Rational Modules for Corings

Jawad Y. Abuhlail
Mathematics Department
Birzeit University
P.O.Box 14, Birzeit - Palestine
jabuhlail@birzeit.edu

Abstract

The so called *dense pairings* were studied mainly by D. Radford in his work on coreflexive coalegbras over fields. They were generalized in a joint paper with J. Gómez-Torricillas and J. Lobillo to the so called *rational pairings* over a commutative ground ring R to study the interplay between the comodules of an R-coalgebra C and the modules of an R-algebra A that admits an R-algebra morphism $\kappa : A \to C^*$. Such pairings, satisfying the so called *α-condition*, were called in the author's dissertation *measuring α-pairings* and can be considered as the corner stone in his study of duality theorems for Hopf algebras over commutative rings. In this paper we lay the basis of the theory of rational modules of corings extending results on rational modules for coalgebras to the case of arbitrary ground rings. We apply these results mainly to categories of entwined modules (e.g. Doi-Koppinen modules, alternative Doi-Koppinen modules) generalizing results of Y. Doi , M. Koppinen and C. Menini et al.

Introduction

Let (H, A, C) be a right-right Doi-Koppinen structure over a commutative ring R, $\mathcal{M}(H)_A^C$ the corresponding category of Doi-Koppinen modules and $A\#^{op}C^*$ the Koppinen opposite smash product. If ${}_RC$ is flat, then $\mathcal{M}(H)_A^C$ is a Grothendieck category with enough injective objects. A sufficient, however not necessary, condition for $\mathcal{M}(H)_A^C$ to embed as a *full* subcategory of $\mathcal{M}_{A\#^{op}C^*}$ is the projectivity of ${}_RC$ [24, Proposition 3.1]. A similar result for a left-right Doi-Koppinen structure (H, A, C) was obtained by Y. Doi [14, 3.1], where the corresponding category of Doi-Koppinen modules ${}_A\mathcal{M}(H)^C$ was shown to be naturally isomorphic to the category of *$\#$-rational $\#(C, A)$-modules.* In this paper we show that these results can be obtained under a weaker condition, that ${}_RC$ is *locally projective*, as corollaries from the more general theory of rational modules for *corings* over a (not necessarily commutative) ring. Moreover, we show that these categories are of type $\sigma[M]$, the theory of which is well developed (e.g. [39]). This extends our results in [3] and [2] on the category of rational modules of an R-coalgebra. A fundamental tool in our work is the so called *α-condition,* introduced in [3] for commutative base rings, which proved also to be very helpful in the author's study of duality theorems for Hopf algebras [2].

The concept of a *coring* over an arbitrary ground ring R is due to M. Sweedler [33] and is a generalization of the concept of a *coalgebra* over a commutative ground ring. In the first section we give the needed definitions as well as the basic properties of the category of comodules of a coring. We introduce also the category of *measuring left* (resp. *measuring right*) *R-pairings* $\mathcal{P}_{ml}$ (resp. $\mathcal{P}_{mr}$) and the category of *measuring R-pairings* $\mathcal{P}_m$. For each $(\mathcal{A},\mathcal{C}) \in \mathcal{P}_{ml}$ (resp. $(\mathcal{A},\mathcal{C}) \in \mathcal{P}_{mr}$) we consider two right (resp. left) linear topologies on $\mathcal{A}$, namely the *weak linear topology* $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$ and the *$\mathcal{C}$-adic topology* $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$ (resp. $\mathcal{A}[\mathfrak{T}^l_{ls}(\mathcal{C})]$ and $\mathcal{T}_{\mathcal{C}-}(\mathcal{A})$) and show that $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})] = \mathcal{T}_{-\mathcal{C}}(\mathcal{A})$ (resp. $\mathcal{A}[\mathfrak{T}^l_{ls}(\mathcal{C})] = \mathcal{T}_{\mathcal{C}-}(\mathcal{A})$).

In the second section we define the *rational modules* of a measuring left (resp. right) R-pairing satisfying the so called *α-condition.* The main result (Theorem 2.9) characterizes the measuring left R-pairings $(\mathcal{A},\mathcal{C})$ satisfying the α-condition as those for which ${}_R\mathcal{C}$ is locally projective and $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense (equivalently, those for which $\mathcal{M}^{\mathcal{C}} = \sigma[\mathcal{C}_{\mathcal{A}}] = \sigma[\mathcal{C}_{^*\mathcal{C}}]$). Theorem 2.11 provides a dual version for measuring right R-pairings. For a measuring left α-pairing $(\mathcal{A},\mathcal{C})$ we prove for $\mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}}) = \sigma[\mathcal{C}_{\mathcal{A}}]$ the important *Finiteness Theorem* (2.24). The properties of the right linear topology $\mathcal{T}_{-\mathcal{C}}(\mathcal{A}) = \mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$ introduced in the first section will be used then to give topological (besides the algebraic) characterizations of the rational modules (Proposition 2.28).

In the third section we give some applications of our results in the first and second sections to the category of entwined modules $\mathcal{M}^C_A(\psi)$ corresponding to an entwining structure (A, C, ψ) with ${}_RC$ locally projective, where R is a commutative ground ring. Our observations generalize results of Y. Doi [14] and M. Koppinen [24] on the category of Doi-Koppinen modules $\mathcal{M}(H)^C_A$ corresponding to a Doi-Koppinen structure (H, A, C) with ${}_RC$ projective and results of C. Menini et al. (e.g. [27], [26]) on the category of relative Hopf modules $\mathcal{M}^H_A$ with ${}_RH$ projective.

Throughout this paper R denotes an associative ring with $1_R \neq 0_R$. We consider R as a right (and a left) linear topological ring with the *discrete topology.* With $\mathcal{M}_R$ (resp. ${}_R\mathcal{M}$, ${}_R\mathcal{M}_R$) we denote the category of right R-modules (resp. left R-modules, R-bimodules). All R-modules are assumed to be *unital.* For every right (resp. left) R-module M we denote by $\vartheta^r_M : M \otimes_R R \to M$ (resp. $\vartheta^l_M : R \otimes_R M \to M$) the canonical isomorphisms. With R^{op} we denote the *opposite ring.* For a right R-Module M and a left R-module N we denote with $\tau : M \otimes_R N \to N \otimes_{R^{op}} M$ the canonical *twist* $\mathbb{Z}$-isomorphism. For a left (resp. a right) R-module K we consider K as a right (resp. a left) module over its ring of endomorphisms $\mathrm{End}({}_RK)^{op}$ (resp. $\mathrm{End}(K_R)$) and a left (resp. a right) module over $\mathrm{Biend}({}_RK) := \mathrm{End}(K_{\mathrm{End}({}_RK)^{op}})$ (resp. $\mathrm{Biend}(K_R) := \mathrm{End}({}_{\mathrm{End}(K_R)}K)^{op}$), the *ring of biendomorphisms* of K (e.g. [39, 6.4]).

For an R-ring $\mathcal{A}$ and an $\mathcal{A}$-module M we call an $\mathcal{A}$-submodule $N \subset M$ *R-cofinite*, if M/N is f.g. as an R-module. If U is an R-bimodule, then for every right (resp. left) R-module L we consider $\mathrm{Hom}_{-R}(U, L)$ (resp. $\mathrm{Hom}_{R-}(U, L)$) as a right (resp. left) R-module through $(fr)(u) := f(ru)$ (resp. $(rf)(u) := f(ur)$). Moreover $U^* := \mathrm{Hom}_{-R}(U, R)$ (resp. ${}^*U := \mathrm{Hom}_{R-}(U, R)$) is an R-bimodule through the right (resp. left) R-action given above and the left (resp. right) R-action given by $(\widetilde{r}f)(u) := \widetilde{r}f(u)$ (resp. $(f\widetilde{r})(u) := f(u)\widetilde{r}$). With ${}^*U^* := \mathrm{Hom}_{R-R}(U, R)$ we denote the R-bimodule of R-bilinear maps from U to R.

1 Preliminaries

By an *associative R-ring* we mean an R-bimodule with an R-bilinear map (*multiplication*) $\mu_{\mathcal{A}} : \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$, such that

$$\mu_{\mathcal{A}} \circ (\mu_{\mathcal{A}} \otimes id_{\mathcal{A}}) = \mu_{\mathcal{A}} \circ (id_{\mathcal{A}} \otimes \mu_{\mathcal{A}}).$$

If there exists an R-bilinear map $\eta_{\mathcal{A}} : R \to \mathcal{A}$, such that

$$\mu_{\mathcal{A}} \circ (id_{\mathcal{A}} \otimes \eta_{\mathcal{A}}) = \vartheta^r_{\mathcal{A}} \text{ and } \mu_{\mathcal{A}} \circ (\eta_{\mathcal{A}} \otimes id_{\mathcal{A}}) = \vartheta^l_{\mathcal{A}},$$

then we call $\eta_{\mathcal{A}}$ the *unity* of $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are R-rings (with unities $\eta_{\mathcal{A}}$, $\eta_{\mathcal{B}}$), then an R-bilinear map $f : \mathcal{A} \to \mathcal{B}$ is called a *morphism of R-rings*, if $f \circ \mu_{\mathcal{A}} = \mu_{\mathcal{B}} \circ (f \otimes f)$ (and $f \circ \eta_{\mathcal{A}} = \eta_{\mathcal{B}}$). The set of morphisms of R-rings form $\mathcal{A}$ to $\mathcal{B}$ is denoted by $\mathrm{Rng}_R(\mathcal{A}, \mathcal{B})$. The category of associative R-rings with unities will be denoted by $\mathbf{Rng}_R$.

Dual to R-rings are R-corings presented by M. Sweedler [33]:

1.1. A *coassociative R-coring* is an R-bimodule $\mathcal{C}$ associated with an R-bilinear map (*comultiplication*) $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$, such that the following diagram is commutative:

$$\begin{CD}
\mathcal{C} @>{\Delta_{\mathcal{C}}}>> \mathcal{C} \otimes_R \mathcal{C} \\
@V{\Delta_{\mathcal{C}}}VV @VV{id \otimes \Delta_{\mathcal{C}}}V \\
\mathcal{C} \otimes_R \mathcal{C} @>>{\Delta_{\mathcal{C}} \otimes id}> \mathcal{C} \otimes_R \mathcal{C} \otimes_R \mathcal{C}
\end{CD}$$

If there exists an R-bilinear map $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$, so that the following diagram

$$\begin{array}{ccccc}
 & & \mathcal{C} & & \\
 & \nearrow^{\vartheta^l_{\mathcal{C}}} & \downarrow{\scriptstyle \Delta_{\mathcal{C}}} & \nwarrow^{\vartheta^r_{\mathcal{C}}} & \\
R \otimes_R \mathcal{C} & \xleftarrow[\varepsilon_{\mathcal{C}} \otimes id]{} & \mathcal{C} \otimes_R \mathcal{C} & \xrightarrow[id \otimes \varepsilon_{\mathcal{C}}]{} & \mathcal{C} \otimes_R R
\end{array}$$

is commutative, then we call $\varepsilon_{\mathcal{C}}$ the *counity* of $\mathcal{C}$. Unless the contrary is assumed, we make the convention that an R-coring has a counit. If R is a commutative ring, then R-corings are called R-coalgebras (see [34]).

Let $(\mathcal{C}, \Delta)$ be an R-coring. For $c \in \mathcal{C}$ we use Sweedler-Heyneman's $\sum$-notation:

$$\Delta(c) = \sum c_1 \otimes c_2 \in \mathcal{C} \otimes_R \mathcal{C}.$$

Moreover we define Δ_n inductively as $\Delta_1 := \Delta$ and

$$\Delta_n := (\Delta \otimes id^{n-1}) \circ \Delta_{n-1} : \mathcal{C} \to \mathcal{C}^{n+1},\ c \longmapsto \sum c_1 \otimes ... \otimes c_{n+1} \text{ for } n \geq 2.$$

1.2. The category of R-corings. For two R-corings $(\mathcal{C}, \Delta_{\mathcal{C}})$, $(\mathcal{D}, \Delta_{\mathcal{D}})$ (with counities $\varepsilon_{\mathcal{C}}, \varepsilon_{\mathcal{D}}$) we call an R-bilinear map $f : \mathcal{D} \to \mathcal{C}$ an *R-coring morphism*, if the following diagram

$$\begin{CD}
\mathcal{D} @>{f}>> \mathcal{C} \\
@V{\Delta_{\mathcal{D}}}VV @VV{\Delta_{\mathcal{C}}}V \\
\mathcal{D} \otimes_R \mathcal{D} @>>{f \otimes f}> \mathcal{C} \otimes_R \mathcal{C}
\end{CD}$$

is commutative (and $\varepsilon_{\mathcal{C}} \circ f = \varepsilon_{\mathcal{D}}$). The set of R-coring morphisms from $\mathcal{D}$ to $\mathcal{C}$ is denoted by $\mathrm{Cog}_R(\mathcal{D}, \mathcal{C})$. The category of coassociative R-corings with counities is denoted by $\mathbf{Corng}_R$.

Definition 1.3. Let M be a right (resp. a left) R-module and $N \subset M$ an R-submodule. We call $N \hookrightarrow M$ W*-pure* for some left (resp. right) R-module W, if $0 \to N \otimes_R W \to M \otimes_R W$ (resp. $0 \to W \otimes_R N \to W \otimes_R M$) is exact in ${}_{\mathbb{Z}}\mathcal{M}$. We call $N \hookrightarrow M$ *pure* (in the sense of Cohn), if $N \hookrightarrow M$ is W-pure for every left (resp. right) R-module W. If M is an R-bimodule and $N \subset M$ is an R-subbimodule, then we call ${}_R N_R \subset {}_R M_R$ pure, if $N \subset M$ is pure as a right as well as a left R-submodule.

1.4. Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an R-coring. We call a *pure* R-subbimodule $\mathcal{D} \subseteq \mathcal{C}$ an R*-subcoring*, if $\Delta_{\mathcal{C}}(\mathcal{D}) \subseteq \mathcal{D} \otimes_R \mathcal{D}$. We call an R-subbimodule $K \subset \mathcal{C}$ a $\mathcal{C}$*-bicoideal* (resp. a $\mathcal{C}$*-coideal*), if $\Delta_{\mathcal{C}}(K) \subseteq \mathrm{Im}(\mathcal{C} \otimes_R K) \cap \mathrm{Im}(K \otimes_R \mathcal{C})$ (resp. $\Delta_{\mathcal{C}}(K) \subseteq K \wedge K := \mathrm{Ke}(\mathcal{C} \otimes_R \mathcal{C} \to \mathcal{C}/K \otimes_R \mathcal{C}/K)$). A right (resp. a left) R-submodule $K \subset \mathcal{C}$ is called a *right* $\mathcal{C}$*-coideal* (resp. a *left* $\mathcal{C}$*-coideal*), if $\Delta_{\mathcal{C}}(K) \subset \mathrm{Im}(K \otimes_R \mathcal{C})$ (resp. $\Delta_{\mathcal{C}}(K) \subset \mathrm{Im}(\mathcal{C} \otimes_R K)$).

1.5. The Dual rings of a coring. ([22]) Let $\mathcal{C}$ be a *coassociative* R-coring. Then ${}^*\mathcal{C} := (\mathrm{Hom}_{R-}(\mathcal{C}, R), \star_l)$ is an *associative* R-ring, where

$$(f \star_l g)(c) = \sum g(c_1 f(c_2)) \text{ for all } f, g \in {}^*\mathcal{C} \text{ and } c \in \mathcal{C};$$

$\mathcal{C}^* := (\mathrm{Hom}_{-R}(\mathcal{C}, R), \star_r)$ is an *associative* R-ring, where

$$(f \star_r g)(c) = \sum f(g(c_1)c_2) \text{ for all } f, g \in \mathcal{C}^* \text{ and } c \in \mathcal{C};$$

${}^*\mathcal{C}^* := (\mathrm{Hom}_{R-R}(\mathcal{C}, R), \star)$ is an *associative* R-ring, where

$$(f \star g)(c) = \sum g(c_1) f(c_2) \text{ for all } f, g \in {}^*\mathcal{C}^* \text{ and } c \in \mathcal{C}.$$

If $\mathcal{C}$ has counity $\varepsilon_{\mathcal{C}}$, then $\varepsilon_{\mathcal{C}}$ is a unity for ${}^*\mathcal{C}$, $\mathcal{C}^*$ and ${}^*\mathcal{C}^*$.

1.6. Let $(\mathcal{A}, \mu)$ be an R-ring (*not necessarily with unity*). A right $\mathcal{A}$-module M will be called *unital* (resp. $\mathcal{A}$*-faithful*), if $M\mathcal{A} = M$ (resp. the canonical map $\rho_M : M \to \mathrm{Hom}_{-R}(\mathcal{A}, M)$ is injective). For right $\mathcal{A}$-modules $(M, \rho_M), (N, \rho_N)$ an R-linear map $f : M \to N$ will be called $\mathcal{A}$*-linear*, if $\rho_N \circ f = (\mathcal{A}, f) \circ \rho_M$. The set of $\mathcal{A}$-linear maps from M to N will be denoted by $\mathrm{Hom}_{-\mathcal{A}}(M, N)$. With $\mathcal{M}_{\mathcal{A}}$ (resp. $\widetilde{\mathcal{M}}_{\mathcal{A}}$) we denote the category of unital (resp. $\mathcal{A}$-faithful) right $\mathcal{A}$-modules. Analogously we define left $\mathcal{A}$-modules. For left $\mathcal{A}$-modules M and N, the set of $\mathcal{A}$-linear maps from M to N is denoted by $\mathrm{Hom}_{\mathcal{A}-}(M, N)$. The category of unital (resp. $\mathcal{A}$-faithful) left $\mathcal{A}$-modules is denoted by ${}_{\mathcal{A}}\mathcal{M}$ (resp. ${}_{\mathcal{A}}\widetilde{\mathcal{M}}$).

Let $\mathcal{A}, \mathcal{B}$ be R-rings. A $(\mathcal{B}, \mathcal{A})$-bimodule M will be called *unital* (resp. $(\mathcal{B}, \mathcal{A})$*-faithful*), if ${}_{\mathcal{B}}M$ and $M_{\mathcal{A}}$ are unital (resp. if M is $\mathcal{B}$-faithful and $\mathcal{A}$-faithful). For $(\mathcal{B}, \mathcal{A})$-bimodules M and N we denote the set of $\mathcal{B}$-linear $\mathcal{A}$-linear maps from M to N (called $(\mathcal{B}, \mathcal{A})$*-bilinear*) by $\mathrm{Hom}_{\mathcal{B}-\mathcal{A}}(M, N)$. The category of unital (resp. $(\mathcal{B}, \mathcal{A})$-faithful) $(\mathcal{B}, \mathcal{A})$-bimodules and $(\mathcal{B}, \mathcal{A})$-bilinear maps is denoted by ${}_{\mathcal{B}}\mathcal{M}_{\mathcal{A}}$ (resp. ${}_{\mathcal{B}}\widetilde{\mathcal{M}}_{\mathcal{A}}$).

Dual to modules of R-rings are comodules of R-corings:

1.7. Let $(\mathcal{C}, \Delta)$ be an R-coring (*not necessarily with counit*). A right $\mathcal{C}$-*comodule* is a right R-module M associated with an R-linear map ($\mathcal{C}$-*coaction*)

$$\varrho_M : M \to M \otimes_R \mathcal{C},\ m \mapsto \sum m_{<0>} \otimes m_{<1>},$$

such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\varrho_M} & M \otimes_R \mathcal{C} \\ {\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle id_M \otimes \Delta} \\ M \otimes_R \mathcal{C} & \xrightarrow[\varrho_M \otimes id_{\mathcal{C}}]{} & M \otimes_R \mathcal{C} \otimes_R \mathcal{C} \end{array}$$

If ϱ_M is injective, then we call M *counital.* For right $\mathcal{C}$-comodules $M, N \in \mathcal{M}^{\mathcal{C}}$ we call an R-linear map $f : M \to N$ a $\mathcal{C}$-*comodule morphism* (or $\mathcal{C}$-*colinear*), if the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle \varrho_M}\downarrow & & \downarrow{\scriptstyle \varrho_N} \\ M \otimes_R \mathcal{C} & \xrightarrow[f \otimes id_{\mathcal{C}}]{} & N \otimes_R \mathcal{C} \end{array}$$

The set of $\mathcal{C}$-colinear maps from M to N is denoted by $\mathrm{Hom}^{\mathcal{C}}(M, N)$. The category of counital right $\mathcal{C}$-comodules and $\mathcal{C}$-colinear maps is denoted by $\mathcal{M}^{\mathcal{C}}$. For a right $\mathcal{C}$-comodule N we call a right R-submodule $K \subset N$ a $\mathcal{C}$-*subcomodule*, if $(K, \varrho_K) \in \mathcal{M}^{\mathcal{C}}$ and the embedding $K \overset{\iota_K}{\hookrightarrow} N$ is $\mathcal{C}$-colinear.

Analogously we define the left $\mathcal{C}$-comodules. For two left $\mathcal{C}$-comodules M, N we denote by ${}^{\mathcal{C}}\mathrm{Hom}(M, N)$ the set of $\mathcal{C}$-colinear maps from M to N. The catgeory of counital left $\mathcal{C}$-comodules will be denoted by ${}^{\mathcal{C}}\mathcal{M}$.

Lemma 1.8. (*Compare* [11, Lemma 1.1.]) *Let $(\mathcal{C}, \Delta)$ be an R-coring. If $\mathcal{C}$ has counity ε, then a right $\mathcal{C}$-comodule (M, ϱ_M) is counital iff $\vartheta^r_M \circ (id_M \otimes \varepsilon) \circ \varrho_M = id_M$. Hence, if M is counital, then ϱ_M is a splitting monomorphism.*

Definition 1.9. Let $\mathfrak{C}$ be a category with finite limits and finite colimits. A functor $F : \mathfrak{C} \to \mathfrak{D}$ is called *left-exact* (resp. *right-exact*), if F preserves finite limits (resp. finite colimits). F is called *exact*, if it's left-exact and right-exact.

For forthcoming reference we list some properties of the category of right comodules of an R-coring C. These can be found in several references (e.g. [8], [12], [36]).

Proposition 1.10. *Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an R-coring.*

1. *We have a covariant functor*

$$- \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}},\ M \mapsto (M \otimes_R \mathcal{C}, id_M \otimes \Delta_{\mathcal{C}}).$$

Moreover $- \otimes_R \mathcal{C}$ is right adjoint to the forgetful functor $\mathcal{F} : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$ and left adjoint to the functor $\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, -) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$.

2. $- \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ *is left-exact and* $\mathcal{F} : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$ *is right-exact (since a right adjoint functor preserves limits and a left adjoint functor preserves colimits (e.g.* [32, Proposition 16.4.6]*).*

3. $\mathcal{F}$ *is exact iff* ${}_R\mathcal{C}$ *is flat iff* $- \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ *preserves injective objects (for the commutative case see* [40, Proposition 8]*).*

4. *The category* $\mathcal{M}^{\mathcal{C}}$ *is cocomplete and has cokernels. The direct limits and direct sums are formed in* $\mathcal{M}_R$*. Moreover* $- \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ *respects direct limits (i.e. direct sums and cokernels).*

5. *If* Q *is a cogenerator in* $\mathcal{M}_R$*, then* $Q \otimes_R \mathcal{C}$ *is a cogenerator in* $\mathcal{M}^{\mathcal{C}}$*. In particular* $\mathcal{M}^{\mathcal{C}}$ *has a cogenerator.*

6. *If* ${}_R\mathcal{C}$ *is flat, then* $\mathcal{M}^{\mathcal{C}}$ *is a Grothendieck catgeory with enough injective objects. In this case* $- \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ *respects inverse limits (i.e. direct products and kernels).*

7. *If* R_R *is a cogenerator, then* $\mathcal{C}$ *is a cogenerator in* $\mathcal{M}^{\mathcal{C}}$*. If* ${}_R\mathcal{C}$ *is flat and* R_R *is injective, then* $\mathcal{C}$ *is injective in* $\mathcal{M}^{\mathcal{C}}$*.*

Remark 1.11. If $\mathcal{C}$ is an R-coring such that $\mathcal{M}^{\mathcal{C}}$ is Grothendieck, then ${}_R\mathcal{C}$ need not be flat. A counter example is [17, Example 1.1.]. This shows that the conjecture of M. Wischnewsky [40, Conjecture 14] is false for corings.

1.12. Bicomodules. Let $(M, \varrho_M^{\mathcal{C}})$ be a right $\mathcal{C}$-comodule, $(M, \varrho_M^{\mathcal{D}})$ be a left $\mathcal{D}$-comodule and consider the left $\mathcal{D}$-comodule $(M \otimes_R \mathcal{C}, \varrho_M^{\mathcal{D}} \otimes id_{\mathcal{C}})$ (resp. the right $\mathcal{C}$-comodule $(\mathcal{D} \otimes_R M, id_{\mathcal{D}} \otimes \varrho_M^{\mathcal{C}})$). We call M a $(\mathcal{D}, \mathcal{C})$-*bicomodule*, if $\varrho_M^{\mathcal{C}} : M \to M \otimes_R \mathcal{C}$ is $\mathcal{D}$-colinear (equivalently, if $\varrho_M^{\mathcal{D}} : M \to \mathcal{D} \otimes_R M$ is $\mathcal{C}$-colinear). For $(\mathcal{D}, \mathcal{C})$-bicomodules M, N we call a $\mathcal{D}$-colinear $\mathcal{C}$-colinear map $f : M \to N$ a $(\mathcal{D}, \mathcal{C})$-*bicomodule morphism* (or $(\mathcal{D}, \mathcal{C})$-*bicolinear*). We say a $(\mathcal{D}, \mathcal{C})$-bicomodule is *counital*, if it's counital as a left $\mathcal{D}$-comodule and as a right $\mathcal{C}$-comodule. The category of counital $(\mathcal{D}, \mathcal{C})$-bicomodules and $(\mathcal{D}, \mathcal{C})$-bicolinear maps is denoted by ${}^{\mathcal{D}}\mathcal{M}^{\mathcal{C}}$.

The weak linear topology

Next we introduce the categories of left (resp. right) R-pairings $\mathcal{P}_l$ (resp. $\mathcal{P}_r$) and the category of R-pairings $\mathcal{P}$. For each left (resp. right) R-pairing $P = (V, W)$ we consider V with a right (resp. a left) linear topology, the *weak linear topology* $V[\mathfrak{T}_{ls}^r(W)]$ (resp. $V[\mathfrak{T}_{ls}^l(W)]$) (e.g. [23, 10.3], [30]).

1.13. The category of R-pairings. With a *left R-pairing* we denote a right R-module V and a left R-module W with an R-linear map $\kappa_P : V \to {}^*W$ (equivalently $\chi_P : W \to V^*$). For left R-pairings (V, W) and (V', W') a *morphism of left R-pairings* $(\xi, \theta) : (V', W') \to (V, W)$ consists of a morphism of right R-modules $\xi : V \to V'$ and a morphism of left R-modules $\theta : W' \to W$, such that

$$< \xi(v), w' >=< v, \theta(w') > \text{ for all } v \in V \text{ and } w' \in W'. \quad (1)$$

The left R-pairings with the morphisms described above (and the usual composition of pairings) build a category, which we denote by $\mathcal{P}_l$. Analogously we define the category of right R-pairings $\mathcal{P}_r$.

With an *R-pairing* we denote R-bimodules V and W with an R-bilinear map $\kappa_P : V \to {}^*W^*$ (equivalently $\chi_P : W \to {}^*V^*$). If (V, W) and (V', W') are R-pairings, then a *morphism of R-pairings* $(\xi, \theta) : (V', W') \to (V, W)$ consists of R-bilinear maps $\xi : V \to V'$ and $\theta : W' \to W$ with the compatibility condition (1). The category of R-pairings is denoted by $\mathcal{P}$.

1.14. The finite topology. Let E be a right (resp. a left) R-module, W a set and identify the direct product E^W with the set of all maps from W to E. If we consider E with the discrete topology and the right (resp. the left) R-module E^W with the *product topology*, then the induced *linear* topology on an R-submodule $Z \subseteq E^W$ is called the *finite topology* and has a neighbourhood system of 0_Z :

$$\mathcal{B}_f(0_Z) := \{\mathrm{An}(F) |\ F = \{w_1, ..., w_k\} \subset W \text{ is a finite subset}\}.$$

1.15. Let $P = (V, W)$ be a left (resp. a right) R-pairing and consider the right (resp. the left) R-submodule ${}^*W \subset R^W$ (resp. $W^* \subset R^W$) with the *finite topology.* Then there is on V a right (resp. a left) linear topology, the *weak linear topology* $V[\mathfrak{T}^r_{ls}(W)]$ (resp. $V[\mathfrak{T}^l_{ls}(W)]$), such that $\kappa_P : V \to {}^*W$ (resp. $\kappa_P : V \to W^*$) is continuous. The neighbourhood system of 0_V w.r.t. this topology is given by

$$\mathcal{B}_f(0_V) = \{F^\perp := \kappa_P^{-1}(\mathrm{An}(F)) |\ F = \{w_1, ..., w_k\} \subset W \text{ is a finite subset}\}.$$

The closure $\overline{X}$ of any subset $X \subseteq V$ is then given by

$$\overline{X} = \bigcap\{X + F^\perp \mid F \subset W \text{ is a finite subset}\}. \tag{2}$$

Hence $V[\mathfrak{T}^r_{ls}(W)]$ (resp. $V[\mathfrak{T}^l_{ls}(W)]$) is *Hausdorff* iff $V \overset{\kappa_P}{\hookrightarrow} {}^*W$ (resp. $V \overset{\kappa_P}{\hookrightarrow} W^*$) is an embedding.

Next we state some properties of the weak linear topology without proof. For the proofs and other details the interested reader may refer to [1]. For the case of a commutative ground ring a reference is [2, Anhang].

Lemma 1.16. *Let $P = (V, W)$ be a left R-pairing and consider V with the weak linear topology $V[\mathfrak{T}^r_{ls}(W)]$.*

1. *$\overline{X} \subseteq X^{\perp\perp}$ for any subset $X \subset V$. Consequently every orthogonally closed right R-submodule of V is closed.*

2. *If R_R is noetherian, then all open R-submodules of V are R-cofinite.*

3. *Let R_R be artinian.*

    (a) Every R-cofinite closed right R-submodule $X \subset V$ is open.

(b) Let $X \subset Y \subset V$ be right R-submodules. If $X \subset V$ is closed and R-cofinite, then $Y \subset V$ is also closed and R-cofinite.

We call the ring R a *QF ring*, if R_R (equivalently ${}_RR$) is noetherian and a cogenerator (e.g. [39, 48.15]).

The following result characterizes the closed and the open R-submodules of V w.r.t. the weak linear topology:

Proposition 1.17. *Let $P = (V, W)$ be a left R-pairing and consider V with the weak linear topology $V[\mathfrak{T}^r_{ls}(W)]$. Assume R_R to be an injective cogenerator.*

1. *The closure of a right R-submodule $X \subseteq V$ is given by $\overline{X} = X^{\perp\perp}$.*
2. *For right R-submodules $X \subset Y \subseteq V$ we have: X is dense in Y iff $X^{\perp} = Y^{\perp}$. If $W \hookrightarrow V^*$, then $X \subset V$ is dense iff $X^{\perp} = 0$.*
3. *The class of closed right R-submodules of V is given by*

$$\{K^{\perp} \mid K \subset W \text{ is an arbitrary left } R\text{-submodule}\}.$$

4. *If R is a QF-ring and $W \overset{\chi_P}{\hookrightarrow} V^*$ is an embedding, then the class of open right R-submodules of V is given by*

$$\{K^{\perp} \mid K \subset W \text{ is a f.g. left } R\text{-submodule}\}.$$

Lemma 1.18. *Let W, W' be left R-modules and consider the right R-modules ${}^*W$ and ${}^*W'$ with the finite topology. Let $\theta \in \mathrm{Hom}_{R-}(W', W)$. If R is a QF-ring, then $\mathrm{Ke}(\theta^*(X)) = \theta^{-1}(\mathrm{Ke}(X))$ for every right R-submodule $X \subset {}^*W$.*

The $\mathcal{C}$-adic topology

We introduce now the category of *measuring left R-pairings* $\mathcal{P}_{ml}$. For every $(\mathcal{A}, \mathcal{C}) \in \mathcal{P}_{ml}$ we define the *$\mathcal{C}$-adic topology* $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$ (see [5], [6]), which we prove to coincide with the linear weak topology $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$.

1.19. The category of measuring R-pairings. If $\mathcal{C}$ is an R-coring and $\mathcal{A}$ is an R-ring with a morphism of R-rings $\kappa : \mathcal{A} \to {}^*\mathcal{C}$, $a \mapsto [c \mapsto < a, c >]$, then we call $P := (\mathcal{A}, \mathcal{C})$ a *measuring left R-pairing* (the terminology is inspired by [34, Definition, Page 138]). For measuring left R-pairings $(\mathcal{A}, \mathcal{C})$, $(\mathcal{B}, \mathcal{D})$ we say a morphism of left R-pairings $(\xi, \theta) : (\mathcal{B}, \mathcal{D}) \to (\mathcal{A}, \mathcal{C})$ is a *morphism of measuring left R-pairings*, if $\xi : \mathcal{A} \to \mathcal{B}$ is a morphism of R-rings and $\theta : \mathcal{D} \to \mathcal{C}$ is a morphism of R-corings. The measuring left R-pairings with the morphisms described above build a subcategory $\mathcal{P}_{ml} \subset \mathcal{P}_l$.

A *measuring right R-pairing* $P = (\mathcal{A}, \mathcal{C})$ consists of an R-ring $\mathcal{A}$ and an R-coring $\mathcal{C}$ with a morphism of R-rings $\kappa_P : \mathcal{A} \to \mathcal{C}^*$. If $\mathcal{A}$ is an R-ring with a morphism of R-rings $\kappa_P : \mathcal{A} \to {}^*\mathcal{C}^*$, then we call $(\mathcal{A}, \mathcal{C})$ a *measuring R-pairing*. The category of measuring right R-pairings (resp. measuring R-pairings) is denoted by $\mathcal{P}_{mr}$ (resp. $\mathcal{P}_m$).

1.20. If $P = (\mathcal{A}, \mathcal{C})$ is a measuring left (right) R-pairing, then $\mathcal{C}$ becomes a right (a left) $\mathcal{A}$-module with $\mathcal{A}$-action given by

$$c \leftharpoonup a := \sum c_1 < a, c_2 > \text{ (resp. } a \rightharpoonup c := \sum < a, c_1 > c_2). \tag{3}$$

If $P = (\mathcal{A}, \mathcal{C})$ is a measuring R-pairing, then $\mathcal{C}$ is an $\mathcal{A}$-bimodule with the right and the left $\mathcal{A}$-actions in (3).

The following example was communicated to the author by Tomasz Brzeziński:

Example 1.21. Let $\mathcal{C}$ be a *coseparable* R-coring (i.e. there exists a $\mathcal{C}$-bicolinear map $\pi : \mathcal{C} \otimes_R \mathcal{C} \to \mathcal{C}$ with $\pi \circ \Delta_{\mathcal{C}} = id_{\mathcal{C}}$ [21], equivalently there exists a *cointegral* $\gamma \in \mathrm{Hom}_{R-R}(\mathcal{C} \otimes_R \mathcal{C}, R)$, such that $\gamma \circ \Delta_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$ and

$$\sum c_1 \gamma(c_2 \otimes c') = \sum \gamma(c \otimes c'_1) c'_2 \text{ for all } c, c' \in \mathcal{C}.$$

[8, Theorem 3.5, Corollary 3.6]). Then $\mathcal{C}$ is a (non unital) R-ring with multiplication

$$\mu : \mathcal{C} \otimes_R \mathcal{C} \to \mathcal{C}, \; c \otimes \widetilde{c} \mapsto \sum c_1 \gamma(c_2 \otimes \widetilde{c})$$

and therefore $P := (\mathcal{C}, \mathcal{C})$ is a measuring left R-pairing with

$$\kappa_P : \mathcal{C} \to {}^*\mathcal{C}, \; c \mapsto [c' \mapsto \gamma(c' \otimes c)].$$

1.22. Subpairings. Let $P = (\mathcal{A}, \mathcal{C})$ and assume $P \in \mathcal{P}_{ml}$ (resp. $P \in \mathcal{P}_{mr}$, $P \in \mathcal{P}_m$), $\mathcal{J} \lhd \mathcal{A}$ an $\mathcal{A}$-ideal, $\mathcal{D} \overset{\iota}{\hookrightarrow} \mathcal{C}$ an R-subcoring with $< \mathcal{J}, \mathcal{D} >= 0$ and put $Q := (\mathcal{A}/\mathcal{J}, \mathcal{D})$. Then $Q \in \mathcal{P}_{ml}$ (resp. $Q \in \mathcal{P}_{mr}$, $Q \in \mathcal{P}_m$) and $(\pi, \iota) : (\mathcal{A}/\mathcal{J}, \mathcal{D}) \to (\mathcal{A}, \mathcal{C})$ is a morphism in $\mathcal{P}_{ml}$ (resp. in $\mathcal{P}_{mr}$, in $\mathcal{P}_m$). We call Q a *measuring R-subpairing* of P and write $Q \subset P$. For every R-subcoring $\mathcal{D} \subseteq \mathcal{C}$ we have the *two-sided* $\mathcal{A}$-ideal $\mathcal{D}^{\perp} \subseteq \mathcal{A}$ and so $(\mathcal{A}/\mathcal{D}^{\perp}, \mathcal{D}) \subseteq (\mathcal{A}, \mathcal{C})$ is a measuring R-subpairing. In particular, to every measuring left (resp. right) R-pairing $(\mathcal{A}, \mathcal{C})$, is $(\mathcal{A}/\mathcal{C}^{\perp}, \mathcal{C}) \subseteq ({}^*\mathcal{C}, \mathcal{C})$ (resp. $(\mathcal{A}/\mathcal{C}^{\perp}, \mathcal{C}) \subseteq (\mathcal{C}^*, \mathcal{C})$) a *non-degenerate* measuring left (resp. right) R-subpairing.

1.23. Subgenerators. Let $\mathcal{A}$ be an R-ring and K an $\mathcal{A}$-module. We say that an $\mathcal{A}$-module N is *K-subgenerated*, if N is isomorphic to a submodule of a K-generated $\mathcal{A}$-module (equivalently, if N is kernel of a morphism between K-generated $\mathcal{A}$-modules). For a right $\mathcal{A}$-module K, we denote by $\sigma[K_{\mathcal{A}}]$ the *full* subcategory of $\mathcal{M}_{\mathcal{A}}$ whose objects are the K-subgenerated right $\mathcal{A}$-modules. For every right $\mathcal{A}$-module M

$$\mathrm{Sp}(\sigma[K_{\mathcal{A}}], M) := \sum \{ f(N) \mid \; f \in \mathrm{Hom}_{-\mathcal{A}}(N, M), \; N \in \sigma[K_{\mathcal{A}}] \} \tag{4}$$

is the largest K-subgenerated right $\mathcal{A}$-submodule of M. Moreover $\sigma[K_{\mathcal{A}}]$ is the *smallest* Grothendieck full subcategory of $\mathcal{M}_{\mathcal{A}}$ that contains K. For the well developed theory of categories of this type the reader is referred to [39].

1.24. The $\mathcal{C}$-adic topology. Let $P = (\mathcal{A}, \mathcal{C}) \in \mathcal{P}_{ml}$ and consider $\mathcal{C}$ with the right $\mathcal{A}$-module structure through $(\leftharpoonup)$ in (3). Then the class of right $\mathcal{A}$-ideals

$$\mathcal{B}_{-\mathcal{C}}(0_{\mathcal{A}}) := \{ (0_{\mathcal{C}} : F) | \; F \subset \mathcal{C} \text{ is a finite subset} \}$$

is a neighbourhood system of $0_{\mathcal{A}}$ for a right linear topology, the *$\mathcal{C}$-adic topology* $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$, and $(\mathcal{A}, \mathcal{T}_{-\mathcal{C}}(\mathcal{A}))$ is a right linear topological R-ring. A right $\mathcal{A}$-ideal $I \lhd_r \mathcal{A}$ is open w.r.t. $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$ iff $\mathcal{A}/I$ is $\mathcal{C}$-subgenerated. If $(\mathcal{A}, \mathfrak{T})$ is a right linear topological ring, then the category of $(\mathcal{A}, \mathfrak{T})$-discrete modules coincides with $\sigma[\mathcal{C}_{\mathcal{A}}]$ iff $\mathfrak{T} = \mathcal{T}_{-\mathcal{C}}(\mathcal{A})$. We refer mainly to [5] and [6] for detailed investigation of this topology.

Lemma 1.25. *Let $P = (\mathcal{A}, \mathcal{C}) \in \mathcal{P}_{ml}$. The weak linear topology $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$ and the $\mathcal{C}$-adic topology $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$ coincide. In particular $(\mathcal{A}, \mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})])$ is a right linear topological R-ring and a right $\mathcal{A}$-module M is $(\mathcal{A}, \mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})])$-discrete iff $M_{\mathcal{A}}$ is $\mathcal{C}$-subgenerated.*

Proof. Let U be a neighbourhood of $0_{\mathcal{A}}$ w.r.t. $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$. Then there exists a f.g. left R-submodule $K \subset \mathcal{C}$, such that $K^{\perp} \subseteq U$. But we have then for every $a \in (0_{\mathcal{C}} : K)$ and $c \in K$:

$$< a, c >=< a, \sum \varepsilon(c_1)c_2 >= \varepsilon(\sum c_1 < a, c_2 >) = \varepsilon(c \leftharpoonup a) = 0,$$

and so $(0_{\mathcal{C}} : K) \subseteq K^{\perp} \subseteq U$, i.e. U is a neighbourhood of $0_{\mathcal{A}}$ w.r.t. $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$. On the other hand, let U be a neighbourhood of $0_{\mathcal{A}}$ w.r.t. $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$. Then there exists a finite set $F = \{c_1, ..., c_k\} \subset \mathcal{C}$, such that $(0_{\mathcal{C}} : F) \subseteq U$. Assume $\Delta(c_i) = \sum\limits_{j=1}^{n_i} c_{ij} \otimes \widetilde{c}_{ij}$ for $i = 1, ..., k$ and put $K := \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{n_i} R\widetilde{c}_{ij}$. Then $K^{\perp} \subseteq (0_{\mathcal{C}} : F) \subseteq U$, i.e. U is a neighbourhood of $0_{\mathcal{A}}$ w.r.t. $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$. So $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})] = \mathcal{T}_{-\mathcal{C}}(\mathcal{A})$ and the last statement follows now from 1.24.■

As a corollary of Proposition 1.17 (2) and Lemma 1.25 we get

Corollary 1.26. *If $P = (\mathcal{A}, \mathcal{C}) \in \mathcal{P}_{ml}$, then for every right R-submodule $Y \subset \mathcal{A}$ the following are equivalent:*

1. *$Y \subset \mathcal{A}$ is dense w.r.t. $\mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$.*
2. *$Y \subset \mathcal{A}$ is dense w.r.t. $\mathcal{T}_{-\mathcal{C}}(\mathcal{A})$.*

   If R_R is an injective cogenerator and $\mathcal{C} \hookrightarrow \mathcal{A}^*$, then (1) & (2) *are equivalent to*
3. $Y^{\perp} := \{c \in \mathcal{C} | < a, c >= 0$ *for every* $a \in Y\} = 0$.

The α-condition

We introduce now the category of left (resp. right) α-pairings $\mathcal{P}^{\alpha}_l$ (resp. $\mathcal{P}^{\alpha}_r$) and the category of α-pairings $\mathcal{P}^{\alpha}$.

1.27. We say that a left R-pairing $P = (V, W)$ satisfies the (*left*) *α-condition*, or is *left α-pairing*, if for every right R-module M the following map is injective:

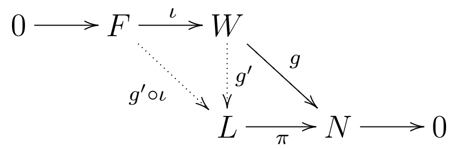
$$\alpha^P_M : M \otimes_R W \to \mathrm{Hom}_{-R}(V, M), \ \sum m_i \otimes w_i \mapsto [v \mapsto \sum m_i < v, w_i >].$$

A right R-pairing $P = (V, W)$ is said to satisfy the (*right*) *α-condition*, or is a *right α-pairing*, if for every left R-module M, the canonical map $\alpha^P_M : W \otimes_R M \to \mathrm{Hom}_{R-}(V, M)$ is injective. With $\mathcal{P}^{\alpha}_l \subset \mathcal{P}_l$ (resp. $\mathcal{P}^{\alpha}_r \subset \mathcal{P}_r$) we denote the full subcategory, whose objects satisfy the α-condition. We call a left (resp. a right) R-pairing $P = (V, W)$ *dense*, if $\kappa_P(V) \subseteq {}^*W$ (resp. $\kappa_P(V) \subseteq W^*$) is dense w.r.t. the finite topology.

With $\mathcal{P}^{\alpha}_{ml} \subset \mathcal{P}_{ml}$ (resp. $\mathcal{P}^{\alpha}_{m} \subset \mathcal{P}_{m}$) we denote the *full* subcategory of *measuring left α-pairings* (resp. *measuring right α-pairings*). If $P \in \mathcal{P}^{\alpha}_{ml}$ (resp. $P \in \mathcal{P}^{\alpha}_{mr}$) and $Q \subset P$ is a measuring R-subpairing, then Q satisfies the α-condition as well (see Proposition 1.32 (1-b) below).

We say a left (resp. a right) R-module W satisfies the *α-condition*, if the left R-pairing $({}^*W, W)$ (resp. the right R-pairing (W^*, W)) satisfies the α-condition, i.e. if ${}_RW$ (resp. W_R) is *universally torsion free* in the sense of G. Garfinkel [18].

1.28. Locally projective modules. An R-module W is called *locally projective* (in the sense of B. Zimmermann-Huisgen [41]), if for every diagram

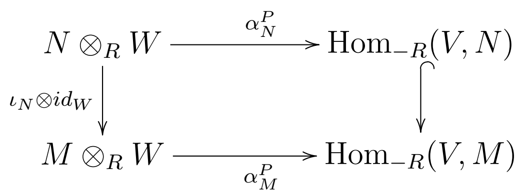


with exact rows and F f.g.: for every R-linear map $g : W \to N$, there exists an R-linear map $g' : W \to L$, such that the entstanding parallelogram is commutative. Note that every projective R-module is locally projective.

Analog to [18, Theorem 3.2] and [41, Theorem 2.1] we get the following characterizations of the R-modules satisfying the α-condition:

Lemma 1.29. *A left (resp. a right) R-module W satisfies the α-condition iff ${}_RW$ (resp. W_R) is locally projective.*

Remark 1.30. Let $P = (V, W) \in \mathcal{P}^{\alpha}_{l}$. Then $W \subset V^*$, in particular ${}_RW$ is R-cogenerated. If M is any right R-module, then we have for every R-submodule $N \subset M$ the commutative diagram



By assumption α^P_N is injective and so $N \subset M$ is W-pure. Since M and N are arbitrary, we conclude that ${}_RW$ is flat. If ${}_RR$ is perfect, then ${}_RW$ is projective. In particular every locally projective left R-module (over a left perfect ring) is flat (projective). So over perfect rings projectivity and local projectivity coincide.

Lemma 1.31. *Let $P = (V, W)$ be a left α-pairing. If L is a right R-module and $K \subset L$ is an R-submodule, then we have for every $\sum l_i \otimes w_i \in L \otimes_R W$:*

$$\sum l_i \otimes w_i \in K \otimes_R W \Longleftrightarrow \sum l_i < v, w_i > \in K \text{ for all } v \in V.$$

Proof. By Remark 1.30 ${}_RW$ is flat and so we get the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K\otimes_R W & \xrightarrow{\iota_K\otimes id_W} & L\otimes_R W & \xrightarrow{\pi\otimes id_W} & L/K\otimes_R W & \longrightarrow & 0 \\ & & \big\downarrow \alpha_K^P & & \big\downarrow \alpha_L^P & & \big\downarrow \alpha_{L/K}^P & & \\ 0 & \longrightarrow & \mathrm{Hom}_{-R}(V,K) & \xrightarrow[(V,\iota_K)]{} & \mathrm{Hom}_{-R}(V,L) & \xrightarrow[(V,\pi)]{} & \mathrm{Hom}_{-R}(V,L/K) & & \end{array}$$

Clearly $\sum l_i < v, w_i >\in K$ for every $v \in V$ iff $\sum l_i \otimes w_i \in \mathrm{Ke}((V,\pi)\circ\alpha_L^P) = \mathrm{Ke}(\alpha_{L/K}^P \circ (\pi\otimes id_W)) = \mathrm{Ke}(\pi\otimes id_W) = K\otimes_R W.$■

Analog to the commutative case [2, Proposition 2.1.7] we get

Proposition 1.32. *1. Let $P = (V,W)$ be a left R-pairing.*

(a) Let $W' \subset W$ be an R-submodule and consider the induced left R-pairing $P' := (V,W')$. If $P' \in \mathcal{P}_l^\alpha$, then $W' \subset W$ is pure. If $P \in \mathcal{P}_l^\alpha$, then $P' \in \mathcal{P}_l^\alpha$ iff $W' \subset W$ is pure.

(b) Let $V' \subset V$, $W' \subset W$ be R-submodules with $< V', W' >= 0$ and consider the left R-pairing $Q := (V/V', W')$. If $P \in \mathcal{P}_l^\alpha$, then $Q \in \mathcal{P}_l^\alpha$ iff $W' \subset W$ is pure.

2. Let $Q = (Y,W)$ be a left R-pairing, V a right R-module, $\xi : V \to Y$ an R-linear map, $P := (V,W)$ the induced left R-pairing and consider the following statements:

(i) $Q \in \mathcal{P}_l^\alpha$ and P is dense;

(ii) $Q \in \mathcal{P}_l^\alpha$ and $\xi(V) \subset Y$ is dense w.r.t. $Y[\mathfrak{T}_{ls}^r(W)]$;

(iii) $P \in \mathcal{P}_l^\alpha$;

(iv) $Q \in \mathcal{P}_l^\alpha$ and $W \overset{\chi_P}{\hookrightarrow} V^*$ is an embedding.

The following implications are always true: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ *(iv). If R_R is an injective cogenerator, then* (i)-(iv) *are equivalent.*

The proof of the following result is similar to that of [3, Proposition 2.5]:

Lemma 1.33. *Let V, W be R-bimodules.*

1. If $P = (V,W)$, $P' = (V',W')$ are left α-pairings, then $P\otimes_l P' := (V'\otimes_R V, W\otimes_R W')$ is a left α-pairing, where

$$\kappa_{P\otimes_l P'}(v'\otimes v)(w\otimes w') =< v, w < v', w' >>=<< v', w' > v, w > .$$

2. If $P = (V,W)$, $P' = (V',W')$ are right α-pairings, then $P\otimes_r P' := (V\otimes_R V', W'\otimes_R W)$ is a right α-pairing, where

$$\kappa_{P'\otimes_r P}(v\otimes v')(w'\otimes w) =< v, < v', w' > w >=< v < v', w' >, w > .$$

2 Rational modules

In this section we define the *$\mathcal{C}$-rational $\mathcal{A}$-modules* associated with a measuring left (resp. right) R-pairing $(\mathcal{A},\mathcal{C})$ satisfying the α-condition and prove the main result in this paper, namely Theorem 2.9 (resp. its dual version Theorem 2.11).

Remark 2.1. Let $P=(\mathcal{A},\mathcal{C})$ be a measuring left R-pairing. For every right R-module M, $\mathrm{Hom}_{-R}(\mathcal{A},M)$ is a right $\mathcal{A}$-module through $(fa)(a')=f(aa')$ for all $a,a'\in\mathcal{A}$ and $\alpha_M^P: M\otimes_R\mathcal{C}\to\mathrm{Hom}_{-R}(\mathcal{A},M)$ is $\mathcal{A}$-linear. If moreover M is a right $\mathcal{A}$-module, then the canonical map $\rho_M: M\to\mathrm{Hom}_{-R}(\mathcal{A},M)$ is $\mathcal{A}$-linear.

2.2. Let $\mathcal{A}$ be an R-ring (*not necessarily with unity*), $P=(\mathcal{A},\mathcal{C})$ a measuring left α-pairing and M an $\mathcal{A}$-faithful right $\mathcal{A}$-module. Put $\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}}):=\rho_M^{-1}(M\otimes_R\mathcal{C})$, i.e. $m\in\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})$ iff there exists a uniquely determined element $\sum m_i\otimes c_i$, such that $ma=\sum m_i<a,c_i>$ for every $a\in\mathcal{A}$. We call $M_{\mathcal{A}}$ *$\mathcal{C}$-rational*, if $\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})=M$. In this case we get an R-linear map $\varrho_M:=(\alpha_M^P)^{-1}\circ\rho_M: M\to M\otimes_R\mathcal{C}$.

Analog to the commutative case (e.g. [19, Proposition 2.9]) we get

Lemma 2.3. *Let $P=(\mathcal{A},\mathcal{C})$ be a measuring left α-pairing* ($\mathcal{A}$ not necessarily with unity). *For every $(M,\rho_M)\in\widetilde{\mathcal{M}}_{\mathcal{A}}$ we have:*

1. $\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})\subseteq M$ *is an $\mathcal{A}$-submodule.*
2. *For every $\mathcal{A}$-submodule $N\subset M$ we have* $\mathrm{Rat}^{\mathcal{C}}(N_{\mathcal{A}})=N\cap\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})$.
3. $\mathrm{Rat}^{\mathcal{C}}(\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}}))=\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})$.
4. *For every $L\in\widetilde{\mathcal{M}}_{\mathcal{A}}$ and $f\in\mathrm{Hom}_{-\mathcal{A}}(M,L)$ we have* $f(\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}}))\subseteq\mathrm{Rat}^{\mathcal{C}}(L_{\mathcal{A}})$.

Analog to the commutative case [2, Folgerung 2.2.10] we have

Remark 2.4. Let $P=(\mathcal{A},\mathcal{C})$ be a measuring left α-pairing and consider the embedding $\mathcal{C}\overset{\chi_P}{\hookrightarrow}\mathcal{A}^*$. Since χ_P is $\mathcal{A}$-linear, we have $\mathcal{C}\overset{\chi_P}{\hookrightarrow}\mathrm{Rat}^{\mathcal{C}}((\mathcal{A}^*)_{\mathcal{A}})$ by Lemma 2.3 (4). If $f\in\mathrm{Rat}^{\mathcal{C}}((\mathcal{A}^*)_{\mathcal{A}})$ with $\varrho(f)=\sum f_i\otimes c_i$, then we have for every $a\in\mathcal{A}$

$$f(a)=(fa)(1_{\mathcal{A}})=\sum f_i(1_{\mathcal{A}})<a,c_i>=\chi_P(\sum f_i(1_{\mathcal{A}})c_i)(a),$$

i.e. $f=\chi_P(\sum f_i(1_{\mathcal{A}})c_i)$. So χ_P is an isomorphism.

For a left (resp. right) measuring α-pairing $(\mathcal{A},\mathcal{C})$ we denote the category of $\mathcal{C}$-rational right (resp. left) $\mathcal{A}$-modules and $\mathcal{A}$-linear maps by $\mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}})$ (resp. ${}^{\mathcal{C}}\mathrm{Rat}({}_{\mathcal{A}}\widetilde{\mathcal{M}})$. The subcategory of unital $\mathcal{C}$-rational right (resp. left) $\mathcal{A}$-modules will be denoted by $\mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}})$ (resp. ${}^{\mathcal{C}}\mathrm{Rat}({}_{\mathcal{A}}\mathcal{M})$).

Lemma 2.5. *Let $P=(\mathcal{A},\mathcal{C})$ be a measuring left R-pairing* ($\mathcal{A}$ not necessarily with unity).

1. *If (M,ϱ_M) is a right $\mathcal{C}$-comodule, then M becomes a right $\mathcal{A}$-module through*

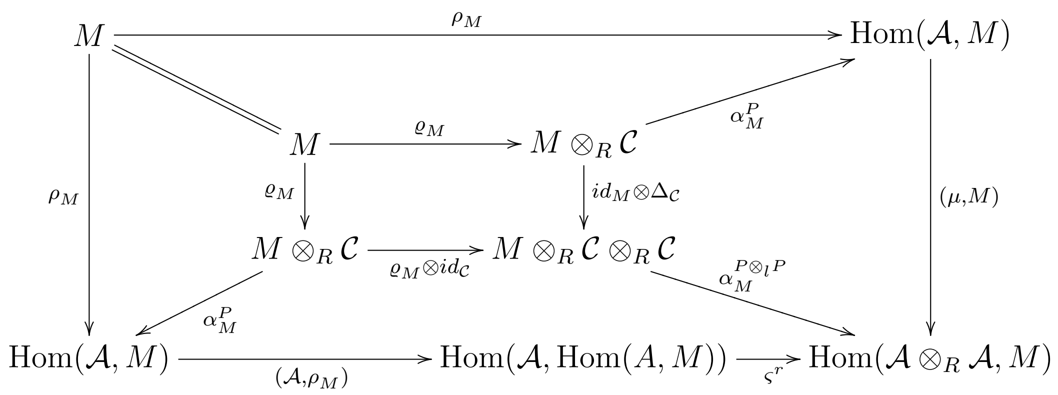
$$\rho_M:=M\overset{\varrho_M}{\to}M\otimes_R\mathcal{C}\overset{\alpha_M^P}{\to}\mathrm{Hom}_{-R}(\mathcal{A},M).$$

If $\mathcal{A}$ has unity and M is counital, then $M_{\mathcal{A}}$ is unital (and so $\mathcal{A}$-faithful).

2. *Let $(M, \varrho_M), (N, \varrho_N)$ be right $\mathcal{C}$-comodules and consider the induced structures of right $\mathcal{A}$-modules (M, ρ_M), (N, ρ_N). If $f : M \to N$ is $\mathcal{C}$-colinear, then f is $\mathcal{A}$-linear.*

3. *Let N be a right $\mathcal{C}$-comodule, $K \subset N$ a $\mathcal{C}$-subcomodule and consider the induced right $\mathcal{A}$-module structures (N, ρ_N), (K, ρ_K). Then $K \subset N$ is an $\mathcal{A}$-submodule.*

Proof. 1. Consider the left R-pairing $P \otimes_l P := (\mathcal{A} \otimes_R \mathcal{A}, \mathcal{C} \otimes_R \mathcal{C})$. For every right $\mathcal{A}$-module (M, ρ_M) we have the diagram



(5)

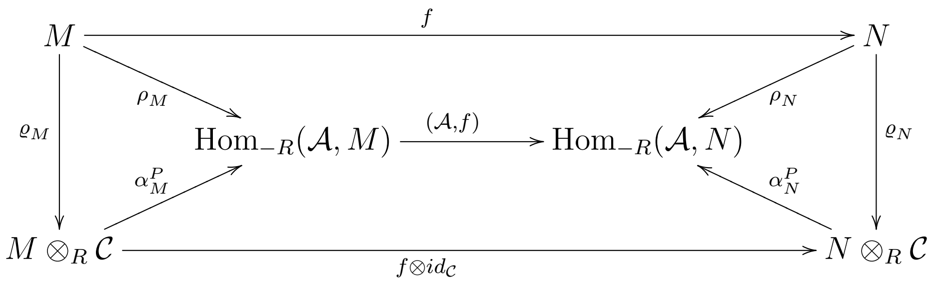
(where ς^r is the canonical isomorphism). By definition of ρ_M and $\alpha_M^{P \otimes_l P}$ all trapezoids are commutative. Since (M, ϱ_M) is a right $\mathcal{C}$-comodule, the inner rectangle is commutative and so the outer rectangle is commutative, i.e. (M, ρ_M) is a right $\mathcal{A}$-module.

Assume M to be counital. If $\mathcal{A}$ has unity, then $\kappa_P(1_{\mathcal{A}}) = \varepsilon_{\mathcal{C}}$ and we have for every $m \in M$

$$m = \sum m_{<0>} \varepsilon_{\mathcal{C}}(m_{<1>}) = m\varepsilon_{\mathcal{C}} = m1_{\mathcal{A}} \in M\mathcal{A},$$

i.e. $M_{\mathcal{A}}$ is unital.

2. Consider the diagram



(6)

The lower trapezoid is obviously commutative. The triangles are commutative by definition of ρ_M, ρ_N. If the outer rectangle is commutative (f is $\mathcal{C}$-colinear), then the upper trapezoid is commutative (f is $\mathcal{A}$-linear).

3. trivial.■

Lemma 2.6. *Let $P = (\mathcal{A}, \mathcal{C})$ be a measuring left α-pairing* ($\mathcal{A}$ not necessarily with unity).

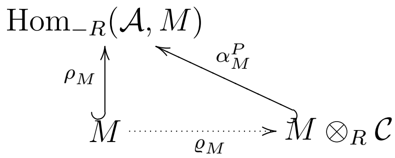
1. *If $(M, \rho_M) \in \widetilde{\mathcal{M}}_{\mathcal{A}}$ is $\mathcal{C}$-rational, then M gets a structure of a* counital *right $\mathcal{C}$-comodule through*

$$\varrho_M : M \overset{\rho_M}{\to} \mathrm{Hom}_{-R}(\mathcal{A}, M) \overset{(\alpha_M^P)^{-1}}{\to} M \otimes_R \mathcal{C}.$$

2. *Let $(M, \rho_M), (N, \rho_N) \in \widetilde{\mathcal{M}}_{\mathcal{A}}$ be $\mathcal{C}$-rational and consider the induced structures of right $\mathcal{C}$-comodules (M, ϱ_M), (N, ϱ_N). Then* $\mathrm{Hom}^{\mathcal{C}}(M, N) = \mathrm{Hom}_{-\mathcal{A}}(M, N)$.

3. *Let $(N, \rho_N) \in \widetilde{\mathcal{M}}_{\mathcal{A}}$ be $\mathcal{C}$-rational and consider the induced counital right $\mathcal{C}$-comodule (N, ϱ_N). If $K \subset N$ is an $\mathcal{A}$-submodule, then K gets a structure of a counital $\mathcal{C}$-subcomodule. Moreover $\varrho_K = (\varrho_N)_{|K}$.*

Proof. 1. If (M, ρ_M) is $\mathcal{C}$-rational, then by definition $\rho_M(M) \subset \alpha_M^P(M \otimes_R \mathcal{C})$, i.e. $\varrho_M := (\alpha_M^P)^{-1} \circ \rho_M$ is well defined and we get the commutative diagram



By assumption $M_{\mathcal{A}}$ is $\mathcal{A}$-faithful (i.e. ρ_M is injective) and so ϱ_M is injective, i.e. the induced right $\mathcal{C}$-coaction on M is counital. Consider now diagram (5). By definition of ϱ_M and $\alpha_M^{P \otimes_l P}$ all trapezoids are commutative. By assumption M is a right $\mathcal{C}$-comodule and so the outer rectangle is commutative. By Lemma 1.33 (1) $\alpha_M^{P \otimes_l P}$ is injective and so the inner rectangle is commutative, i.e. (M, ϱ_M) is a counital right $\mathcal{C}$-comodule.

2. Consider diagram (6). The lower trapezoid is obviously commutative and the triangles are commutative by definition of ϱ_M and ϱ_N. Moreover α_N^P is injective and so the outer rectangle is commutative (f is $\mathcal{C}$-colinear) iff the upper trapezoid is commutative (f is $\mathcal{A}$-linear), i.e. $\mathrm{Hom}^{\mathcal{C}}(M, N) = \mathrm{Hom}_{-\mathcal{A}}(M, N)$.

3. Let (N, ρ_N) be a $\mathcal{C}$-rational right $\mathcal{A}$-module. If $K \subset N$ is an $\mathcal{A}$-submodule, then by Lemma 2.3 (2) $\mathrm{Rat}^{\mathcal{C}}(K_{\mathcal{A}}) = K \cap \mathrm{Rat}^{\mathcal{C}}(N_{\mathcal{A}}) = K$ and so $K_{\mathcal{A}}$ is $\mathcal{C}$-rational, hence K is a counital right $\mathcal{C}$-comodule by (1). Moreover $K \overset{\iota_K}{\hookrightarrow} N$ is $\mathcal{A}$-linear and so $\mathcal{C}$-colinear by (2), i.e. $K \subset N$ is a $\mathcal{C}$-subcomodule. Note that by assumption and Remark 1.30 ${}_R\mathcal{C}$ is flat, hence $\varrho_K = (\varrho_N)_{|K}$.■

2.7. For every R-coring $\mathcal{C}$ we have an isomorphism of R-rings $(\mathcal{C}^*, \star_r) \simeq \mathrm{End}^{\mathcal{C}}(\mathcal{C})$ via $f \mapsto [c \mapsto \sum f(c_1)c_2]$ with inverse $g \mapsto \varepsilon_{\mathcal{C}} \circ g$ (compare Proposition 1.10 (1)). Analogously $({}^*\mathcal{C}, \star_l) \simeq {}^{\mathcal{C}}\mathrm{End}(\mathcal{C})^{op}$ as R-rings.

If $P = (\mathcal{A}, \mathcal{C}) \in \mathcal{P}_{ml}^{\alpha}$, then by Lemma 2.6 (2) $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = \mathrm{End}(\mathcal{C}_{\mathcal{A}})$ and so

$${}^*\mathcal{C} \simeq {}^{\mathcal{C}}\mathrm{End}(\mathcal{C})^{op} \subseteq \mathrm{End}({}_{\mathcal{C}^*}\mathcal{C})^{op} = \mathrm{End}({}_{\mathrm{End}^{\mathcal{C}}(\mathcal{C})}\mathcal{C})^{op} = \mathrm{End}({}_{\mathrm{End}(\mathcal{C}_{\mathcal{A}})}\mathcal{C})^{op} := \mathrm{Biend}(\mathcal{C}_{\mathcal{A}}),$$

i.e. $({}^*\mathcal{C}, \star_l)$ is isomorphic to an R-subring of $\mathrm{Biend}(\mathcal{C}_{\mathcal{A}})$. If moreover $\mathcal{C}_R$ is locally projective, then ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C}) = \mathrm{End}({}_{\mathcal{C}^*}\mathcal{C})$, hence ${}^*\mathcal{C} \simeq \mathrm{Biend}(\mathcal{C}_{\mathcal{A}})$.

On the other hand, if $P \in \mathcal{P}^{\alpha}_{mr}$, then we have analogously ${}^{\mathcal{C}}\mathrm{End}(\mathcal{C}) = \mathrm{End}({}_{\mathcal{A}}\mathcal{C})$ and so

$$\mathcal{C}^* \simeq \mathrm{End}^{\mathcal{C}}(\mathcal{C}) \subseteq \mathrm{End}(\mathcal{C}_{{}^*\mathcal{C}}) = \mathrm{End}(\mathcal{C}_{{}^{\mathcal{C}}\mathrm{End}(\mathcal{C})^{op}}) = \mathrm{End}(\mathcal{C}_{\mathrm{End}({}_{\mathcal{A}}\mathcal{C})^{op}}) := \mathrm{Biend}({}_{\mathcal{A}}\mathcal{C}),$$

i.e. $(\mathcal{C}^*, \star_r)$ is isomorphic to an R-subring of $\mathrm{Biend}({}_{\mathcal{A}}\mathcal{C})$. If moreover ${}_R\mathcal{C}$ is locally projective, then $\mathrm{End}^{\mathcal{C}}(\mathcal{C}) = \mathrm{End}(\mathcal{C}_{{}^*\mathcal{C}})$, hence $\mathcal{C}^* \simeq \mathrm{Biend}({}_{\mathcal{A}}\mathcal{C})$.

Note that it follows from above, that in case ${}_R\mathcal{C}$ and $\mathcal{C}_R$ are locally projective we have ${}^*\mathcal{C} \simeq \mathrm{Biend}(\mathcal{C}_{{}^*\mathcal{C}})$ and $\mathcal{C}^* \simeq \mathrm{Biend}({}_{\mathcal{C}^*}\mathcal{C})$ as R-rings.

The following result generalizes [2, Theorem 2.2.13] from the case of commutative base rings to the case of arbitrary rings:

Proposition 2.8. *Let $P = (\mathcal{A}, \mathcal{C})$ be a measuring left R-pairing ($\mathcal{A}$ not necessarily with unity). If ${}_R\mathcal{C}$ is locally projective and $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense, then*

$$\begin{array}{lllllll} \mathcal{M}^{\mathcal{C}} & \simeq & \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}}) & = & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}}) & = & \sigma[\mathcal{C}_{\mathcal{A}}] \\ & \simeq & \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{{}^*\mathcal{C}}) & = & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{{}^*\mathcal{C}}) & = & \sigma[\mathcal{C}_{{}^*\mathcal{C}}]. \end{array} \tag{7}$$

Proof. Step (1). By Proposition 1.32 (2) $(\mathcal{A}, \mathcal{C})$ satisfies the left α-condition. By Lemmata 2.5 and 2.6 we have covariant functors

$$\begin{array}{llll} (-)_{\mathcal{A}} : & \mathcal{M}^{\mathcal{C}} & \to & \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}}), \\ & (M, \varrho_M) & \mapsto & (M, \alpha_M \circ \varrho_M), \end{array} \qquad \begin{array}{llll} (-)^{\mathcal{C}} : & \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}}) & \to & \mathcal{M}^{\mathcal{C}}, \\ & (M, \rho_M) & \mapsto & (M, \alpha_M^{-1} \circ \rho_M), \end{array}$$

that act as the identity on morphisms. Clearly we have

$$(-)^{\mathcal{C}} \circ \ (-)_{\mathcal{A}} = id_{\mathcal{M}^{\mathcal{C}}}, \ (-)_{\mathcal{A}} \circ (-)^{\mathcal{C}} = id_{\mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}})},$$

i.e. $\mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}}) \simeq \mathcal{M}^{\mathcal{C}}$.

Step (2). We show now that every $\mathcal{C}$-rational right $\mathcal{A}$-module is unital. Let $(N, \rho_N) \in \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{\mathcal{A}})$ and $n \in N$ with $\varrho_N(n) = \sum_{i=1}^{k} n_i \otimes c_i$. By assumption $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense and so there exists some $a \in \mathcal{A}$, such that $\kappa_P(a)(c_i) = \varepsilon_{\mathcal{C}}(c_i)$ for $i = 1, ..., k$. So

$$n = \sum_{i=1}^{k} n_i \varepsilon_{\mathcal{C}}(c_i) = \sum_{i=1}^{k} n_i < a, c_i >= na \in N\mathcal{A}.$$

Step (3). Let $(N, \rho_N) \in \mathcal{M}^{\mathcal{C}}$. For every $n \in N$ with $\varrho_N(n) = \sum_{i=1}^{k} n_i \otimes c_i$ we have $\{c_1, ..., c_k\}^{\perp} \subseteq (0_N : n)$, hence $N_{\mathcal{A}}$ is $\mathcal{C}$-subgenerated. By Proposition 1.10 (3), Lemma 2.6 and Step (1) $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck full subcategory of $\mathcal{M}_{\mathcal{A}}$ and so $\mathcal{M}^{\mathcal{C}} = \sigma[\mathcal{C}_{\mathcal{A}}]$ (since $\sigma[\mathcal{C}_{\mathcal{A}}]$ is the *smallest* such subcategory of $\mathcal{M}_{\mathcal{A}}$ containing $\mathcal{C}$).

Step (4). For $\mathcal{A} = {}^*\mathcal{C}$ we get as above

$$\mathcal{M}^{\mathcal{C}} \simeq \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}}_{{}^*\mathcal{C}}) = \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{{}^*\mathcal{C}}) = \sigma[\mathcal{C}_{{}^*\mathcal{C}}].\blacksquare$$

We are now ready to present the main result in this article, namely

Theorem 2.9. *Let $P = (\mathcal{A}, \mathcal{C})$ be a measuring left R-pairing. Then the following are equivalent:*

(i) ${}_R\mathcal{C}$ is locally projective and $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense;

(ii) ${}_R\mathcal{C}$ satisfies the α-condition and $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense;

(iii) $(\mathcal{A}, \mathcal{C})$ satisfies the left α-condition;

(iv) $\sigma[\mathcal{C}_{^*\mathcal{C}}] \simeq \mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{\mathcal{A}}]$.

If the equivalent conditions (i)-(iv) *are satisfied, then we have isomorphisms of categories*

$$\begin{array}{ccccccc} \mathcal{M}^{\mathcal{C}} & \simeq & \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}_{\mathcal{A}}}) & = & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}}) & = & \sigma[\mathcal{C}_{\mathcal{A}}] \\ & \simeq & \mathrm{Rat}^{\mathcal{C}}(\widetilde{\mathcal{M}_{^*\mathcal{C}}}) & = & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{^*\mathcal{C}}) & = & \sigma[\mathcal{C}_{^*\mathcal{C}}]. \end{array} \tag{8}$$

Proof. (i) $\Longleftrightarrow$ (ii) By Lemma 1.29 ${}_R\mathcal{C}$ is locally projective iff it satisfies the left α-condition.

(ii) $\Rightarrow$ (iii) follows from Proposition 1.32 (2).

(iii) $\Rightarrow$ (iv) If $(\mathcal{A}, \mathcal{C})$ satisfies the α-condition, then clearly ${}_R\mathcal{C}$ satisfies the left α-condition. Moreover, since (by our convention) $\mathcal{A}$ has unity with $\kappa_P(1_{\mathcal{A}}) = \varepsilon_{\mathcal{C}}$, we get by an analogous argument to that in the proof of Theorem 2.8 the isomorphisms of categories $\mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{\mathcal{A}}] = \sigma[\mathcal{C}_{^*\mathcal{C}}]$.

(iv) $\Rightarrow$ (i) By Assumption $\sigma[\mathcal{C}_{^*\mathcal{C}}] \simeq \mathcal{M}^{\mathcal{C}}$ and it follows analog to [36, 3.5] that ${}_R\mathcal{C}$ is locally projective. Moreover, for all $c_1, ..., c_k \in \mathcal{C}$, the right $\mathcal{A}$-module $(c_1, ..., c_k)\mathcal{A} \subset \mathcal{C}^k$ is a right ${}^*\mathcal{C}$-submodule (because $\sigma[\mathcal{C}_{\mathcal{A}}] \simeq \sigma[\mathcal{C}_{^*\mathcal{C}}]$). Hence $(c_1, ..., c_k)^*\mathcal{C} = ((c_1, ..., c_k)\mathcal{A})^*\mathcal{C} \subseteq (c_1, ..., c_k)\mathcal{A}$, i.e. $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense (see [39, 15.8]).

If the equivalent conditions (i)-(iv) are satisfied, then the assumptions of Theorem 2.8 are satisfied and so the isomorphisms of categories (8) are evident.■

Remark 2.10. Note that if $\mathcal{A}$ has no unity, then the implication (iv) $\Rightarrow$ (i) in the previous theorem is still evident, if ${}^*\mathcal{C}$ is unital as a left $\mathcal{A}$-module. In this case the four statements become equivalent, if we add "$\kappa_P(\mathcal{A}) \subset {}^*\mathcal{C}$ is dense" to statement (iii).

A dual version of Theorem 2.9 is valid for *measuring right R-pairings:*

Theorem 2.11. *For a measuring right R-pairing $P = (\mathcal{A}, \mathcal{C})$ the following are equivalent:*

(i) $\mathcal{C}_R$ is locally projective and $\kappa_P(\mathcal{A}) \subseteq \mathcal{C}^*$ is dense;

(ii) $\mathcal{C}_R$ satisfies the α-condition and $\kappa_P(\mathcal{A}) \subseteq \mathcal{C}^*$ is dense;

(iii) $(\mathcal{A}, \mathcal{C})$ is a right α-pairing;

(iv) $\sigma[{}_{\mathcal{C}^*}\mathcal{C}] \simeq {}^{\mathcal{C}}\mathcal{M} \simeq \sigma[{}_{\mathcal{A}}\mathcal{C}]$.

If the equivalent conditions (i)-(iv) *are satisfied, then we have isomorphisms of categories*

$$\begin{array}{ccccccc} {}^{\mathcal{C}}\mathcal{M} & \simeq & {}^{\mathcal{C}}\mathrm{Rat}(\widetilde{{}_{\mathcal{A}}\mathcal{M}}) & = & {}^{\mathcal{C}}\mathrm{Rat}({}_{\mathcal{A}}\mathcal{M}) & = & \sigma[{}_{\mathcal{A}}\mathcal{C}] \\ & \simeq & {}^{\mathcal{C}}\mathrm{Rat}(\widetilde{{}_{\mathcal{C}^*}\mathcal{M}}) & = & {}^{\mathcal{C}}\mathrm{Rat}({}_{\mathcal{C}^*}\mathcal{M}) & = & \sigma[{}_{\mathcal{C}^*}\mathcal{C}]. \end{array} \tag{9}$$

Remark 2.12. We should mention here that the implications (iii) $\Rightarrow$ (iv) in Theorem 2.9 resp. 2.11 were achieved independently in [17] Theorem 2.6' resp. Theorem 2.6 (note the interchange between *left* and *right* pairings in our notation).

As a corollary of Proposition 1.32 and Theorem 2.9 we get

Corollary 2.13. *Let $Q = (\mathcal{B}, \mathcal{C}) \in \mathcal{P}_{ml}$, $\xi : \mathcal{A} \to \mathcal{B}$ a morphism of R-rings and consider the induced measuring left pairing $P := (\mathcal{A}, \mathcal{C})$. Then the fallowing are equivalent:*

(i) $P \in \mathcal{P}^{\alpha}_{ml}$;

(ii) $Q \in \mathcal{P}^{\alpha}_{ml}$ and $\xi(\mathcal{A}) \subseteq \mathcal{B}$ is dense w.r.t. $\mathcal{B}[\mathfrak{T}_{ls}(\mathcal{C})]$;

(iii) ${}_R\mathcal{C}$ is locally projective and $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ is dense;

(iv) $\sigma[\mathcal{C}_{^*\mathcal{C}}] \simeq \mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{\mathcal{A}}] \simeq \sigma[\mathcal{C}_{\mathcal{B}}]$.

If these equivalent conditions are satisfied, then we have isomorphisms of categories:

$$\begin{array}{lclcl} \mathcal{M}^{\mathcal{C}} & \simeq & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}}) & = & \sigma[\mathcal{C}_{\mathcal{A}}] \\ & \simeq & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{^*\mathcal{C}}) & = & \sigma[\mathcal{C}_{^*\mathcal{C}}] \\ & \simeq & \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{B}}) & = & \sigma[\mathcal{C}_{\mathcal{B}}]. \end{array} \tag{10}$$

Definition 2.14. We call a measuring left (right) α-pairing $(\mathcal{A}, \mathcal{C})$ coproper, if $\mathcal{A}^{rat} := \mathrm{Rat}^{\mathcal{C}}(\mathcal{A}_{\mathcal{A}})$ (${}^{rat}\mathcal{A} := \mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{A}}\mathcal{A})$) is dense in $\mathcal{A}$. An R-coring with ${}_R\mathcal{C}$ ($\mathcal{C}_R$) locally projective will be called *left coproper* (*right coproper*), if ${}^{\Box}\mathcal{C} := \mathrm{Rat}^{\mathcal{C}}({}^*\mathcal{C}_{^*\mathcal{C}}) \subset {}^*\mathcal{C}$ (resp. $\mathcal{C}^{\Box} := {}^{\mathcal{C}}\mathrm{Rat}(\mathcal{C}^*_{\mathcal{C}^*}) \subseteq \mathcal{C}^*$) is dense. If ${}_R\mathcal{C}$ and $\mathcal{C}_R$ are locally projective, then we call $\mathcal{C}$ *coproper*, if it is left and right coproper.

Proposition 2.15. ([1]) *Let $P = (\mathcal{A}, \mathcal{C})$ be a coproper left measuring α-pairing (i.e. $\mathcal{T} := \mathrm{Rat}^{\mathcal{C}}(\mathcal{A}_{\mathcal{A}}) \subset \mathcal{A}$ is dense).*

1. $\mathcal{C}$ is left coproper, i.e. ${}^{\Box}\mathcal{C} \subset {}^*\mathcal{C}$ is dense.

2. For every $f \in \mathcal{T}$, there exists some $e \in \mathcal{T}$, such that $fe = f$.

3. For every right $\mathcal{A}$-module M we have $\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}}) = M\mathcal{T}$.

4. There is an isomorphism of categories $\mathcal{M}_{^{\Box}\mathcal{C}} \simeq \mathcal{M}^{\mathcal{C}} \simeq \mathcal{M}_{\mathcal{T}}$.

2.16. Birational modules. Let $P = (\mathcal{A}, \mathcal{C})$ be a measuring left α-pairing and $Q = (\mathcal{B}, \mathcal{D})$ be a measuring right α-pairing ($\mathcal{A}$, $\mathcal{B}$ not necessarily with unities). For a $(\mathcal{B}, \mathcal{A})$-faithful $(\mathcal{B}, \mathcal{A})$-bimodule $(M, \rho^{\mathcal{A}}_M, \rho^{\mathcal{B}}_M)$ it's obvious that ${}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{B}}M)$ is a right $\mathcal{A}$-module, $\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})$ is a left $\mathcal{B}$-module, and

$$\mathrm{Rat}^{\mathcal{C}}(({}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{B}}M))_{\mathcal{A}}) = {}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{B}}M) \cap \mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}}) = {}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{B}}(\mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}}))) \tag{11}$$

is a $(\mathcal{B}, \mathcal{A})$-subbimodule of M, which we call the $(\mathcal{D}, \mathcal{C})$-*birational* $(\mathcal{B}, \mathcal{A})$-*subbimodule* of M. If $M = \mathrm{Rat}^{\mathcal{C}}(({}^{\mathcal{D}}\mathrm{Rat}({}_{\mathcal{B}}M))_{\mathcal{A}})$, then we call ${}_{\mathcal{B}}M_{\mathcal{A}}$ $(\mathcal{D}, \mathcal{C})$-*birational.*

With ${}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{B}}\widetilde{\mathcal{M}}_{\mathcal{A}}) \subset {}_{\mathcal{B}}\widetilde{\mathcal{M}}_{\mathcal{A}}$ we denote the full subcategory of $(\mathcal{D}, \mathcal{C})$-birational $(\mathcal{B}, \mathcal{A})$-bimodules. The subcategory of unital $(\mathcal{D}, \mathcal{C})$-birational $(\mathcal{B}, \mathcal{A})$-bimodules is denoted with ${}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{B}}\mathcal{M}_{\mathcal{A}})$.

As a generalization of the corresponding result for coalgebras over base fields (e.g. [13, Theorem 2.3.3]) resp. over commutative rings ([2, Folgerung 2.2.19]) we get

Theorem 2.17. *Let $P = (\mathcal{A}, \mathcal{C})$ be a measuring left R-pairing and $Q = (\mathcal{B}, \mathcal{D})$ be a measuring right R-pairing ($\mathcal{A}$, $\mathcal{B}$ not necessarily with unities). If $\mathcal{C}, \mathcal{D}$ are locally projective, $\kappa_P(\mathcal{A}) \subseteq {}^*\mathcal{C}$ and $\kappa_Q(\mathcal{B}) \subseteq \mathcal{D}^*$ are dense, then there are isomorphisms of categories*

$$\begin{array}{lclcl} {}^{\mathcal{D}}\mathcal{M}^{\mathcal{C}} & \simeq & {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{B}}\widetilde{\mathcal{M}}_{\mathcal{A}}) & = & {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{B}}\mathcal{M}_{\mathcal{A}}) \\ & \simeq & {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{D}^*}\widetilde{\mathcal{M}}_{^*\mathcal{C}}) & = & {}^{\mathcal{D}}\mathrm{Rat}^{\mathcal{C}}({}_{\mathcal{D}^*}\mathcal{M}_{^*\mathcal{C}}) \end{array}$$

Proof. Let M be an arbitrary R-bimodule. In view of the previous results in this section it's enough to show that M is a counital $(\mathcal{D},\mathcal{C})$-bicomodule iff it's a $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{B},\mathcal{A})$-bimodule. If M is a counital $(\mathcal{D},\mathcal{C})$-bicomodule, then M is by Lemma 2.5 (1) a $\mathcal{C}$-rational right $\mathcal{A}$-module and analogously a $\mathcal{D}$-rational left $\mathcal{B}$-module. Moreover α_M^Q is obviously $\mathcal{A}$-linear, $\varrho_M^{\mathcal{D}}$ is by assumption $\mathcal{C}$-colinear, hence $\mathcal{A}$-linear by Lemma 2.5 (2). Consequently $\rho_M^{\mathcal{B}} = \alpha_M^Q \circ \varrho_M^{\mathcal{D}}$ is $\mathcal{A}$-linear, i.e. M is a $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{B},\mathcal{A})$-bimodule.

On the other hand, let M be a $(\mathcal{D},\mathcal{C})$-birational $(\mathcal{B},\mathcal{A})$-bimodule. By Lemma 2.6 M is a counital right $\mathcal{C}$-comodule and analogously a counital left $\mathcal{D}$-comodule. Since M is a $(\mathcal{B},\mathcal{A})$-bimodule, $\rho_M^{\mathcal{B}}$ is $\mathcal{A}$-linear and so we have for all $a \in \mathcal{A}$ and $m \in M$:

$$\alpha_M^Q(\varrho_M^{\mathcal{D}}(ma)) = \rho_M^{\mathcal{B}}(ma) = \rho_M^{\mathcal{B}}(m)a = (\alpha_M^Q(\varrho_M^{\mathcal{D}}(m))a = \alpha_M^Q(\varrho_M^{\mathcal{D}}(m)a),$$

hence $\varrho_M^{\mathcal{D}}$ is $\mathcal{A}$-linear by the injectivity of α_M^Q. By Lemma 2.6 (2), $\varrho_M^{\mathcal{D}}$ is $\mathcal{C}$-colinear, i.e. M is a counital $(\mathcal{D},\mathcal{C})$-bicomodule.■

As a consequence of Theorems 2.9 and 2.11 we get

Proposition 2.18. 1. *Let $P = (\mathcal{A},\mathcal{C}) \in \mathcal{P}_{ml}$. If $K \subset \mathcal{C}$ is a right $\mathcal{C}$-coideal (resp. a left $\mathcal{C}$-coideal, a $\mathcal{C}$-bicoideal), then $K^{\perp}$ is a left $\mathcal{A}$-ideal (resp. a right $\mathcal{A}$-ideal, an $\mathcal{A}$-ideal). If K is a $\mathcal{C}$-coideal, then $K^{\perp} \subset \mathcal{A}$ is an R-subring with unity $1_{\mathcal{A}}$. If $P \in \mathcal{P}_{ml}^{\alpha}$ and $I \subset \mathcal{A}$ is a left $\mathcal{A}$-ideal, then $I^{\perp} \subset \mathcal{C}$ is a right $\mathcal{C}$-coideal.*

2. *Let $P = (\mathcal{A},\mathcal{C}) \in \mathcal{P}_{mr}$. If $K \subset \mathcal{C}$ is a left $\mathcal{C}$-coideal (resp. a right $\mathcal{C}$-coideal, a $\mathcal{C}$-bicoideal), then $K^{\perp}$ is a right $\mathcal{A}$-ideal (resp. a left $\mathcal{A}$-ideal, an $\mathcal{A}$-ideal). If K is a $\mathcal{C}$-coideal, then $K^{\perp} \subset \mathcal{A}$ is an R-subring with unity $1_{\mathcal{A}}$. If $P \in \mathcal{P}_{mr}^{\alpha}$ and $I \subset \mathcal{A}$ is a right $\mathcal{A}$-ideal, then $I^{\perp} \subset \mathcal{C}$ is a left $\mathcal{C}$-coideal.*

Lemma 2.19. *Let X be set and XR the free R-module with basis X. If R_R is noetherian, then for every right R-module M, the following R-linear map is injective*

$$\beta_M : M \otimes_R R^X \to M^X,\ m \otimes f \longmapsto [x \longmapsto mf(x)]. \tag{12}$$

Hence $\widetilde{P} := (XR, R^X)$ is a left α-pairing.

Proof. Let M be an arbitrary right R-module and write M as a direct limit of its f.g. R-submodules $M = \varinjlim_{\Lambda} M_\lambda$ ([39, 24.7]). For every $\lambda \in \Lambda$, M_λ is f.p. in $\mathcal{M}_R$ and we have by ([39, 25.4]) the isomorphisms

$$\beta_{M_\lambda} : M_\lambda \otimes_R R^X \to M_\lambda^X,\ m \otimes f \mapsto [x \mapsto mf(x)]$$

Moreover for each $\lambda \in \Lambda$ the restriction of β_M on M_λ is equal to β_{M_λ} and so the following map is injective:

$$\beta_M = \varinjlim \beta_{M_\lambda} : \varinjlim M_\lambda \otimes_R R^X \to \varinjlim M_\lambda^X \subset M^X.$$

Obviously $\widetilde{P} \in \mathcal{P}_l^{\alpha}$ iff β_M is injective for every $M \in \mathcal{M}_R$.■

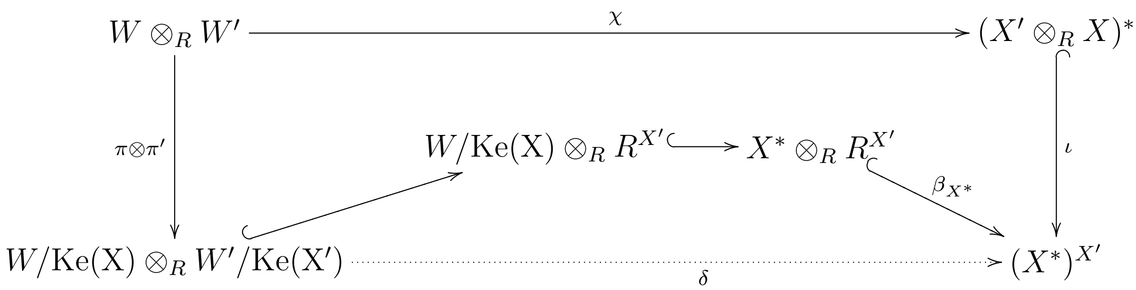
Corollary 2.20. *Let W, W' be R-bimodules, $X \subset {}^*W, X' \subset {}^*W'$ be R-subbimodules and consider the canonical R-linear maps*

$$\kappa : X' \otimes_R X \to {}^*(W \otimes_R W') \text{ and } \chi : W \otimes_R W' \to (X' \otimes_R X)^*.$$

If R_R is noetherian, W_R is flat and $\mathrm{Ke}(X)_R \subset W_R$ is pure, then

$$\mathrm{Ke}(\kappa(X' \otimes_R X)) \simeq \mathrm{Ke}(X) \otimes_R W' + W \otimes_R \mathrm{Ke}(X'). \tag{13}$$

Proof. Consider the embeddings $E := W/\mathrm{Ke}(X) \hookrightarrow X^*$, $E' := W'/\mathrm{Ke}(X') \hookrightarrow R^{X'}$ and the commutative diagram



It follows by assumptions that $W/\mathrm{Ke}(X)$ is flat in $\mathcal{M}_R$ and ${}_RR^{X'}$ is flat (e.g. [39, 36.5, 26.6]). Moreover β_{X^*} is injective by Lemma 2.19, hence δ is injective. It follows then by [10, II-3.6] that

$$\begin{array}{lcl} \mathrm{Ke}(\kappa(X \otimes_R X')) & := & \mathrm{Ke}(\chi) \quad = \quad \mathrm{Ke}(\delta \circ (\pi \otimes \pi')) \\ & = & \mathrm{Ke}(\pi_X \otimes \pi_{X'}) \quad = \quad \mathrm{Ke}(X) \otimes_R W' + W \otimes_R \mathrm{Ke}(X').\blacksquare \end{array}$$

Proposition 2.21. *Let R be a QF ring and $\mathcal{C}$ an R-coring. If $\mathcal{A} \subseteq {}^*\mathcal{C}$ is an R-subring (with $\varepsilon_{\mathcal{C}} \in \mathcal{A}$), $\mathcal{C}_R$ is flat and $\mathrm{Ke}(\mathcal{A})_R \subset \mathcal{C}_R$ is pure, then $\Delta_{\mathcal{C}}(\mathrm{Ke}(\mathcal{A})) \subseteq \mathrm{Ke}(\mathcal{A}) \otimes_R \mathcal{C} + \mathcal{C} \otimes_R \mathrm{Ke}(\mathcal{A})$ ($\mathrm{Ke}(\mathcal{A}) \subset \mathcal{C}$ is a $\mathcal{C}$-coideal).*

Proof. Let $\mathcal{A} \subseteq {}^*\mathcal{C}$ be an R-subring and consider the R-linear map

$$\kappa : \mathcal{A} \otimes_R \mathcal{A} \to {}^*(\mathcal{C} \otimes_R \mathcal{C}),\ a \otimes b \mapsto [c \otimes d \mapsto < b, c < a, d >>].$$

If $\mathcal{C}_R$ is flat and $\mathrm{Ke}(\mathcal{A})_R \subset \mathcal{C}_R$ is pure, then we have by Corollary 2.20 and Lemma 1.18:

$$\mathrm{Ke}(\mathcal{A}) \subseteq \mathrm{Ke}(\Delta_{\mathcal{C}}^*(\kappa(\mathcal{A} \otimes_R \mathcal{A}))) = \Delta_{\mathcal{C}}^{-1}(\mathrm{Ke}(\mathcal{A}) \otimes_R \mathcal{C} + \mathcal{C} \otimes_R \mathrm{Ke}(\mathcal{A})), \tag{14}$$

i.e. $\Delta_{\mathcal{C}}(\mathrm{Ke}(\mathcal{A})) \subseteq \mathrm{Ke}(\mathcal{A}) \otimes_R \mathcal{C} + \mathcal{C} \otimes_R \mathrm{Ke}(\mathcal{A})$. If $\varepsilon_{\mathcal{C}} \in \mathcal{A}$, then $\varepsilon_{\mathcal{C}}(\mathrm{Ke}(\mathcal{A})) = 0$, hence $\mathrm{Ke}(\mathcal{A}) \subset \mathcal{C}$ is a $\mathcal{C}$-coideal.$\blacksquare$

Corollary 2.22. *Let $\mathcal{C}$ be an R-coring and assume that ${}_R\mathcal{C}$ is locally projective. For every R-coring $\mathcal{D}$ with an injective morphism of R-corings $\iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C}$ we have:*

1. *${}_R\mathcal{D} \subseteq {}_R\mathcal{C}$ is pure iff $Q := ({}^*\mathcal{C}, \mathcal{D}) \in \mathcal{P}^{\alpha}_{ml}$. In this case ${}_R\mathcal{D}$ is locally projective, $\iota^*_{\mathcal{D}}({}^*\mathcal{C}) \subseteq {}^*\mathcal{D}$ is dense and there are isomorphisms of categories*

$$\begin{array}{rcccl}\mathcal{M}^{\mathcal{D}} & \simeq & \mathrm{Rat}^{\mathcal{D}}(\mathcal{M}_{{}^*\mathcal{D}}) & = & \sigma[\mathcal{D}_{{}^*\mathcal{D}}] \\ & \simeq & \mathrm{Rat}^{\mathcal{D}}(\mathcal{M}_{{}^*\mathcal{C}}) & = & \sigma[\mathcal{D}_{{}^*\mathcal{C}}].\end{array}$$

2. *Let ${}_RR$ be $\mathcal{C}$-injective. Then ${}_R\mathcal{D} \subseteq {}_R\mathcal{C}$ is pure iff ${}_R\mathcal{D}$ is locally projective.*

3. *If ${}_R\mathcal{D} \subseteq {}_R\mathcal{C}$ is pure, then*

$$\mathcal{D} = \mathcal{C} \Longleftrightarrow \mathcal{M}^{\mathcal{D}} = \mathcal{M}^{\mathcal{C}} \Longleftrightarrow \mathcal{C}_{{}^*\mathcal{C}} \text{ is } \mathcal{D}\text{-rational.}$$

Proof. 1. Since $\iota_{\mathcal{D}}$ is a morphism of R-corings, it follows that $\iota^*_{\mathcal{D}} : {}^*\mathcal{C} \to {}^*\mathcal{D}$ is a morphism of R-rings, i.e. Q is a measuring left R-pairing. The result follows now by Theorem 2.9 and the commutativity of the following diagram for every right R-module M

$$\begin{array}{ccc} M \otimes_R \mathcal{D} & \xrightarrow{id_M \otimes \iota_{\mathcal{D}}} & M \otimes_R \mathcal{C} \\ \downarrow{\scriptstyle \alpha^{\mathcal{D}}_M} & \searrow{\scriptstyle \alpha^{Q}_M} & \downarrow{\scriptstyle \alpha^{\mathcal{C}}_M} \\ \mathrm{Hom}_{-R}({}^*\mathcal{D}, M) & \xrightarrow[(\iota^*_{\mathcal{D}}, M)]{} & \mathrm{Hom}_{-R}({}^*\mathcal{C}, M) \end{array} \qquad (15)$$

2. If ${}_RR$ is $\mathcal{C}$-injective, then $\iota^*_{\mathcal{D}} : {}^*\mathcal{C} \to {}^*\mathcal{D}$ is surjective. Hence, for every right R-module M the map $(\iota^*_{\mathcal{D}}, M)$ in diagram (15) is injective and the result follows.

3. It is enough to prove: $\mathcal{C} \in \mathrm{Rat}^{\mathcal{D}}(\mathcal{M}_{{}^*\mathcal{C}}) \Longrightarrow \mathcal{D} = \mathcal{C}$.

   Assume $\mathcal{C} \in \mathrm{Rat}^{\mathcal{D}}(\mathcal{M}_{{}^*\mathcal{C}}) \simeq \mathcal{M}^{\mathcal{D}}$. Then there exists a right R-linear map

$$\varrho : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{D},\ c \longmapsto \sum_{i=1}^{k_c} c_i \otimes d_i,$$

   such that $c \leftharpoonup f = \sum\limits_{i=1}^{k_c} c_i \iota^*_{\mathcal{D}}(f)(d_i)$ for every $f \in {}^*\mathcal{C}$. Consider the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varrho} & \mathcal{C} \otimes_R \mathcal{D} \\ & \searrow{\scriptstyle \Delta_{\mathcal{C}}} & \downarrow{\scriptstyle id_{\mathcal{C}} \otimes \iota_{\mathcal{D}}} \\ & & \mathcal{C} \otimes_R \mathcal{C} \end{array} \qquad (16)$$

   and the left α-pairing $P \otimes_l P := ({}^*\mathcal{C} \otimes_R {}^*\mathcal{C}, \mathcal{C} \otimes_R \mathcal{C})$ (see Lemma 1.33 (1)). Then we have for all $c \in \mathcal{C}$ and $f, g \in {}^*\mathcal{C}$:

$$\begin{array}{rclcl}\chi_{P \otimes_l P}(\sum c_1 \otimes c_2)(f \otimes g) & = & \sum g(c_1 f(c_2)) & = & g(c \leftharpoonup f) \\ & = & g(\sum\limits_{i=1}^{k_c} c_i \iota^*_{\mathcal{D}}(f)(d_i)) & = & \chi_{P \otimes_l P}(\sum\limits_{i=1}^{k_c} c_i \otimes \iota(d_i))(f \otimes g),\end{array}$$

and so $\sum c_1 \otimes c_2 = \sum_{i=1}^{k_c} c_i \otimes \iota(d_i)$, i.e. diagram (16) is commutative. Hence for every $c \in \mathcal{C}$ we have

$$c = \sum \varepsilon_{\mathcal{C}}(c_1)c_2 = \sum_{i=1}^{k_c} \varepsilon_{\mathcal{C}}(c_i)\iota(d_i) \in \iota(\mathcal{D}),$$

i.e. $\mathcal{C} = \mathcal{D}$.■

Remark 2.23. Even if $\mathcal{C}$ is an R-coring and $\mathcal{D} \subset \mathcal{C}$ is an R-subbimodule with $\Delta_{\mathcal{C}}(\mathcal{D}) \subseteq \mathrm{Im}(\mathcal{D} \otimes_R \mathcal{D})$, $\mathcal{D}$ may have no R-coring structure such that the natural embedding $\iota_{\mathcal{D}} : \mathcal{D} \hookrightarrow \mathcal{C}$ is a morphism of R-corings. For $\mathcal{D}$ to be an R-subcoring of $\mathcal{C}$ we need ${}_R\mathcal{D}_R \subset {}_R\mathcal{C}_R$ to be *pure* (in the sense of Cohn). A counterexample for coalgebras over commutative rings can be found in [28, Page 56].

An important role by studying the category of rational representations of a left measuring pairing $P \in \mathcal{P}^{\alpha}_{ml}$ is played by the

2.24. Finiteness Theorem. Let $P = (\mathcal{A}, \mathcal{C}) \in \mathcal{P}^{\alpha}_{ml}$.

1. If $M \in \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}})$, then there exists for every finite subset $\{m_1, ..., m_k\} \subset M$ some $N \in \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}})$, such that $N \subset M$ and N_R is finitely generated.

2. Every finite subset of $\mathcal{C}$ is contained in a right $\mathcal{C}$-coideal, which is f.g. in $\mathcal{M}_R$.

Proof. 1. Let $M \in \mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}})$ and $\{m_1, ..., m_k\} \subset M$. Then $m_i\mathcal{A} \subset M$ is an $\mathcal{A}$-submodule, hence a $\mathcal{C}$-subcomodule. Moreover $m_i \in m_i\mathcal{A}$ and consequently there exists a subset $\{(m_{ij}, c_{ij})\}_{j=1}^{n_i} \subset m_i\mathcal{A} \times \mathcal{C}$, so that $\varrho_M(m_i) = \sum_{j=1}^{n_i} m_{ij} \otimes c_{ij}$ for $i = 1, ..., k$. Obviously $N := \sum_{i=1}^{k} m_i\mathcal{A} = \sum_{i=1}^{k}\sum_{j=1}^{n_i} m_{ij}R \subset M$ is a $\mathcal{C}$-subcomodule and contains $\{m_1, ..., m_k\}$.

2. This is a special case of (1).■

For every $(\mathcal{A}, \mathcal{C}) \in \mathcal{P}^{\alpha}_{ml}$ we get from the isomorphism of categories $\mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}}) \simeq \sigma[\mathcal{C}_{\mathcal{A}}]$ and [38, 2.9]:

Corollary 2.25. *Let* $(\mathcal{A}, \mathcal{C}) \in \mathcal{P}^{\alpha}_{ml}$.

1. $\mathrm{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{A}})$ *is* $(\mathcal{A}, R)$*-finite (i.e. a* $\mathcal{C}$*-rational right* $\mathcal{A}$*-module is f.g. in* $\mathcal{M}_{\mathcal{A}}$ *iff it's f.g. in* $\mathcal{M}_R$*).*

2. *If* ${}_RR$ *is perfect, then every* $\mathcal{C}$*-rational right* $\mathcal{A}$*-module satisfies the descending chain condition w.r.t. the f.g.* $\mathcal{A}$*-submodules.*

3. *If* R_R *is noetherian, then every* $\mathcal{C}$*-rational right* $\mathcal{A}$*-module is locally noetherian.*

4. *If* R_R *is artinian, then every f.g.* $\mathcal{C}$*-rational right* $\mathcal{A}$*-module has finite length.*

Proposition 2.26. *For every dense measuring left R-pairing $P = (\mathcal{A}, \mathcal{C})$ the following are equivalent:*

1. $\mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{\mathcal{A}}] = \mathcal{M}_{\mathcal{A}/\mathrm{An}_{\mathcal{A}}(\mathcal{C})}$;
2. *The functor* $-\otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}_{\mathcal{A}/\mathrm{An}_{\mathcal{A}}(\mathcal{C})}$ *has a left adjoint;*
3. ${}_R\mathcal{C}$ *is f.g. and projective;*
4. ${}_{\mathcal{C}^*}\mathcal{C}$ *is f.g. and* ${}_R\mathcal{C}$ *is locally projective;*
5. $\mathcal{A}/\mathrm{An}_{\mathcal{A}}(\mathcal{C})$ *is f.g. in* $\mathcal{M}_R$ *and* ${}_R\mathcal{C}$ *is locally projective.*

Proof. With the help of Proposition 1.10 and Theorem 2.9, the equivalence of the first four statements can be established as in [36, 3.6].

If we assume (1) or (5), then we conclude that $\mathcal{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{\mathcal{A}}] \simeq \sigma[\mathcal{C}_{^*\mathcal{C}}]$ by Proposition 2.8 and is $(\mathcal{A}, R)$-finite by Corollary 2.25. The result follows then by the fact that in this case $\sigma[\mathcal{C}_{\mathcal{A}}] = \mathcal{M}_{\mathcal{A}/\mathrm{An}_{\mathcal{A}}(\mathcal{C})}$ iff $\mathcal{A}/\mathrm{An}_{\mathcal{A}}(\mathcal{C})$ is f.g. in $\mathcal{M}_R$ [38, 2.9 (3)].■

Remark 2.27. It follows from Proposition 2.26 that for an R-coring $\mathcal{C}$ with ${}_R\mathcal{C}$ locally projective and every dense R-subring $\mathcal{A} \subseteq {}^*\mathcal{C}$:

$$\mathcal{A}_R \text{ is f.g.} \Longleftrightarrow {}_R\mathcal{C} \text{ is f.g.} \Longleftrightarrow {}_R\mathcal{C} \text{ is f.g. and projective.}$$

In particular a f.g. locally projective R-coring is projective.

The following result gives topological characterizations of the $\mathcal{C}$-rational right $\mathcal{A}$-modules. Here we generalize some of those characterizations given by D. Radford in [30, 2.2] in the case of coalgebras over base fields to the case of corings over arbitrary (artinian) ground rings. See also [2, Proposition 2.2.26] for the case of coalgebras over commutative rings.

Proposition 2.28. *Let* $P = (\mathcal{A}, \mathcal{C}) \in \mathcal{P}^{\alpha}_{ml}$ *and consider* $\mathcal{A}$ *with the* $\mathcal{C}$*-adic topology* $\mathcal{T}_{-\mathcal{C}}(\mathcal{A}) = \mathcal{A}[\mathfrak{T}^r_{ls}(\mathcal{C})]$. *If* M *is a unital right* $\mathcal{A}$*-module, then for every* $m \in M$ *the following are equivalent:*

1. *there exists a finite subset* $F = \{c_1, ..., c_k\} \subset \mathcal{C}$*, such that* $(0_{\mathcal{C}} : F) \subseteq (0_M : m)$;
2. $m\mathcal{A}$ *is* $\mathcal{C}$*-subgenerated;*
3. $m \in \mathrm{Rat}^{\mathcal{C}}(M_{\mathcal{A}})$;
4. *there exists a f.g. left* R*-submodule* $K \subset \mathcal{C}$*, such that* $K^{\perp} \subseteq (0_M : m)$.

   If R_R *is artinian, then* (1)-(4) *are moreover equivalent to:*

5. $(0_M : m)$ *contains an* R*-cofinite closed* R*-submodule;*
6. $(0_M : m)$ *is an* R*-cofinite closed right* $\mathcal{A}$*-ideal.*

Proof. (1) $\Longrightarrow$ (2) By assumption and 1.24 $m \in N := \mathrm{Sp}(\sigma[\mathcal{C}_\mathcal{A}], M)$. Moreover $m\mathcal{A} \subset N$ is a right $\mathcal{A}$-submodule and is consequently $\mathcal{C}$-subgenerated.

(2) $\Longrightarrow$ (3) By assumption and Theorem 2.9 $m \in m\mathcal{A} \subset \mathrm{Rat}^{\mathcal{C}}(M_\mathcal{A})$.

(3) $\Longrightarrow$ (4) Let $\varrho(m) = \sum_{i=1}^{k} m_i \otimes c_i$ and put $K := \sum_{i=1}^{k} Rc_i \subset \mathcal{C}$. Then clearly $K^\perp \subseteq (0_M : m)$.

(4) $\Longrightarrow$ (1) For every left R-submodule $K \subseteq \mathcal{C}$ we have $(0_\mathcal{C} : K) \subseteq K^\perp$.

Let R_R be *artinian.*

(4) $\Longrightarrow$ (5). Assume $K^\perp \subseteq (0_M : m)$ for some $K = \sum_{i=1}^{k} Rc_i \subset \mathcal{C}$. Since $\mathcal{A}/K^\perp \hookrightarrow {}^*K$ and R_R is noetherian, we conclude that $K^\perp \subset \mathcal{A}$ is R-cofinite. Moreover $K^\perp$ is by Lemma 1.16 (1) closed.

The implications (5) $\Rightarrow$ (6) $\Rightarrow$ (1) follow from Lemma 1.16 (3).■

3 Applications

In what follows R is a *commutative* ring and $\mathcal{M}_R$ is the category of R-(bi)modules. For an R-algebra (A, μ_A, η_A) and an R-coalgebra $(C, \Delta_C, \varepsilon_C)$ we consider $(\mathrm{Hom}_R(C, A), \star) := \mathrm{Hom}_R(C, A)$ as an R-algebra with the so called *convolution product* $(f \star g)(c) := \sum f(c_1)g(c_2)$ and unity $\eta_A \circ \varepsilon_C$. With this definition C becomes a C^*-bimodule through the left and the right C^*-action $f \rightharpoonup c = \sum c_1 f(c_2)$ and $c \leftharpoonup f = \sum f(c_1)c_2$.

Entwined Modules

Next we apply our results in the previous sections to the category of entwined modules corresponding to a right-right entwining structure (A, C, ψ). These were introduced by T. Brzeziński and S. Majid [7] as a generalization of the Doi-Koppinen modules corresponding to a right-right Doi-Koppinen structure (see 3.15).

3.1. A *right-right entwining structure* (A, C, ψ) over R consists of an R-algebra A, an R-coalgebra C and an R-linear map

$$\psi : C \otimes_R A \to A \otimes_R C,\ c \otimes a \mapsto \sum a_\psi \otimes c^\psi,$$

such that

$$\begin{array}{rclrcl} \sum (a\widetilde{a})_\psi \otimes c^\psi & = & \sum a_\psi \widetilde{a}_\Psi \otimes c^{\psi\Psi}, & \sum (1_A)_\psi \otimes c^\psi & = & 1_A \otimes c, \\ \sum a_\psi \otimes \Delta_C(c^\psi) & = & \sum a_{\psi\Psi} \otimes c_1^\Psi \otimes c_2^\psi, & \sum a_\psi \varepsilon_C(c^\psi) & = & \varepsilon_C(c)a. \end{array} \qquad (17)$$

3.2. Let (A, C, ψ) be a right-right entwining structure. An *entwined module* corresponding to (A, C, ψ) is a right A-module M, which is also a right C-comodule through ϱ_M, such that

$$\varrho_M(ma) = \sum m_{<0>} a_\psi \otimes m_{<1>}^\psi \text{ for all } m \in M \text{ and } a \in A.$$

The category of right-right entwined modules and A-linear C-colinear morphisms is denoted by $\mathcal{M}_A^C(\psi)$. For $M, N \in \mathcal{M}_A^C(\psi)$ we denote by $\mathrm{Hom}_A^C(M, N)$ the set of A-linear C-colinear morphisms from M to N. By a remark of M. Takeuchi (e.g. [8, Proposition 2.2]) $\mathcal{C} := A \otimes_R C$ is an A-coring with A-bimodule structure given by

$$a(\widetilde{a} \otimes c) := a\widetilde{a} \otimes c,\ (\widetilde{a} \otimes c)a := \sum \widetilde{a}a_\psi \otimes c^\psi, \tag{18}$$

comultiplication

$$\Delta_{\mathcal{C}} : A \otimes_R C \to (A \otimes_R C) \otimes_A (A \otimes_R C),\ a \otimes c \mapsto \sum (a \otimes c_1) \otimes_A (1_A \otimes c_2)$$

and counity $\varepsilon_{\mathcal{C}} := \vartheta_A^r \circ (id_A \otimes \varepsilon_C)$. Moreover $\mathcal{M}_A^C(\psi) \simeq \mathcal{M}^{\mathcal{C}}$.

Lemma 3.3. (See [37, 4.2]) *Let (A, C, ψ) be a right-right entwining structure over R and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

1. $\#_\psi^{op}(C, A) := \mathrm{Hom}_R(C, A)$ *is an A-ring with A-bimodule structure given by* $(af)(c) := \sum a_\psi f(c^\psi)$, $(fa)(c) := f(c)a$, *multiplication*

$$(f \cdot g)(c) = \sum f(c_2)_\psi g(c_1^\psi), \tag{19}$$

and unity $\eta_A \circ \varepsilon_C$.

2. $\#_\psi^{op}(C, A) \simeq {}^*\mathcal{C}$ *as A-rings via*

$$\varphi : \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_{A-}(A \otimes_R C, A),\ f \mapsto [a \otimes c \mapsto af(c)] \tag{20}$$

with inverse $h \mapsto [c \mapsto h(1_A \otimes c)]$.

3.4. A *left-right entwining structure* is a triple (A, C, ψ), where A is an R-algebra, C is an R-coalgebra and

$$\psi : A \otimes_R C \to A \otimes_R C,\ a \otimes c \mapsto \sum a_\psi \otimes c^\psi,$$

is an R-linear map such that the conditions in (17) are satisfied with the first of them replaced by

$$\sum (a\widetilde{a})_\psi \otimes c^\psi = \sum a_\psi \widetilde{a}_\Psi \otimes c^{\Psi\psi} \text{ for all } a, \widetilde{a} \in A, c \in C.$$

3.5. Let (A, C, ψ) be a left-right entwining structure. With an entwined module corresponding to (A, C, ψ) we mean a left A-module M, which is also a right C-comodule through ϱ_M, s.t.

$$\varrho_M(am) = \sum a_\psi m_{<0>} \otimes m_{<1>}^\psi \text{ for all } a \in A \text{ and } m \in M.$$

The category of left-right entwined modules and A-linear C-colinear morphisms is denoted by ${}_A\mathcal{M}^C(\psi)$. For $M, N \in {}_A\mathcal{M}^C(\psi)$ we denote by ${}_A\mathrm{Hom}^C(M, N)$ the set of all A-linear C-colinear morphisms from M to N. It's easy to see that $(A^{op}, C, \psi \circ \tau)$ is a right-right entwining structure, hence $\mathcal{D} := A^{op} \otimes_R C$ is an A^{op}-coring and ${}_A\mathcal{M}^C(\psi) \simeq \mathcal{M}_{A^{op}}^C(\psi \circ \tau) \simeq \mathcal{M}^{A^{op} \otimes_R C}$. Moreover $\#_{\psi\circ\tau}^{op}(C, A^{op}) \simeq {}^*\mathcal{D}$ as A^{op}-rings and $\#_\psi(C, A) := (\#_{\psi\circ\tau}^{op}(C, A^{op}))^{op}$ is an A-ring with multiplication

$$(f \cdot g)(c) = \sum f(c_1^\psi) g(c_2)_\psi \text{ for all } f, g \in \mathrm{Hom}_R(C, A) \text{ and } c \in C \tag{21}$$

and unity $\eta_A \circ \varepsilon_C$.

3.6. Let (A, C, ψ) be a right-right entwining structure over R and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$. We say that (A, C, ψ) satisfies the *α-condition*, if for every right A-module M the following map is injective

$$\alpha_M^{\psi} : M \otimes_R C \to \mathrm{Hom}_R(\#_{\psi}^{op}(C, A), M),\ m \otimes c \mapsto [f \mapsto mf(c)]$$

(equivalently, if ${}_A\mathcal{C}$ is locally projective).

Inspired by [14, 3.1] we present

Definition 3.7. 1. Let (A, C, ψ) be a right-right entwining structure satisfying the α-condition. Let $M \in \mathcal{M}_{\#_{\psi}^{op}(C,A)}$, $\rho_M : M \to \mathrm{Hom}_{-A}(\#_{\psi}^{op}(C, A), M)$ the canonical map and $\mathrm{Rat}^C(M) := \rho_M^{-1}(M \otimes_R C)$. If $\mathrm{Rat}^C(M_{\#_{\psi}^{op}(C,A)}) = M$, then we call M *$\#$-rational* and set $\varrho_M := (\alpha_M^{\psi})^{-1} \circ \rho_M : M \to M \otimes_R C$. The class of *$\#$-rational* right $\#_{\psi}^{op}(C, A)$-modules build a full subcatgery of $\mathcal{M}_{\#_{\psi}^{op}(C,A)}$, which we denote with $\mathrm{Rat}^C(\mathcal{M}_{\#_{\psi}^{op}(C,A)})$. For a left-right entwining structure (A, C, ψ), the α-condition and the category of *$\#$-rational left $\#_{\psi}(C, A)$-modules* are analogously defined.

Lemma 3.8. *Let (A, C, ψ) be a right-right entwining structure over R and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

1. *If ${}_RC$ is flat (resp. projective, f.g.), then ${}_A\mathcal{C}$ is flat (resp. projective, f.g.).*
2. *If ${}_RC$ is locally projective, then ${}_A\mathcal{C}$ is locally projective (i.e. (A, C, ψ) satisfies the left α-condition).*

Proof. 1. Clear.

2. For every right A-module M we have the commutative diagram

$$\begin{array}{ccc} M \otimes_A (A \otimes_R C) & \xrightarrow{\alpha_M^{\mathcal{C}}} & \mathrm{Hom}_{-A}({}^*\mathcal{C}, M) \\ \| & & \downarrow{\scriptstyle (\gamma, M)} \\ M \otimes_R C & \xrightarrow[\alpha_M^C]{} & \mathrm{Hom}_R(C^*, M) \end{array}$$

where $\gamma : C^* \to {}^*\mathcal{C},\ f \mapsto [a \otimes c \mapsto af(c)]$. The result follows then by Lemma 1.29.■

Lemma 3.9. *Let (A, C, ψ) be a right-right entwining structure.*

1. *If $M \in \mathcal{M}_A^C(\psi)$ and $N \subset M$ is a C-subcomodule, then NA is a subobject of M in $\mathcal{M}_A^C(\psi)$.*
2. *Assume ${}_RC$ to be locally projective.*

   (a) *For every right $\#_{\psi}^{op}(C, A)$ module M we have $\mathrm{Rat}^C(M_{\#_{\psi}^{op}(C,A)}) \in \mathcal{M}_A^C(\psi)$.*

(b) *If $M \in \mathcal{M}_A^C(\psi)$, then M becomes a #-rational right $\#_\psi^{op}(C,A)$-module through*

$$mf = \sum m_{<0>} f(m_{<1>}) \text{ for all } m \in M \text{ and } f \in \#_\psi^{op}(C,A).$$

(c) *Assume $A \in \mathcal{M}_A^C(\psi)$, so that $\sum 1_{<0>} \otimes 1_{<1>} \in \mathcal{C}$ is a group-like element. Let $M \in \mathcal{M}_{\#_\psi^{op}(C,A)}$ be C-rational and put $M^{co\mathcal{C}} := \{m \in M \mid \varrho_M(m) = \sum m1_{<0>} \otimes 1_{<1>}\}$. If $\Psi_M : M^{co\mathcal{C}} \otimes_B A \to M$, $m \otimes a \mapsto ma$ is surjective, then M is #-rational.*

Proof. 1. For every $n \in N$ and $a \in A$ we have

$$\varrho_M(na) = \sum n_{<0>} a_\psi \otimes n_{<1>}^\psi \in NA \otimes_R C.$$

Consequently $NA \subset M$ is a C-subcomodule with structure map $(\varrho_M)_{|_{NA}}$, hence $NA \subset M$ is a subobject in $\mathcal{M}_A^C(\psi)$.

2. Assume ${}_RC$ to be locally projective.

(a) Every right $\#_\psi^{op}(C,A)$-module M becomes a right A-module through the canonical algebra morphism $\iota_A : A \to \#_\psi^{op}(C,A)$ and a left C^*-module through the algebra anti-morphism $\iota_{C^*} : C^* \to \#_\psi^{op}(C,A)$. Clearly $\mathrm{Rat}^C(M_{\#_\psi^{op}(C,A)}) \subset M$ is a C-rational left C^*-module. Moreover, for all $m \in \mathrm{Rat}^C(M_{\#_\psi^{op}(C,A)})$, $a \in A$ and $g \in \#_\psi^{op}(C,A)$ we have

$$\begin{aligned}
[ma]g &= (m\iota_A(a))g \\
&= m(\iota_A(a) \cdot g) \\
&= \sum m_{<0>}(\iota_A(a) \cdot g))(m_{<1>}) \\
&= \sum m_{<0>} \iota_A(a)(m_{<1>2})_\psi g(m_{<1>1}^\psi) \\
&= \sum m_{<0>}(a\varepsilon_C(m_{<1>2}))_\psi g(m_{<1>1}^\psi) \\
&= \sum m_{<0>} a_\psi g(m_{<1>}^\psi),
\end{aligned}$$

i.e. $ma \in \mathrm{Rat}^C(M_{\#_\psi^{op}(C,A)})$ with $\varrho(ma) = \sum m_{<0>} a_\psi \otimes m_{<1>}^\psi$ and the result follows.

(b) Let $M \in \mathcal{M}_A^C(\psi)$. Then for every $m \in M$ and $f, g \in \#_\psi^{op}(C,A)$ we have

$$\begin{aligned}
m(f \cdot g) &= \sum m_{<0>}(f \cdot g)(m_{<1>}) \\
&= \sum m_{<0>} f(m_{<1>2})_\psi g(m_{<1>1}^\psi) \\
&= \sum m_{<0><0>} f(m_{<1>})_\psi g(m_{<0><1>}^\psi) \\
&= (\sum m_{<0>} f(m_{<1>}))g \\
&= (mf)g.
\end{aligned}$$

(c) Let $m \in M$ be arbitrary. By assumption $m = \Psi_M(\sum_{i=1}^k n_i \otimes a_i)$ for some $\sum_{i=1}^k n_i \otimes a_i \in M^{co\mathcal{C}} \otimes_R A$, hence we have for all $f \in \#_\psi^{op}(C,A)$:

$$\begin{aligned}
mf &= \sum_{i=1}^k n_i(a_i f) &&= \sum_{i=1}^k n_i 1_{<0>} a_{i_\psi} f(1_{<1>}^\psi) \\
&= \sum_{i=1}^k (n_i a_i)_{<0>} f((n_i a_i)_{<1>}) &&= \sum_{i=1}^k m_{<0>} f(m_{<1>}). \blacksquare
\end{aligned}$$

The main result in this section is

Theorem 3.10. *Let (A, C, ψ) be a right-right entwining structure and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

1. *If $_RC$ is flat, then $\mathcal{M}_A^C(\psi)$ is a Grothendieck category with enough injective objects.*
2. *If $_RC$ is locally projective (resp. f.g. and projective), then*

$$\mathcal{M}_A^C(\psi) \simeq \mathrm{Rat}^C(\mathcal{M}_{\#_\psi^{op}(C,A)}) \simeq \sigma[(A \otimes_R C)_{\#_\psi^{op}(C,A)}] \text{ (resp. } \mathcal{M}_A^C(\psi) \simeq \mathcal{M}_{\#_\psi^{op}(C,A)}). \tag{22}$$

Proof. 1. If $_RC$ is flat, then $_A\mathcal{C}$ is flat and the result follows by the isomorphism $\mathcal{M}_A^C(\psi) \simeq \mathcal{M}^{\mathcal{C}}$ and Proposition 1.10 (3) (this is a generalization of [12, Section 2.8, Corollary 4], where (A, C, ψ) is *monoidal* and R is a base field).

2. If $_RC$ is locally projective, then $_A\mathcal{C}$ satisfies the α-condition by Lemma 3.8 (2), i.e. $({}^*\mathcal{C}, \mathcal{C}) \in \mathcal{P}_{ml}^\alpha$. The result follows now by Theorem 2.9 and Lemma 3.9. If $_RC$ is f.g. and projective, then $_A\mathcal{C}$ is also f.g. and projective and the result follows by Proposition 2.26.■

Corollary 3.11. *Let (A, C, ψ) be a left-right entwining structure and consider the corresponding A^{op}-coring $\mathcal{D} := A^{op} \otimes_R C$. If $_RC$ is flat, then by Theorem 3.10 $_A\mathcal{M}^C(\psi) \simeq \mathcal{M}_{A^{op}}^C(\psi \circ \tau)$ is a Grothendieck category with enough injective objects. If moreover $_RC$ is locally projective (resp. f.g. and projective), then*

$$_A\mathcal{M}^C(\psi) \simeq \mathrm{Rat}^C({}_{\#_\psi(C,A)}\mathcal{M}) = \sigma[{}_{\#_\psi(C,A)}\mathcal{D}] \text{ (resp. } {}_A\mathcal{M}^C(\psi) \simeq {}_{\#_\psi(C,A)}\mathcal{M}).$$

3.12. Let (A, C, ψ) be a right-right entwining structure.

1. By [9, Corollaries 3.4, 3.7] $- \otimes_R^c A : \mathcal{M}^C \to \mathcal{M}_A^C(\psi)$ is a functor, where for every $N \in \mathcal{M}^C$ we consider the canonical right A-module $N \otimes_R^c A := N \otimes_R A$ with the C-coaction $n \otimes a \mapsto [\sum n_{<0>} \otimes a_\psi \otimes n_{<1>}^\psi]$. Moreover $- \otimes_R A$ is left adjoint to the forgetful functor $\mathcal{F}_A : \mathcal{M}_A^C(\psi) \to \mathcal{M}^C$, where for every $N \in \mathcal{M}^C$ and $M \in \mathcal{M}_A^C(\psi)$

$$\mathrm{Hom}_A^C(N \otimes_R^c A, M) \to \mathrm{Hom}^C(N, M), \ g \mapsto g(- \otimes 1_A) \tag{23}$$

is a functorial isomorphism with inverse $f \mapsto [n \otimes a \mapsto f(n)a]$.

2. By the isomorphism $\mathcal{M}_A^C(\psi) \simeq \mathcal{M}^{\mathcal{C}}$ and Proposition 1.10 (1) $- \otimes_R C \simeq - \otimes_A \mathcal{C} : \mathcal{M}_A \to \mathcal{M}_A^C(\psi)$ is a functor, where for every $N \in \mathcal{M}_A$ we consider the canonical right C-comodule $N \otimes_R C$ with the A-action $(n \otimes c)a \mapsto \sum na_\psi \otimes c^\psi$. Moreover $- \otimes_R C$ is right adjoint to the forgetful functor $\mathcal{F}^C : \mathcal{M}_A^C(\psi) \to \mathcal{M}_A$ and left adjoint to $\mathrm{Hom}_A^C(\mathcal{C}, -) : \mathcal{M}_A^C(\psi) \to \mathcal{M}_A$.

Definition 3.13. Let C be an R-coalgebra.

1. C is said to be *left (right) Quasi-co-Frobenius*, if C is cogenerated by C^* as a left (a right) C^*-module (i.e. C is a torsionless C^*-module [20]).

2. Assume ${}_RC$ to be locally projective. After [4] we call C *left coproper* (resp. *right coproper*), if $C^{\Box} := \mathrm{Rat}^C({}_{C^*}C^*)$ (resp. ${}^{\Box}C := \mathrm{Rat}^C(C^*_{C^*})$) is dense in C^*. We call C *coproper*, if C is left and right coproper.

Corollary 3.14. *Let (A, C, ψ) be a right-right entwining structure and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

1. *Let C be projective in $\mathcal{M}^C$ (e.g. R is a QF ring and C is left Quasi-co-Frobenius). Then $C \otimes^c_R A$ is projective in $\mathcal{M}^C_A(\psi)$. If moreover ψ is bijective, then $A \otimes_R C$ is also projective in $\mathcal{M}^C_A(\psi)$.*

2. *If ${}_RC$ is locally projective and left coproper, then $C^{\Box} \otimes^c_R A$ is a generator in $\mathcal{M}^C_A(\psi)$.*

3. *If A is a cogenerator in $\mathcal{M}_A$, then $A \otimes_R C$ is a cogenerator in $\mathcal{M}^C_A(\psi)$. If ${}_RC$ is flat and A_A is injective, then $A \otimes_R C$ is injective in $\mathcal{M}^C_A(\psi)$.*

Proof. 1. This follows from the functorial isomorphism (23): $\mathrm{Hom}^C_A(C \otimes^c_R A, M) \simeq \mathrm{Hom}^C(C, M)$ for every $M \in \mathcal{M}^C_A(\psi)$. If R is a QF ring and C is left Quasi-co-Frobenius, then C is projective in $\mathcal{M}^C$ by [27]. Note that ψ is a morphism in $\mathcal{M}^C_A(\psi)$, hence $A \otimes_R C \simeq C \otimes^c_R A$ in $\mathcal{M}^C_A(\psi)$, if ψ is bijective.

2. If ${}_RC$ is locally projective and left coproper, then $C^{\Box}$ is a generator in $\sigma[{}_{C^*}C] \simeq \mathcal{M}^C$ by [38, 2.6]. The result follows then by the functorial isomorphism (23): $\mathrm{Hom}^C_A(C^{\Box} \otimes^c_R A, M) \simeq \mathrm{Hom}^C(C^{\Box}, M)$ for every $M \in \mathcal{M}^C_A(\psi)$.

3. Consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$. If A is a cogenerator in $\mathcal{M}_A$, then by Lemma 1.10 (2) $\mathcal{C}$ is a cogenerator in $\mathcal{M}^{\mathcal{C}} \simeq \mathcal{M}^C_A(\psi)$. If ${}_RC$ is flat, then ${}_A\mathcal{C}$ is flat and the second statement follows by Proposition 1.10 (7).■

Doi-Koppinen Modules

In what follows we consider a fundamental class of entwined modules, namely the class of Doi-Koppinen modules introduced independently by Y. Doi [15] and M. Koppinen [24].

3.15. A *right-right Doi-Koppinen structure* over R is a triple (H, A, C) consisting of an R-bialgebra H, a right H-comodule algebra A and a right H-module coalgebra C. A *right-right Doi-Koppinen module* for (H, A, C) is a right A-module M, which is also a right C-comodule through ϱ_M, such that

$$\varrho_M(ma) = \sum m_{<0>}a_{<0>} \otimes m_{<1>}a_{<1>} \text{ for all } m \in M \text{ and } a \in A.$$

With $\mathcal{M}(H)^C_A$ we denote the category of right-right Doi-Koppinen modules and A-linear C-colinear morphisms. By [9, Page 295] (A, C, ψ) is a right-right entwining structure and $\mathcal{M}(H)^C_A \simeq \mathcal{M}^C_A(\psi)$, where

$$\psi : C \otimes_R A \to A \otimes_R C,\ c \otimes a \mapsto \sum a_{<0>} \otimes ca_{<1>}. \tag{24}$$

By Lemma 3.3 $\#^{op}(C, A) := \mathrm{Hom}_R(C, A)$ is an A-ring with A-bimodule structure

$$(af)(c) := \sum a_{<0>} f(ca_{<1>}) \text{ and } (fa)(c) := f(c)a,$$

multiplication

$$(f \cdot g)(c) = \sum f(c_2)_{<0>} g(c_1 f(c_2)_{<1>}) \tag{25}$$

and unity $\eta_A \circ \varepsilon_C$. Moreover we have, with $\mathcal{C} := A \otimes_R C$ the corresponding A-coring, an isomorphism of A-rings $\#^{op}(C, A) \simeq {}^*\mathcal{C}$. The R-algebra $\#^{op}(C, A)$ was introduced by M. Koppinen [24, 2.2].

3.16. Let H be an R-bialgebra. Since H itself is a right H-module coalgebra with structure map μ_H, it turns out that, for every right H-comodule algebra A, the triple (H, A, H) is a right-right Doi-Koppinen structure and $\mathcal{M}(H)_A^H = \mathcal{M}_A^H$, the category of *relative Hopf modules* investigated in [16]. Note also that H is a right H-comodule algebra with structure map Δ_H and it turns out that, for every right H-module coalgebra C, the triple (H, H, C) is a right-right Doi-Koppinen structure and $\mathcal{M}(H)_H^C = \mathcal{M}_{[C,H]}$, the category of *Doi's* $[C, H]$-*modules* introduced in [16]. Finally (H, H, H) is a right-right Doi-Koppinen structure and $\mathcal{M}(H)_H^H = \mathcal{M}_H^H$, the category of *Hopf modules* studied by M. Sweedler [34, 4.1].

The following result is easy to prove.

Lemma 3.17. *Let (H, A, C) be a right-right Doi-Koppinen structure over R, $\mathcal{C} := A \otimes_R C$ the corresponding A-coring and $T \subseteq C^*$ a left H-module subalgebra.*

1. *$A\#^{op}T := A \otimes_R T$ is an A-ring with A-bimodule structure*

$$\widetilde{a}(a\#f) := \sum \widetilde{a}_{<0>} a\#\widetilde{a}_{<1>} f \text{ and } (a\#f)\widetilde{a} := a\widetilde{a}\#f \tag{26}$$

and multiplication

$$(a\#f) \cdot (b\#g) := \sum a_{<0>} b\#(a_{<1>} g) \star f. \tag{27}$$

If $\varepsilon_C \in T$, then $1_A\#\varepsilon_C$ is a unity for $A\#^{op}T$ and $A \to A\#^{op}T$, $a \mapsto a\#\varepsilon_C$ is a morphism of A-rings.

2. *We have a morphism of A-rings*

$$\beta : A\#^{op}T \to \#^{op}(C, A),\ a\#f \mapsto [c \mapsto af(c)].$$

Hence $Q := (A\#^{op}T, \mathcal{C})$ is a measuring left A-pairing with

$$\kappa_Q := \varphi \circ \beta : A\#^{op}T \to {}^*\mathcal{C},\ a\#f \mapsto [\widetilde{a} \otimes c \mapsto \widetilde{a}af(c)].$$

Theorem 3.18. *Let (H, A, C) be a right-right Doi-Koppinen structure and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$.*

1. *If ${}_RC$ is flat, then $\mathcal{M}(H)_A^C$ is a Grothendieck catgeory with enough injective objects.*

2. *Let $T \subseteq C^*$ be an A-pure left H-module subalgebra and $Q := (A\#^{op}T, \mathcal{C})$. If ${}_RC$ is locally projective (resp. f.g. and projective) and $T \subseteq C^*$ is dense, then $\beta(A\#^{op}T) \subseteq \#^{op}(C,A)$ is dense, $Q \in \mathcal{P}^{\alpha}_{ml}$ and we have isomorphisms of categories*

$$\mathcal{M}(H)^C_A \simeq \sigma[\mathcal{C}_{\#^{op}(C,A)}] = \sigma[\mathcal{C}_{A\#^{op}T}] \text{ (resp. } \mathcal{M}(H)^C_A \simeq \mathcal{M}_{\#^{op}(C,A)} \simeq \mathcal{M}_{A\#^{op}C^*}).$$

Proof. 1. Since $\mathcal{M}(H)^C_A \simeq \mathcal{M}^C_A(\psi)$, where ψ is defined in (24), the result follows by Proposition 3.10 (1).

2. Consider the left measuring A-pairing $P := (A\#^{op}C^*, \mathcal{C})$ and let $\phi : C^* \to A \otimes_R C^*$, $f \mapsto 1_A \otimes f$. Then we have for every right A-module M the following commutative diagram

$$\begin{array}{ccc} M \otimes_A (A \otimes_R C) & \xrightarrow{\alpha^P_M} & \mathrm{Hom}_{-A}(A \otimes_R C^*, M) \\ \| & & \downarrow{\scriptstyle (\phi,M)} \\ M \otimes_R C & \xrightarrow[\alpha^C_M]{} & \mathrm{Hom}_R(C^*, M) \end{array}$$

Let ${}_RC$ be locally projective. Then α^C_M is injective and so α^P_M is injective. Since M is an arbitrary right A-module, P satisfies the α-condition and we get by Theorem 3.10 the category isomorphisms $\mathcal{M}(H)^C_A \simeq \sigma[\mathcal{C}_{\#^{op}(C,A)}] \simeq \sigma[\mathcal{C}_{A\#^{op}C^*}]$. It follows then by Theorem 2.9 that $\kappa_P(A\#^{op}C^*) \subseteq {}^*\mathcal{C}$ is dense. If $T \subseteq C^*$ is an A-pure dense left H-module subalgebra, then obviously $A\#^{op}T \subseteq A\#^{op}C^*$ is dense, hence $\kappa_Q(A\#^{op}T) \subseteq {}^*\mathcal{C}$ is dense. Since ${}^*\mathcal{C} \simeq \#^{op}(C,A)$ it follows then that $\beta(A\#^{op}T) \subseteq \#^{op}(C,A)$ is dense.

If ${}_RC$ is f.g. and projective, then $\mathcal{M}(H)^C_A \simeq \mathcal{M}_{\#^{op}(C,A)}$ by Theorem 3.10 (2). Note that in this case $A\#^{op}C^* \simeq \#^{op}(C,A)$ and the result follows.■

3.19. A left-right Doi-Koppinen structure is a triple (H, A, C), where H is an R-bialgebra, A is a right H-comodule algebra and C is a left H-module coalgebra. A Doi-Koppinen module corresponding to (H, A, C) is a left A-module M, which is also a right C-comodule through ϱ_M, s.t. $\varrho_M(am) = \sum a_{<0>}m_{<0>} \otimes a_{<1>}m_{<1>}$. The category of left-right Doi-Koppinen modules and A-linear C-colinear morphisms is denoted by ${}_A\mathcal{M}^C(H)$. It turns out that (H^{op}, A^{op}, C) is a right-right Doi-Koppinen structure, hence $\#(C,A) := (\#^{op}(C, A^{op}))^{op}$ is an A-ring with multiplication

$$(f \cdot g)(c) = \sum f(g\,(c_2)_{<1>}\, c_1) g(c_2)_{<0>}. \tag{28}$$

and unity $\eta_A \circ \varepsilon_C$. For every right H-module subalgebra $T \subseteq C^*$ (with $\varepsilon_C \in T$) the smash product $A\#T := (A^{op}\#^{op}T)^{op}$ is an A-ring with multiplication

$$(a\#f) \cdot (b\#g) := \sum ab_{<0>}\#(fb_{<1>}) \star g \tag{29}$$

(and unity $1_A\#\varepsilon_C$). In fact the R-algebra $\#(C,A)$ (resp. $A\#T$) was introduced in [24, 2.1] (resp. in [15, Page 375]).

Corollary 3.20. *Let (H, A, C) be a left-right Doi-Koppinen structure, $\mathcal{D} := A^{op} \otimes_R C$ the corresponding A^{op}-coring and $\beta : A\#C^* \to \#(C, A)$ the canonical morphism. If $_RC$ is flat, then $_A\mathcal{M}(H)^C \simeq \mathcal{M}(H^{op})^C_{A^{op}}$ is a Grothendieck category with enough injective objects. If $T \subseteq C^*$ is an A-pure dense right H-module subalgebra and $_RC$ is locally projective (resp. f.g. and projective), then $\beta(A\#T) \subseteq \#(C, A)$ is dense and*

$$_A\mathcal{M}(H)^C \simeq \sigma[_{\#(C,A)}\mathcal{D}] \simeq \sigma[_{A\#T}\mathcal{D}] \text{ (resp. } _A\mathcal{M}(H)^C \simeq {}_{\#(C,A)}\mathcal{M} \simeq {}_{A\#C^*}\mathcal{M}).$$

Next we extend some results of [26] on relative Hopf modules to the general case of right-right Doi-Koppinen modules.

Proposition 3.21. *Let (H, A, C) be a right-right Doi-Koppinen structure such that $_RC$ is locally projective, $\mathcal{C} := A \otimes_R C$ the corresponding A-coring and $T \subseteq C^*$ an A-pure dense left H-module subalgebra.*

1. *Let M be a right A-module and a left T-module. If for all $f \in T$, $a \in A$ and $m \in M$ we have*
$$f[ma] = \sum((a_{<1>}f)m)a_{<0>},$$
then $\mathrm{Rat}^C(_TM) \in \mathcal{M}(H)^C_A$. Consequently $M \in \mathcal{M}(H)^C_A$ iff $M = \mathrm{Rat}^C(_TM)$.

2. *If $\varepsilon_C \in T$, then for every right $A\#^{op}T$-module M we have: $\mathrm{Rat}^C(_TM) = \mathrm{Rat}^{\mathcal{C}}(M_{A\#^{op}T})$ and $M \in \mathcal{M}(H)^C_A$ iff $M = \mathrm{Rat}^C(_TM)$.*

Proof. 1. Since $_RC$ is locally projective and $T \subseteq C^*$ is dense, it follows by [2, Satz 2.2.13] that $\mathrm{Rat}^C(_T\mathcal{M}) \simeq \mathcal{M}^C$. Moreover we have for all $m \in \mathrm{Rat}^C(_TM)$, $f \in T$ and $a \in A$:

$$\begin{aligned} f[ma] &= \sum((a_{<1>}f)m)a_{<0>} &&= \sum(m_{<0>}(a_{<1>}f)(m_{<1>}))a_{<0>} \\ &= \sum f(m_{<1>}a_{<1>})m_{<0>}a_{<0>}, \end{aligned}$$

i.e. $ma \in \mathrm{Rat}^C(_TM)$ with $\varrho_M(ma) = \sum m_{<0>}a_{<0>} \otimes m_{<1>}a_{<1>}$, hence $\mathrm{Rat}^C(_TM) \in \mathcal{M}(H)^C_A$. On the other hand, if $M \in \mathcal{M}(H)^C_A$, then M is in particular a right C-comodule and so $M = \mathrm{Rat}^C(_TM)$.

2. Clearly $\mathrm{Rat}^{\mathcal{C}}(M_{A\#^{op}T}) \subseteq \mathrm{Rat}^C(_TM)$. On the other hand we have for all $f \in T$, $a \in A$ and $m \in \mathrm{Rat}^C(_TM)$:

$$\begin{aligned} m(a\#f) &= m((1_A\#f)\cdot(a\#\varepsilon_C)) &&= (m(1_A\#f))(a\#\varepsilon_C) \\ &= (fm)(a\#\varepsilon_C) &&= (\sum m_{<0>}f(m_{<1>}))(a\#\varepsilon_C)) \\ &= \sum m_{<0>}af(m_{<1>}), \end{aligned}$$

i.e. $m \in \mathrm{Rat}^{\mathcal{C}}(M_{A\#^{op}T})$ with $\varrho_M(m) = \sum m_{<0>} \otimes_A (1_A \otimes m_{<1>})$. Note that for all $f \in T$, $a \in A$ and $m \in \mathrm{Rat}^C(_TM)$ we have by a similar argument that $f[ma] = \sum((a_{<1>}f)m)a_{<0>}$ and the result follows by (1).■

As a direct consequence of Proposition 3.21 we get

Corollary 3.22. *Let (H, A, C) be a right-right (resp. a left-right) Doi-Koppinen structure and assume ${}_RC$ to be locally projective. If $\mathcal{M}^C$ is closed under extensions in ${}_{C^*}\mathcal{M}$, then $\mathcal{M}(H)_A^C$ (resp. ${}_A\mathcal{M}(H)^C$) is closed under extensions in $\mathcal{M}_{A\#^{op}C^*}$ (resp. in ${}_{A\#C^*}\mathcal{M}$).*

The proof of the following result is with slight modifications along the lines of [26, 1.9].

Corollary 3.23. *Let (H, A, C) be a right-right (resp. a left-right) Doi-Koppinen structure with ${}_RC$ locally projective and consider the corresponding A-coring $\mathcal{C} := A \otimes_R C$ (resp. the A^{op}-coring $\mathcal{D} := A^{op} \otimes_R C$). If C is left coproper and $C^\Box \subseteq C^*$ is A-pure, then*

$$\mathcal{M}(H)_A^C = \mathcal{M}_{A\#^{op}C^\Box} = \mathcal{M}_{{}^\Box\mathcal{C}} \text{ (resp. } {}_A\mathcal{M}(H)^C \simeq {}_{A\#C^\Box}\mathcal{M} \simeq {}_{({}^\Box\mathcal{D})^{op}}\mathcal{M}).$$

Proof. Let (H, A, C) be a right-right Doi-Koppinen structure, $\mathcal{A} := A\#^{op}C^*$, $P := (\mathcal{A}, \mathcal{C}) \in \mathcal{P}_{ml}$ and assume ${}_RC$ to be locally projective. Since C is left coproper, it follows by the isomorphism of categories $\mathrm{Rat}^C({}_{C^*}\mathcal{M}) = \sigma[{}_{C^*}C]$ and analog to [38, 2.6] that $\mathrm{Rat}^C({}_{C^*}N) = C^\Box N$ for every left C^*-module N. By Lemma 3.21 (2) we have then

$$\mathcal{T} := \mathrm{Rat}^{\mathcal{C}}(\mathcal{A}_{\mathcal{A}}) = \mathrm{Rat}^C({}_{C^*}\mathcal{A}) = C^\Box\mathcal{A} = \mathcal{A}(1_A\#^{op}C^\Box) = A\#^{op}C^\Box.$$

Since $C^\Box \subset C^*$ is dense, $\mathcal{T} = A \otimes_R C^\Box \subset A \otimes_R C^*$ is dense by Theorem 3.18 and the result follows by Proposition 2.15. The corresponding result for left-right Doi-Koppinen structures follows by symmetry.■

Next we consider three examples of right-right Doi-Koppinen structures (see [12]).

3.24. Yetter-Drinfel'd modules. Let (H, K, A, C) be a *Yetter-Drinfel'd datum,* $(K^{op} \otimes_R H, A, C)$ the corresponding right-right Doi-Koppinen structure and consider the category of Yetter-Drinfel'd modules $\mathcal{YD}(K, H)_A^C \simeq \mathcal{M}(K^{op} \otimes_R H)_A^C$. If ${}_RC$ is flat, then $\mathcal{YD}(K, H)_A^C$ is a Grothendieck category with enough injective objects (this generalizes [12, Section 4.4., Corollary 31], where a base field is assumed). If ${}_RC$ is locally projective (resp. f.g. and projective), then we have with $\mathcal{C} := A \otimes_R C$ the corresponding A-coring

$$\mathcal{YD}(K, H)_A^C \simeq \sigma[\mathcal{C}_{\#^{op}(C,A)}] \simeq \sigma[\mathcal{C}_{A\#^{op}C^*})] \text{ (resp. } \mathcal{YD}(K, H)_A^C \simeq \mathcal{M}_{\#^{op}(C,A)} \simeq \mathcal{M}_{A\#^{op}C^*}).$$

3.25. Long dimodules. Let A be an R-algebra, C an R-coalgebra, (R, A, C) the trivial right-right Doi-Koppinen structure and consider the category of *Long dimodules* $\mathcal{L}_A^C \simeq \mathcal{M}(R)_A^C$. If ${}_RC$ is flat, then $\mathcal{L}_A^C$ is a Grothendieck category with enough injective objects. If moreover ${}_RC$ is locally projective (resp. f.g. and projective), then we have with $\mathcal{C} := A\otimes_R C$ the corresponding A-coring

$$\mathcal{L}_A^C \simeq \sigma[\mathcal{C}_{\#^{op}(C,A)}] \simeq \sigma[\mathcal{C}_{A\#^{op}C^*}] \text{ (resp. } \mathcal{L}_A^C \simeq \mathcal{M}_{\#^{op}(C,A)} = \mathcal{M}_{A\#^{op}C^*}).$$

3.26. Modules graded by G-sets. Let G be a group, A a G-graded R-algebra, X a right G-set (e.g. $X = G$), (RG, A, RX) the corresponding right-right Doi-Koppinen structure and denote by gr-$(G, A, X) \simeq \mathcal{M}(RG)_A^{RX}$ the category of RX-graded right A-modules. Since the free R-module RX is in particular locally projective, we get by Theorem 3.18 (2) isomorphisms of categories

$$gr\text{-}(G, A, X) \simeq \sigma[(A \otimes_R RX)_{\#^{op}(RX,A)}] \simeq \sigma[(A \otimes_R RX)_{A\#^{op}(RX)^*}].$$

If moreover X is finite, then RX is in particular f.g. and projective, hence

$$gr\text{-}(G, A, X) \simeq \mathcal{M}_{\#^{op}(RX,A)} \simeq \mathcal{M}_{A\#^{op}(RX)^*}.$$

Alternative Doi-Koppinen modules

It turns out from work of D. Tambara [35] that every entwining structure (A, C, ψ), for which ${}_RA$ is f.g. and projective, can be obtained from a Doi-Koppinen structure with a suitable auxiliary R-bialgebra giving rise to the entwining map ψ. P. Schauenburg has shown in [31] that this is not the case in general. However, if ${}_RC$ is f.g. and projective, then he remarks that (A, C, ψ) can be derived form what he calls an *alternative Doi-Koppinen structure.*

3.27. Let H be an R-bialgebra, A a right H-module algebra and C a right H-comodule coalgebra. Then (H, A, C) is called a *right-right alternative Doi-Koppinen structure*. It turns out, that (A, C, ψ) is a right-right entwining structure, where

$$\psi : C \otimes_R A \to A \otimes_R C,\ c \otimes a \mapsto \sum ac_{<1>} \otimes c_{<0>}.$$

We denote the corresponding category of entwined modules (called *alternative right-right Doi-Koppinen modules*) by $a\mathcal{M}(H)_A^C$. As for other categories of entwined moudles, if ${}_RC$ is flat, then $a\mathcal{M}(H)_A^C$ is a Grothendieck category with enough injective objects. If moreover ${}_RC$ is locally projective (resp. f.g. and projective), then

$$a\mathcal{M}(H)_A^C \simeq \sigma[(A \otimes_R C)_{\#^{op}(C,A)}] \text{ (resp. } a\mathcal{M}(H)_A^C \simeq \mathcal{M}_{\#^{op}(C,A)}).$$

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