the R -cofinite left ideals, i.e.,

$$\mathcal{L}_{\text{cof}} = \{I \leq {}_A A \mid A/I \text{ is finitely generated as an } R\text{-module}\}$$

4.1. The category $\text{P}_\ell \text{Alg}_R$.

Definition 4.1 (Property \mathbf{P}_ℓ). An R -algebra A has property \mathbf{P}_ℓ in case the set

$$\mathcal{P}_{\text{cof}} = \{I \leq {}_A A \mid A/I \text{ is finitely generated and projective as an } R\text{-module}\}$$

is a basis for \mathcal{L}_{cof} , i.e. for every left cofinite ideal I of A , there exists a left ideal $I_0 \subseteq I$, with A/I_0 finitely generated and projective as an R -module.

We denote by $\text{P}_\ell \text{Alg}_R$ the full subcategory of Alg_R whose objects are all R -algebras which have property \mathbf{P}_ℓ .

Proposition 4.2. *If $A \in \text{P}_\ell \text{Alg}_R$, then $(A^\circ, A, \langle -, - \rangle)$ is a rational system.*

Proof. Let $M \in {}_R \mathcal{M}$ and let $\sum m_i \otimes f_i \in M \otimes A^\circ$. Assume $\sum m_i \langle f_i, a \rangle = 0$ for every $a \in A$. Notice that for each i , $f_i \in (A/I_i)^*$ for some cofinite ideal I_i of A . Put $J = \bigcap_i I_i$. Then J is cofinite. Since $A \in \text{P}_\ell \text{Alg}_R$ there exists some ideal $J_0 \subseteq J$ such that A/J_0 is finitely generated and projective as R -module (and so $(A/J_0)^{**} \simeq A/J_0$). Let $\{a_\lambda + J_0, \phi_\lambda\}_\Lambda$ be a finite dual basis for $(A/J_0)^*$. Since $f_i \in (A/J_0)^*$ for all i , we get

$$\begin{aligned} \sum m_i \otimes f_i &= \sum m_i \otimes \sum \langle f_i, a_\lambda + J_0 \rangle \phi_\lambda \\ &= \sum \left(\sum \langle f_i, a_\lambda + J_0 \rangle m_i \right) \otimes \phi_\lambda \\ &= \sum 0 \otimes \phi_\lambda = 0 \quad (\text{notice that } f_i(J_0) = 0). \end{aligned}$$

Hence $(A^\circ, A, \langle -, - \rangle)$ is a rational system by Proposition 2.1. \square

Corollary 4.3. *If $A \in \text{P}_\ell \text{Alg}_R$ then*

- (1) A° is an R -coalgebra. If in addition A is a bialgebra (resp. Hopf algebra) then A° is a bialgebra (resp. Hopf algebra).
- (2) $(A^\circ, A, \langle -, - \rangle)$ is a rational pairing.

Proof. It follows directly from Propositions 2.7, 2.15 and 4.2. \square

Remark 4.4. By (4), the proof of Proposition 4.2 remains true if we replace left ideals in property \mathbf{P}_ℓ by right or two sided ones. So we can speak of property \mathbf{P}_r or property \mathbf{P} .

Remark 4.5. If $A \in \mathbf{P}_\ell \text{Alg}_R$, then

$$\begin{aligned} A^{\circ*} &= \text{Hom}_R(A^\circ, R) = \text{Hom}_R\left(\varinjlim_{I \in \mathcal{L}_{\text{cof}}} (A/I)^*, R\right) \\ &\simeq \text{Hom}_R\left(\varinjlim_{I \in \mathcal{P}_{\text{cof}}} (A/I)^*, R\right) \\ &\simeq \varprojlim_{I \in \mathcal{P}_{\text{cof}}} (A/I)^{**} \simeq \varprojlim_{I \in \mathcal{P}_{\text{cof}}} A/I \\ &\simeq \varinjlim_{I \in \mathcal{L}_{\text{cof}}} A/I = \hat{A}, \end{aligned}$$

which means that $A^{\circ*} \simeq \hat{A}$, the completion of A with respect to the cofinite topology.

Proposition 4.6. *Let A be in $\mathbf{P}_\ell \text{Alg}_R$ and let B be an R -algebra extension of A such that B is finitely generated and projective in \mathcal{M}_A . Then B belongs to $\mathbf{P}_\ell \text{Alg}_R$.*

Proof. Let $J \leq B$ be a cofinite left ideal. Then $J \cap A \leq A$ is a cofinite left ideal because R is noetherian. Since A belongs to $\mathbf{P}_\ell \text{Alg}_R$, there exists $I_0 \subseteq J \cap A$ such that A/I_0 is finitely generated and projective in ${}_R \mathcal{M}$. By the natural isomorphism

$$\text{Hom}_R\left(B \otimes_A \frac{A}{I_0}, -\right) \cong \text{Hom}_A\left(B, \text{Hom}_R\left(\frac{A}{I_0}, -\right)\right)$$

$B \otimes_A \frac{A}{I_0}$ is finitely generated and projective in ${}_R \mathcal{M}$. Since $BI_0 \subseteq J$ and $B \otimes_A \frac{A}{I_0} \cong \frac{B}{BI_0}$ we get B is in $\mathbf{P}_\ell \text{Alg}_R$. \square

Example 4.7. Let G be a group. An R -algebra is called G -graded if for every $\sigma \in G$ there exists an R -submodule $A_\sigma \subseteq A$ such that $A = \bigoplus_{\sigma \in G} A_\sigma$ and $A_\sigma A_\tau \subseteq A_{\sigma\tau}$. If in addition $A_\sigma A_\tau = A_{\sigma\tau}$, A is called strongly graded. Let G be finite with neutral element e and let A be a strongly G -graded R -algebra. By [6, I.3.3 Corollary] it is clear that A is finitely generated and projective as right A_e -module, so if A_e is in $\mathbf{P}_\ell \text{Alg}_R$ then A also belongs to $\mathbf{P}_\ell \text{Alg}_R$. In particular, if A is in $\mathbf{P}_\ell \text{Alg}_R$ and G is a finite group, then a crossed product $A * G$ also belongs to $\mathbf{P}_\ell \text{Alg}_R$. Crossed products cover the following cases: if $A \in \mathbf{P}_\ell \text{Alg}_R$ then $A[G], A^t[G], AG \in \mathbf{P}_\ell \text{Alg}_R$ where $A[G]$ is the group algebra, $A^t[G]$ is the twisted group algebra and AG is the skew group algebra. See [7] for an introduction on crossed products.

Our aim is the proof of Theorem 4.10, which was shown in [10, Lemma 6.0.1.] for algebras over fields and in [3] for algebras over Dedekind domains. However we need some technical statements.

Lemma 4.8. *Let M and N be two R -modules and consider submodules $M' \subseteq M$ and $N' \subseteq N$. Assume M' to be N -pure and N' to be M -pure (this is in particular valid if M and N are flat in ${}_R \mathcal{M}$). Then*

$$M/M' \otimes N/N' \simeq (M \otimes N)/(M' \otimes N + M \otimes N').$$

Proof. By purity $M' \otimes N$ and $M \otimes N'$ are R -submodules of $M \otimes N$. Since the diagram

$$\begin{array}{ccc} M \otimes N & \longrightarrow & M/M' \otimes N \\ \downarrow & & \downarrow \\ M \otimes N/N' & \longrightarrow & M/M' \otimes N/N' \end{array}$$

is a pushout diagram, the result follows. \square

Proposition 4.9. *Let A, B be algebras in $\text{P}_\ell\text{Alg}_R$.*

- (1) *If $K \leq A \otimes B$ is a cofinite left ideal then there exist $I_0 \leq A$ and $J_0 \leq B$ such that A/I_0 and B/J_0 are finitely generated and projective in ${}_R\mathcal{M}$, and so that $I_0 \otimes B + A \otimes J_0 \subseteq K$.*
- (2) *The R -algebra $A \otimes B$ belongs to $\text{P}_\ell\text{Alg}_R$.*

Proof. Consider the canonical maps

$$\begin{aligned} \alpha : A &\longrightarrow A \otimes B \\ a &\longmapsto a \otimes 1 \end{aligned}$$

and

$$\begin{aligned} \beta : B &\longrightarrow A \otimes B \\ b &\longmapsto 1 \otimes b \end{aligned}$$

Put $I = \alpha^{-1}(K)$ and $J = \beta^{-1}(K)$. Since R is noetherian, I and J are cofinite left ideals of A and B , respectively. Since $A, B \in \text{P}_\ell\text{Alg}_R$ there exist $I_0 \subseteq I$ and $J_0 \subseteq J$ such that A/I_0 and B/J_0 are finitely generated and projective in ${}_R\mathcal{M}$. Let $K_0 = I_0 \otimes B + A \otimes J_0$. Since $I_0 \leq A$ and $J_0 \leq B$ are pure submodules we have $K_0 \subseteq K$ as desired.

By Lemma 4.8

$$\frac{A \otimes B}{K_0} \simeq \frac{A}{I_0} \otimes \frac{B}{J_0},$$

hence $(A \otimes B)/K_0$ is finitely generated and projective and $A \otimes B$ is in $\text{P}_\ell\text{Alg}_R$. \square

Theorem 4.10. *Let A, B be in $\text{P}_\ell\text{Alg}_R$. Then there is a canonical isomorphism $A^\circ \otimes B^\circ \simeq (A \otimes B)^\circ$.*

Proof. Since A, B are in $\text{P}_\ell\text{Alg}_R$, A° is pure in R^A and B° is pure in R^B by Propositions 4.2 and 2.7. So $A^\circ \otimes B^\circ \subseteq R^A \otimes R^B$. Let π be the morphism:

$$\begin{aligned} \pi : R^A \otimes R^B &\longmapsto R^{A \times B} \\ f \otimes g &\longmapsto [(a, b) \mapsto f(a)g(b)] \end{aligned}$$

By [1, Proposition 1.2] this map is injective, so the statement will be clear once we have seen $\pi(A^\circ \otimes B^\circ) = (A \otimes B)^\circ$. So let $f \otimes g \in A^\circ \otimes B^\circ$ and let $I \subseteq A$ and $J \subseteq B$ be left ideals contained in $\ker f$ and $\ker g$ respectively and such that $A/I, B/J$ are finitely generated and projective (they exist because A, B belong to $\text{P}_\ell\text{Alg}_R$). Since $I \subseteq A$ and $J \subseteq B$ are pure, by Lemma 4.8 $I \otimes B + A \otimes J \subseteq A \otimes B$ is a cofinite left ideal, which is contained in $\ker(\pi(f \otimes g))$. As $\pi(f \otimes g)$ is bilinear it is clear that $\pi(A^\circ \otimes B^\circ) \subseteq (A \otimes B)^\circ$.

Let $h \in (A \otimes B)^\circ$, and assume $K \subseteq A \otimes B$ to be a cofinite left ideal contained in $\ker h$. By Proposition 4.9 there exist left ideals $I_0 \leq A$ and $J_0 \leq B$ such that A/I_0 and B/J_0 are finitely generated and projective in ${}_R\mathcal{M}$ and so that $I_0 \otimes B + A \otimes J_0 \subseteq K$. By Lemma 4.8 there is an epimorphism

$$\frac{A}{I_0} \otimes \frac{B}{J_0} \rightarrow \frac{A \otimes B}{K} \rightarrow 0$$

which induces a monomorphism

$$0 \rightarrow \left(\frac{A \otimes B}{K} \right)^* \rightarrow \left(\frac{A}{I_0} \otimes \frac{B}{J_0} \right)^* \simeq \left(\frac{A}{I_0} \right)^* \otimes \left(\frac{B}{J_0} \right)^* \subseteq A^\circ \otimes B^\circ$$

So there exist elements $f_1, \dots, f_n \in (A/I_0)^* \subseteq A^\circ$ and $g_1, \dots, g_n \in (B/J_0)^* \subseteq B^\circ$ such that $\pi(\sum f_i \otimes g_i) = h$. This completes the proof. \square

We finish with some examples.

4.2. The R -bialgebra $R[x_1, \dots, x_n]^\circ$. By [1, Proposition 3.1], every cofinite ideal $I \leq R[x]$ contains a monic polynomial $f(x)$. Put $I_0 = (f(x)) \subseteq I$. Then $R[x]/I_0$ is finitely generated and projective (in fact free). Hence $R[x]$ is in $\text{P}_\ell\text{Alg}_R$ and so $R[x]^\circ$ is an R -coalgebra by Corollary 4.3. Moreover, $R[x_1, \dots, x_n]$ belongs to $\text{P}_\ell\text{Alg}_R$ by Proposition 4.9. There are two canonical bialgebra structures on $R[x_1, \dots, x_n]$. The first one comes from the semigroup algebra structure of $R[x_1, \dots, x_n]$ (i.e. every x_i is a group-like element), and the second one appears when we see $R[x_1, \dots, x_n]$ as the enveloping algebra of an abelian Lie algebra (i.e. every x_i is a primitive element). The latter one is a Hopf algebra structure. By Corollary 4.3, $R[x_1, \dots, x_n]^\circ$ is a bialgebra (resp. Hopf algebra).

It follows from Proposition 4.9 that if A belongs to $\text{P}_\ell\text{Alg}_R$ then $A[x_1, \dots, x_n]$ is in $\text{P}_\ell\text{Alg}_R$.

4.3. The Hopf R -algebra of Laurent polynomials.

Definition 4.11. A monic polynomial $q(x) \in R[x]$ is called reversible if $q(0)$ is a unit in R . An ideal $I \subseteq R[x, x^{-1}]$ is called reversible if it contains a reversible polynomial $q(x)$.

Lemma 4.12. *Let $q(x) \in R[x]$ be a reversible polynomial. Then*

$$R[x]/(q(x)) \simeq R[x, x^{-1}]/(q(x)).$$

Proof. Let $q(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ be a reversible polynomial (i.e. a_0 is a unit in R). Notice

$$R[x, x^{-1}]/(q(x)) \simeq R[x, y]/(xy - 1, q(x)).$$

Put $I = (xy - 1, q(x))$ and consider the R -linear map

$$\begin{aligned} \Psi : R[x] &\longrightarrow R[x, y]/I \\ x &\longmapsto x + I. \end{aligned}$$

Clearly Ψ is an R -algebra homomorphism and $\ker(\Psi) = I \cap R[x]$. Moreover, $I \cap R[x] = R[x]q(x)$. Clearly Ψ is surjective if and only if $y + I \in \text{im}(\Psi)$. Notice

$$\begin{aligned} yq(x) - x^{n-1}(xy - 1) &= a_0y + a_1yx + \cdots + a_{n-1}yx^{n-1} + x^{n-1} \\ &= a_0y + a_1 + a_2x + \cdots + a_{n-1}x^{n-2} + x^{n-1} \pmod{(I)} \end{aligned}$$

So

$$y = -a_0^{-1} [x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1] \pmod{(I)}$$

Hence $y \in \text{im}(\Psi)$ and we conclude that Ψ is surjective. \square

Proposition 4.13. (1) *Let $I \subseteq R[x, x^{-1}]$ be a reversible ideal. Then $R[x, x^{-1}]/I$ is finitely generated as an R -module.*

(2) *Let R be noetherian. Assume $R[x, x^{-1}]/I$ to be finitely generated as an R -module. Then I is reversible ideal.*

Proof. (1) Let $I \subseteq R[x, x^{-1}]$ be a reversible ideal. Then I contains a reversible polynomial $q(x)$. By Lemma 4.12, $R[x, x^{-1}]/(q(x)) \simeq R[x]/(q(x))$ which implies, by [1, Proposition 3.1]), that $R[x, x^{-1}]/(q(x))$ is finitely generated as an R -module. Therefore, $R[x, x^{-1}]/I$ is finitely generated as an R -module.

(2) Since $R[x]/(R[x] \cap I)$ embeds in the finitely generated R -module $R[x, x^{-1}]/I$, we get that $R[x]/(R[x] \cap I)$ is finitely generated as an R -module. By [1], there exists a monic polynomial $f_1(x) = a_0 + a_1x + \cdots + x^n \in I \cap R[x]$. We know $R[x, x^{-1}]$ is a Hopf R -algebra with antipode

$$\begin{aligned} S : R[x, x^{-1}] &\longrightarrow R[x, x^{-1}] \\ x &\longmapsto x^{-1}, \end{aligned}$$

Since S is bijective, $R[x, x^{-1}]/I \simeq R[x, x^{-1}]/S(I)$ as R -modules. So there exists a monic $f_2(x) = b_0 + \cdots + b_{m-1}x^{m-1} + x^m \in S(I) \cap R[x]$. Hence we have that $q(x) = x^m(f_1(x) + S(f_2(x))) \in I$. An easy computation shows that

$$q(x) = 1 + b_{m-1}x + \cdots + (b_0 + a_0)x^m + \cdots + a_{n-1}x^{n+m-1} + x^{n+m}$$

and I contains the reversible polynomial $q(x)$. By Lemma 4.12

$$R[x, x^{-1}]/(q(x)) \simeq R[x]/(q(x))$$

and so is finitely generated and projective (in fact free) as an R -module. \square

Theorem 4.14. *Let R be noetherian and let A be in $\text{P}_\ell\text{Alg}_R$. Then $A[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ belongs to $\text{P}_\ell\text{Alg}_R$. In particular $R[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]^\circ$ is a Hopf algebra.*

Proof. By Proposition 4.13 and Lemma 4.12 it is easy to see that $R[x, x^{-1}]$ is in $\text{P}_\ell\text{Alg}_R$, so the first statement follows from Proposition 4.9. Since $R[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ is a group algebra, the last assertion follows from Corollary 4.3. \square

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