

A Note On Coinduction Functors between Categories of Comodules for Corings*

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Abstract

In this note we consider different versions of coinduction functors between categories of comodules for corings induced by a morphism of corings. In particular we introduce a new version of the coinduction functor in the case of *locally projective* corings as a composition of suitable “Trace” and “Hom” functors and show how to derive it from a more *general* coinduction functor between categories of type $\sigma[M]$. In special cases (e.g. the corings morphism is part of a morphism of *measuring α -pairings* or the corings have the same base ring), a version of our functor is shown to be isomorphic to the usual coinduction functor obtained by means of the cotensor product. Our results in this note generalize previous results of the author on coinduction functors between categories of comodules for coalgebras over commutative base rings.

1 Introduction

With R we denote a commutative ring with $1_R \neq 0_R$, with A an arbitrary R -algebra and with \mathbb{M}_A (respectively ${}_A\mathbb{M}$) the category of right (respectively left) A -modules. The unadorned $- \otimes -$, $\text{Hom}(-, -)$ mean $- \otimes_R -$, $\text{Hom}_R(-, -)$ respectively. For a right (respectively a left) A -module M we denote by

$$\vartheta_M^r : M \otimes_A A \rightarrow M \text{ (respectively } \vartheta_M^l : A \otimes_A M \rightarrow M)$$

the canonical A -isomorphism.

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By an *A-ring* we mean an *A*-bimodule \mathcal{T} with *A*-bilinear maps $\mu_{\mathcal{T}} : \mathcal{T} \otimes_A \mathcal{T} \rightarrow \mathcal{T}$ and $\eta_{\mathcal{T}} : A \rightarrow \mathcal{T}$, such that $\mu_{\mathcal{T}} \circ (\mu_{\mathcal{T}} \otimes_A id_{\mathcal{T}}) = \mu_{\mathcal{T}} \circ (id_{\mathcal{T}} \otimes_A \mu_{\mathcal{T}})$, $\mu_{\mathcal{T}} \circ (id_{\mathcal{T}} \otimes_A \eta_{\mathcal{T}}) = \vartheta_{\mathcal{T}}^r$ and $\mu_{\mathcal{T}} \circ (\eta_{\mathcal{T}} \otimes_A id_{\mathcal{T}}) = \vartheta_{\mathcal{T}}^l$. If \mathcal{T} and \mathcal{S} are *A*-rings with unities $\eta_{\mathcal{T}}, \eta_{\mathcal{S}}$ respectively, then an *A*-bilinear map $f : \mathcal{T} \rightarrow \mathcal{S}$ is called a *morphism of A-rings*, if $f \circ \mu_{\mathcal{T}} = \mu_{\mathcal{S}} \circ (f \otimes_A f)$ and $f \circ \eta_{\mathcal{T}} = \eta_{\mathcal{S}}$. For an *A*-ring \mathcal{T} and two left (respectively right) \mathcal{T} -modules M and N we denote with $\text{Hom}_{\mathcal{T}\text{-}}(M, N)$ (respectively $\text{Hom}_{\text{-}\mathcal{T}}(M, N)$) the set of all \mathcal{T} -linear maps from M to N . If M and N are \mathcal{T} -bimodules, then $\text{Hom}_{\mathcal{T}\text{-}\mathcal{T}}(M, N)$ denotes the set of all \mathcal{T} -bilinear maps from M to N .

Let M and N be right *A*-modules and $f : M \rightarrow N$ be *A*-linear. For a left *A*-module L , the morphism f will be called *L-pure*, if the following sequence is exact

$$0 \longrightarrow \text{Ker}(f) \otimes_A L \xrightarrow{\iota \otimes_A id_L} M \otimes_A L \xrightarrow{f \otimes_A id_L} N \otimes_A L.$$

If f is *L-pure* for all left *A*-modules L then f is called a *pure morphism* (e.g. [BW03, 40.13]). If $M \subseteq N$ is a right *A*-submodule then it is called a *pure submodule*, provided that the embedding $M \hookrightarrow N$ is a pure morphism (equivalently, if $\iota \otimes_A id_L : M \otimes_A L \rightarrow N \otimes_A L$ is injective for every left *A*-module L). Pure morphisms and pure submodules in the category of left *A*-modules are defined analogously. A morphism of *A*-bimodules is said to be pure, if it is pure in \mathbb{M}_A as well as in ${}_A\mathbb{M}$. For an *A*-bimodule N , we call an *A*-subbimodule $M \subset N$ *pure*, if $M_A \subset N_A$ and ${}_A M \subset {}_A N$ are pure.

A left (respectively right) *A*-module W is called *locally projective* (in the sense of B. Zimmermann-Huisgen [Z-H76]), if for every diagram of left (respectively right) *A*-modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & F & \xrightarrow{\iota} & W & & \\ & & \searrow & & \downarrow g' & \searrow g & \\ & & & & L & \xrightarrow{\pi} & N \longrightarrow 0 \\ & & g' \circ \iota & & & & \end{array}$$

with exact rows and F f.g.: for every *A*-linear map $g : W \rightarrow N$, there exists an *A*-linear map $g' : W \rightarrow L$, such that $g \circ \iota = \pi \circ g' \circ \iota$. Note that every projective left (respectively right) *A*-module is locally projective.

Let A, B be *R*-algebras with an *R*-algebra morphism $\beta : A \rightarrow B$ and M (respectively N) be a right (respectively a left) *B*-module. We consider M (respectively N) as a right (respectively as a left) *A*-module through

$$m \leftarrow a := m\beta(a) \text{ (respectively } a \rightarrow n := \beta(a)n)$$

and denote with

$$\chi_{M,N} : M \otimes_A N \rightarrow M \otimes_B N$$

the canonical *R*-linear morphism.

(Co)induction functors between categories of entwined modules induced by a morphism of entwined structures were studied in the case of base fields by T. Brzeziński et al. (e.g. [Brz99], [BCMZ01]). In the general case of a morphism of corings $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} :$

B), J. Gómez-Torrecillas presented in [G-T02] (under some purity conditions and using the cotensor product) a coinduction functor $\mathcal{G} : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}$ and proved it is right adjoint to the canonical induction functor $- \otimes_A B : \mathbb{M}^{\mathcal{C}} \rightarrow \mathbb{M}^{\mathcal{D}}$ ([G-T02, Proposition 5.3]).

In his dissertation [Abu01], the author presented and studied coinduction functors between categories of comodules induced by morphisms between *measuring α -pairings* for coalgebras (over a common commutative base rings). In this note the author generalizes his previous results to the case of coinduction functors between categories of comodules for corings over (possibly different) arbitrary base rings. Although in our setting some technical assumptions are assumed and the corings under consideration are restricted to the locally projective ones, the main advantages of the new version of the coinduction functor is that it is presented as a composition of well-studied “Trace” and “Hom” functors and that it avoids the use of the cotensor product. Another advantage of the new version of the coinduction functor we introduce is that it is derived from a more general coinduction functor between categories of type $\sigma[M]$, the theory of which is well developed (e.g. [Wis88], [Wis96]). With these two main advantages in mind, this note aims to enable a more intensive study of the coinduction functors between categories of comodules and their properties which will be carried out elsewhere.

After this introductory section, we present in the second section some definitions and lemmas, that will be used later. In the third section we consider coinduction functors between categories of type $\sigma[M]$ as a general model from which we derive later our version of coinduction functors between categories of comodules for locally projective corings. In the fourth section we consider the case of coinduction functors induced by morphisms of measuring α -pairings, which turn out to be special forms of the coinduction functors introduced in the third section. We also handle the special case, where the corings are defined over a common base ring. In particular we prove that our coinduction functor in these cases is isomorphic to the coinduction functor presented by J. Gómez-Torrecillas in [G-T02] by means of the cotensor product. In the fifth and last section we handle the general case of coinduction functors induced by morphisms of corings (over possibly different) arbitrary base rings.

2 Preliminaries

In this section we present the needed definitions and lemmas.

2.1. Subgenerators. Let A be an R -algebra, \mathcal{T} be an A -ring and K be a right \mathcal{T} -module. We say a right \mathcal{T} -module N is *K -subgenerated* if N is isomorphic to a submodule of a K -generated right \mathcal{T} -module, equivalently if N is the kernel of a morphism between K -generated right \mathcal{T} -modules. With $\sigma[K_{\mathcal{T}}] \subseteq \mathbb{M}_{\mathcal{T}}$ we denote the *full* subcategory of $\mathbb{M}_{\mathcal{T}}$ whose objects are K -subgenerated. In fact $\sigma[K_{\mathcal{T}}] \subseteq \mathbb{M}_{\mathcal{T}}$ is the *smallest* Grothendieck full subcategory that contains K . Moreover we have the *trace functor*

$$\mathrm{Sp}(\sigma[K_{\mathcal{T}}], -) : \mathbb{M}_{\mathcal{T}} \rightarrow \sigma[K_{\mathcal{T}}], M \mapsto \sum \{f(N) : f \in \mathrm{Hom}_{-\mathcal{T}}(N, M), N \in \sigma[K_{\mathcal{T}}]\},$$

which is right-adjoint to the inclusion functor $\iota : \sigma[K_{\mathcal{T}}] \hookrightarrow \mathbb{M}_{\mathcal{T}}$ ([Wis88, 45.11]). In fact $\text{Sp}(\sigma[K_{\mathcal{T}}], M)$ is the largest right \mathcal{T} -submodule of $M_{\mathcal{T}}$ that belongs to $\sigma[K_{\mathcal{T}}]$ and $m \in \text{Sp}(\sigma[K_{\mathcal{T}}], M)$ if and only if there exists a finite subset $W \subset K$ with $\text{Ann}_{-A}(W) \subseteq \text{Ann}_{-A}(m)$. For a left \mathcal{T} -module L , the full category $\sigma[\mathcal{T}L] \subseteq {}_{\mathcal{T}}\mathbb{M}$ is defined analogously. The reader is referred to [Wis88] and [Wis96] for the well developed theory of categories of this type.

Definition 2.2. Let A be an R -algebra. A *coassociative A -coring* $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ is an A -bimodule \mathcal{C} with A -bilinear maps

$$\Delta_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C} \otimes_A \mathcal{C}, \quad c \mapsto \sum c_1 \otimes_A c_2 \quad \text{and} \quad \varepsilon_{\mathcal{C}} : \mathcal{C} \rightarrow A,$$

such that the following diagrams are commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \\ \Delta_{\mathcal{C}} \downarrow & & \downarrow \text{id}_{\mathcal{C}} \otimes_A \Delta_{\mathcal{C}} \\ \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}} \otimes_A \text{id}_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array} \quad \begin{array}{ccccc} & & \mathcal{C} & & \\ & \nearrow \vartheta_{\mathcal{C}}^l & \downarrow \Delta_{\mathcal{C}} & \nwarrow \vartheta_{\mathcal{C}}^r & \\ A \otimes_A \mathcal{C} & \xleftarrow{\varepsilon_{\mathcal{C}} \otimes_A \text{id}_{\mathcal{C}}} & \mathcal{C} \otimes_A \mathcal{C} & \xrightarrow{\text{id}_{\mathcal{C}} \otimes_A \varepsilon_{\mathcal{C}}} & \mathcal{C} \otimes_A A \end{array}$$

The A -bilinear maps $\Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}}$ are called the *comultiplication* and the *counit* of \mathcal{C} , respectively. All corings under consideration are assumed to be coassociative and have counits. The objects in the *category of corings* \mathbf{Crg} of coassociative corings are understood to be pairs $(\mathcal{C} : A)$, where A is an R -algebra and \mathcal{C} is an A -coring. A *morphism between corings* $(\mathcal{C} : A)$ and $(\mathcal{D} : B)$ consists of a pair of mappings $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$, where $\beta : A \rightarrow B$ is an R -algebra morphism (so one can consider \mathcal{D} as an A -bimodule) and $\theta : \mathcal{C} \rightarrow \mathcal{D}$ is an (A, A) -bilinear map with

$$\chi_{\mathcal{D}, \mathcal{D}} \circ (\theta \otimes_A \theta) \circ \Delta_{\mathcal{C}} = \Delta_{\mathcal{D}} \circ \theta \quad \text{and} \quad \varepsilon_{\mathcal{D}} \circ \theta = \beta \circ \varepsilon_{\mathcal{C}}. \quad (1)$$

A morphism of corings $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ is said to be *pure* if for every right \mathcal{D} -comodule N , the following right A -module morphism is \mathcal{C} -pure (e.g. ${}_A\mathcal{C}$ is flat):

$$\varrho_N \otimes_A \text{id}_{\mathcal{C}} - (\chi_{N, \mathcal{D}} \otimes_A \text{id}_{\mathcal{C}}) \circ (\text{id}_N \otimes_A \theta \otimes \text{id}_{\mathcal{C}}) \circ (\text{id}_N \otimes_A \Delta_{\mathcal{C}}) : N \otimes_A \mathcal{C} \rightarrow N \otimes_B \mathcal{D} \otimes_A \mathcal{C}.$$

If $A = B$ and β is not mentioned explicitly, then we mean $\beta = \text{id}_A$.

Definition 2.3. Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ and $(\mathcal{D}, \Delta_{\mathcal{D}}, \varepsilon_{\mathcal{D}})$ be A -corings with ${}_A\mathcal{D}_A \subseteq {}_A\mathcal{C}_A$. We call \mathcal{D} an A -subcoring of \mathcal{C} , if the embedding $\iota_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{C}$ is a morphism of A -corings. If ${}_A\mathcal{D} \subseteq {}_A\mathcal{C}$ and $\mathcal{D}_A \subseteq \mathcal{C}_A$ are pure, then \mathcal{D} is a subcoring of \mathcal{C} if and only if $\Delta_{\mathcal{C}}(\mathcal{D}) \subseteq \mathcal{D} \otimes_A \mathcal{D}$.

2.4. The dual rings of a coring. ([Guz85]) Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an A -coring. Then ${}^*\mathcal{C} := \text{Hom}_{A-}(\mathcal{C}, A)$ is an associative A -ring with multiplication given by the *left convolution product*

$$(f \star_l g)(c) := \sum g(c_1 f(c_2)) \quad \text{for all } f, g \in {}^*\mathcal{C}, \quad c \in \mathcal{C},$$

unit $\varepsilon_{\mathcal{C}}$, and $\mathcal{C}^* := \text{Hom}_{-A}(\mathcal{C}, A)$ is an associative A -ring with multiplication given by the *right convolution product*

$$(f \star_r g)(c) := \sum f(g(c_1)c_2) \text{ for all } f, g \in \mathcal{C}^*, c \in \mathcal{C},$$

unit $\varepsilon_{\mathcal{C}}$. Moreover ${}^*\mathcal{C}^* := \text{Hom}_{A-A}(\mathcal{C}, A)$ is an associative A -ring with multiplication given by the *convolution product*

$$(f \star g)(c) := \sum g(c_1)f(c_2) \text{ for all } f, g \in {}^*\mathcal{C}^*, c \in \mathcal{C}.$$

2.5. Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an A -coring. A right \mathcal{C} -comodule is a right A -module M associated with a right A -linear map (\mathcal{C} -coaction)

$$\varrho_M : M \rightarrow M \otimes_A \mathcal{C}, m \mapsto \sum m_{\langle 0 \rangle} \otimes_A m_{\langle 1 \rangle},$$

such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{\varrho_M} & M \otimes_A \mathcal{C} \\ \varrho_M \downarrow & & \downarrow id_M \otimes_A \Delta_{\mathcal{C}} \\ M \otimes_A \mathcal{C} & \xrightarrow{\varrho_M \otimes_A id_{\mathcal{C}}} & M \otimes_A \mathcal{C} \otimes_A \mathcal{C} \end{array}$$

We call M *counital*, if

$$\vartheta_M^r \circ (id_M \otimes \varepsilon_{\mathcal{C}}) \circ \varrho_M = id_M.$$

For right \mathcal{C} -comodules $(M, \varrho_M), (N, \varrho_N) \in \mathbb{M}^{\mathcal{C}}$ we call a right A -linear map $f : M \rightarrow N$ a \mathcal{C} -comodule morphism (or *right \mathcal{C} -colinear*), if the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \varrho_M \downarrow & & \downarrow \varrho_N \\ M \otimes_A \mathcal{C} & \xrightarrow{f \otimes_A id_{\mathcal{C}}} & N \otimes_A \mathcal{C} \end{array}$$

The set of \mathcal{C} -colinear maps from M to N is denoted by $\text{Hom}^{\mathcal{C}}(M, N)$. For a right \mathcal{C} -comodule N , we call a right \mathcal{C} -comodule (K, ϱ_K) with $K_A \subset N_A$ a \mathcal{C} -subcomodule if the embedding $K \xrightarrow{\iota_K} N$ is right \mathcal{C} -colinear. The category of counital right \mathcal{C} -comodules and right \mathcal{C} -colinear maps is denoted by $\mathbb{M}^{\mathcal{C}}$. The left \mathcal{C} -comodules and the *left \mathcal{C} -colinear* morphisms are defined analogously. For left \mathcal{C} -comodules M, N we denote with ${}^{\mathcal{C}}\text{Hom}(M, N)$ the set of *left \mathcal{C} -colinear maps* from M to N . The category of counital *left \mathcal{C} -comodules* and *left \mathcal{C} -colinear* morphisms will be denoted with ${}^{\mathcal{C}}\mathbb{M}$. If ${}_A\mathcal{C}$ (respectively \mathcal{C}_A) is flat, then $\mathbb{M}^{\mathcal{C}}$ (respectively ${}^{\mathcal{C}}\mathbb{M}$) is a Grothendieck category.

2.6. Measuring A -pairings. A *measuring left pairing* $P = (\mathcal{T}, \mathcal{C} : A)$ consists of an A -coring \mathcal{C} and an A -ring \mathcal{T} with a morphism of A -rings $\kappa_P : \mathcal{T} \rightarrow {}^* \mathcal{C}$. In this case \mathcal{C} is a right \mathcal{T} -module through

$$c \leftarrow t := \sum c_1 \langle t, c_2 \rangle \text{ for all } t \in \mathcal{T} \text{ and } c \in \mathcal{C}.$$

Let $(\mathcal{T}, \mathcal{C} : A)$ and $(\mathcal{S}, \mathcal{D} : B)$ be measuring left pairings. A (*pure*) *morphism between measuring pairings*

$$(\xi, \theta : \beta) : (\mathcal{T}, \mathcal{C} : A) \rightarrow (\mathcal{S}, \mathcal{D} : B),$$

consists of an R -algebra morphism $\beta : A \rightarrow B$, a (*pure*) morphism of corings $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ and a morphism of A -rings $\xi : \mathcal{S} \rightarrow \mathcal{T}$, such that

$$\langle s, \theta(c) \rangle = \beta(\langle \xi(s), c \rangle) \text{ for all } s \in \mathcal{S} \text{ and } c \in \mathcal{C}.$$

If $A = B$ and β is not mentioned explicitly, then we mean $\beta = id_A$ and write $(\xi, \theta) : (\mathcal{T}, \mathcal{C}) \rightarrow (\mathcal{S}, \mathcal{D})$. The category of measuring left pairings and morphisms described above will be denoted by \mathcal{P}_{ml} .

2.7. The α -condition. We say a measuring left pairing $P = (\mathcal{T}, \mathcal{C} : A)$ satisfies the α -condition (or is a *measuring left α -pairing*), if for every right A -module M the following map is injective

$$\alpha_M^P : M \otimes_A \mathcal{C} \rightarrow \text{Hom}_{-A}(\mathcal{T}, M), \quad \sum m_i \otimes_A c_i \mapsto [t \mapsto \sum m_i \langle t, c_i \rangle]. \quad (2)$$

We say an A -coring \mathcal{C} satisfies the *left α -condition*, if the canonical measuring left pairing $({}^* \mathcal{C}, \mathcal{C} : A)$ satisfies the α -condition (equivalently if ${}_A \mathcal{C}$ is *locally projective* in the sense of Zimmermann-Huisgen [Z-H76, Theorem 2.1], [Gar76, Theorem 3.2]). With $\mathcal{P}_{ml}^\alpha \subset \mathcal{P}_{ml}$ we denote the *full* subcategory of measuring left pairings satisfying the α -condition. The category of *measuring right pairings* \mathcal{P}_{mr} and the full subcategory of *measuring right α -pairings* $\mathcal{P}_{mr}^\alpha \subset \mathcal{P}_{mr}$ are analogously defined.

Remark 2.8. ([Abu03, Remark 1.30]) Let $P = (\mathcal{T}, \mathcal{C} : A)$ be a measuring left α -pairing. Then ${}_A \mathcal{C}$ is flat and A -cogenerated. If moreover ${}_A \mathcal{C}$ is finitely presented or ${}_A A$ is perfect, then ${}_A \mathcal{C}$ turns out to be projective.

2.9. Let $P = (\mathcal{T}, \mathcal{C} : A)$ be a measuring left α -pairing and M be a unital right \mathcal{T} -module. Consider the canonical A -linear mapping $\rho_M : M \rightarrow \text{Hom}_{-A}(\mathcal{T}, M)$ and put $\text{Rat}^{\mathcal{C}}(M_{\mathcal{T}}) := \rho_M^{-1}(M \otimes_A \mathcal{C})$. We call $M_{\mathcal{T}}$ *\mathcal{C} -rational*, if $\text{Rat}^{\mathcal{C}}(M_{\mathcal{T}}) = M$ (equivalently, if for every $m \in M$ there exists a uniquely determined element $\sum m_i \otimes_A c_i$ such that $mt = \sum m_i \langle t, c_i \rangle$ for every $t \in \mathcal{T}$). In this case we get a well-defined \mathcal{T} -linear map $\varrho_M := (\alpha_M^P)^{-1} \circ \rho_M : M \rightarrow M \otimes_A \mathcal{C}$. The category of \mathcal{C} -rational right \mathcal{T} -modules and right \mathcal{T} -linear maps is denoted by $\text{Rat}^{\mathcal{C}}(\mathbb{M}_{\mathcal{T}})$. For a measuring right α -pairing $P = (\mathcal{T}, \mathcal{C} : A)$ the \mathcal{C} -rational left \mathcal{T} -modules are analogously defined and we denote by ${}^{\mathcal{C}}\text{Rat}(\mathcal{T}\mathbb{M}) \subseteq {}_{\mathcal{T}}\mathbb{M}$ the subcategory of \mathcal{C} -rational left \mathcal{T} -modules.

Proposition 2.10. ([Abu03, Proposition 2.8]) *Let $P = (\mathcal{T}, \mathcal{C} : A)$ be a measuring left pairing. If ${}_A\mathcal{C}$ is locally projective and $\kappa_P(\mathcal{T}) \subseteq {}^*\mathcal{C}$ is dense, then*

$$\begin{aligned} \mathcal{M}^{\mathcal{C}} &\simeq \text{Rat}^{\mathcal{C}}(\mathcal{M}_{\mathcal{T}}) = \sigma[\mathcal{C}_{\mathcal{T}}] \\ &\simeq \text{Rat}^{\mathcal{C}}(\mathcal{M}_{{}^*\mathcal{C}}) = \sigma[\mathcal{C}^*\mathcal{C}]. \end{aligned} \quad (3)$$

3 Coinduction Functors between Categories of Type $\sigma[M]$

Our coinduction functor between categories of comodules over corings is derived from a more general coinduction functor between categories of type $\sigma[M]$. In this section we present this general coinduction functor and study some of its properties. Throughout this section we fix R -algebras A, B with a morphism of R -algebras $\beta : A \rightarrow B$, and an A -ring \mathcal{T} , a B -ring \mathcal{S} with a morphism of A -rings $\xi : \mathcal{S} \rightarrow \mathcal{T}$.

3.1. For every right \mathcal{T} -module M , the canonical right B -module $M \otimes_A B$ is a right \mathcal{S} -module through the right \mathcal{S} -action

$$\sum m_i \otimes_A b_i \leftarrow s := \sum m_i \xi(s) \otimes_A b_i.$$

It's easy to see that we have in fact a *covariant functor*

$$- \otimes_A B : \mathbb{M}_{\mathcal{T}} \rightarrow \mathbb{M}_{\mathcal{S}}.$$

3.2. For every right \mathcal{S} -module N , the canonical right A -module $\text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, N)$ is a right \mathcal{T} -module through

$$(\varphi \leftarrow \tilde{t}) \left(\sum t_i \otimes_A b_i \right) = \varphi \left(\sum \tilde{t} t_i \otimes_A b_i \right). \quad (4)$$

It's easily seen that we have in fact a *covariant functor*

$$\text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, -) : \mathbb{M}_{\mathcal{S}} \rightarrow \mathbb{M}_{\mathcal{T}}.$$

Moreover $(- \otimes_A B, \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, -))$ is an adjoint pair through the functorial canonical isomorphisms

$$\text{Hom}_{-\mathcal{S}}(M \otimes_A B, N) \simeq \text{Hom}_{-\mathcal{S}}(M \otimes_{\mathcal{T}} (\mathcal{T} \otimes_A B), N) \simeq \text{Hom}_{-\mathcal{T}}(M, \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, N)). \quad (5)$$

3.3. The coinduction functor $\text{Coind}_K^L(-)$. Let K be a right \mathcal{T} -module and consider the covariant functor

$$\text{HOM}_{-\mathcal{S}}^{(K)}(\mathcal{T} \otimes_A B, -) := \text{Sp}(\sigma[K_{\mathcal{T}}], -) \circ \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, -) : \mathbb{M}_{\mathcal{S}} \rightarrow \sigma[K_{\mathcal{T}}]. \quad (6)$$

For every right \mathcal{S} -module L (6) restricts to the covariant functor

$$\text{Coind}_L^K(-) := \sigma[L_{\mathcal{S}}] \rightarrow \sigma[K_{\mathcal{T}}], N \mapsto \text{Sp}(\sigma[K_{\mathcal{T}}], \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, N)) \quad (7)$$

defined through the commutativity of the following diagram

$$\begin{array}{ccc}
\mathcal{M}_{\mathcal{S}} & \xrightarrow{\text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, -)} & \mathcal{M}_{\mathcal{T}} \\
\uparrow & \searrow \text{HOM}_{-\mathcal{S}}^{(K)}(\mathcal{T} \otimes_A B, -) & \downarrow \text{Sp}(\sigma[K_{\mathcal{T}}, -]) \\
\sigma[L_{\mathcal{S}}] & \xrightarrow{\text{Coind}_L^K(-)} & \sigma[K_{\mathcal{T}}]
\end{array}$$

Proposition 3.4. *Let K be a right \mathcal{T} -module and L be a right \mathcal{S} -module. If the induced right \mathcal{S} -module $K \otimes_A B$ is L -subgenerated, then $- \otimes_A B : \mathbb{M}_{\mathcal{T}} \rightarrow \mathbb{M}_{\mathcal{S}}$ restricts to a functor*

$$- \otimes_A B : \sigma[K_{\mathcal{T}}] \rightarrow \sigma[L_{\mathcal{S}}].$$

Proof. Assume $K \otimes_A B$ to be L -subgenerated. Let $M \in \sigma[K_{\mathcal{T}}]$ and $\sum_{i=1}^q m_i \otimes_A b_i \in M \otimes_A B$ be an arbitrary element. Since $M_{\mathcal{T}}$ is K -subgenerated, there exists for each $i = 1, \dots, q$ a finite subset $\{k_1^{(i)}, \dots, k_{n_i}^{(i)}\} \subset K$ with $\text{An}_{-\mathcal{T}}(\{k_1^{(i)}, \dots, k_{n_i}^{(i)}\}) \subseteq \text{An}_{-\mathcal{T}}(m_i)$. By assumption $K \otimes_A B$ is L -subgenerated and so for every $i = 1, \dots, q$ and $j = 1, \dots, n_i$, there exists a finite subset $W_j^{(i)} \subseteq L$, such that $\text{An}_{-\mathcal{S}}(W_j^{(i)}) \subseteq \text{An}_{-\mathcal{S}}(\{k_j^{(i)} \otimes_A 1_B\})$. Consider the finite subset $W := \bigcup_{i=1}^q (\bigcup_{j=1}^{n_i} W_j^{(i)}) \subseteq L$. If $s \in \text{An}_{-\mathcal{S}}(W)$, then obviously $\xi(s) \in \text{An}_{-\mathcal{T}}(\{m_1, \dots, m_q\})$ and we have $\sum_{i=1}^q m_i \otimes_A b_i \leftarrow s := \sum_{i=1}^q m_i \xi(s) \otimes_A b_i = 0$, i.e. $\text{An}_{-\mathcal{S}}(W) \subseteq \text{An}_{-\mathcal{S}}(\sum_{i=1}^q m_i \otimes_A b_i)$. We conclude then that $M \otimes_A B$ is L -subgenerated. We are done, since $\sigma[K_{\mathcal{T}}] \subseteq \mathbb{M}_{\mathcal{T}}$ and $\sigma[L_{\mathcal{S}}] \subseteq \mathbb{M}_{\mathcal{S}}$ are full subcategories. ■

Lemma 3.5. *Let K be a right \mathcal{T} -module and L be a right \mathcal{S} -module with a right A -linear map $\theta : K \rightarrow L$. If*

$$\xi((0_L : \theta(k))) \subseteq (0_K : k) \text{ for every } k \in K, \quad (8)$$

then the induced right \mathcal{T} -module $K \otimes_A B$ is L -subgenerated. This is the case in particular, if θ is injective and

$$\theta(k \leftarrow \xi(s)) = \theta(k) \leftarrow s \text{ for every } s \in \mathcal{S} \text{ and } k \in K. \quad (9)$$

Proof. Let $\sum_{i=1}^n k_i \otimes_A b_i \in K \otimes_A B$ be an arbitrary element, $V := \{k_i : i = 1, \dots, n\}$ and $W := \{\theta(k_i) : i = 1, \dots, n\}$. If $s \in \text{An}_{-\mathcal{S}}(W)$ is arbitrary, then it follows by assumption that $\xi(s) \in \text{An}_{-\mathcal{T}}(V)$ and so

$$\sum_{i=1}^n k_i \otimes_A b_i \leftarrow s := \sum_{i=1}^n k_i \xi(s) \otimes_A b_i = \sum_{i=1}^n 0 \otimes_A b_i = 0.$$

So $\text{An}_{-\mathcal{S}}(W) \subseteq (0_{K \otimes_A B} : \sum_{i=1}^n k_i \otimes_A b_i)$ and consequently $\sum_{i=1}^n k_i \otimes_A b_i \in \text{Sp}(\sigma[L_{\mathcal{S}}], K \otimes_A B)$.

Since $\sum_{i=1}^n k_i \otimes_A b_i \in K \otimes_A B$ is arbitrary, we conclude that $K \otimes_A B \in \sigma[L_{\mathcal{S}}]$.

Assume now that θ is injective and that the compatibility condition (9) is satisfied. Then we have for arbitrary $k \in K$ and $s \in (0_L : \theta(k))$:

$$\theta(k \leftarrow \xi(s)) = \theta(k) \leftarrow s = 0,$$

hence $k \leftarrow \xi(s) = 0$ (since θ is injective by our assumption). So condition (8) is satisfied and consequently the right \mathcal{S} -module $K \otimes_A B$ is L -subgenerated. ■

We are now ready to present the main result in this section

Theorem 3.6. *Let K be a right \mathcal{T} -module, L be a right \mathcal{S} -module and assume the right \mathcal{S} -module $K \otimes_A B$ to be L -subgenerated. Then we have an adjoint pair of covariant functors $(- \otimes_A B, \text{Coind}_L^K(-))$.*

Proof. By [Wis88, 45.11] $(\iota(-), \text{Sp}(\sigma[K_{\mathcal{T}}], -))$ is an adjoint pair of covariant functors, and so (5) provides us with isomorphisms functorial in $M \in \sigma[K_{\mathcal{T}}]$ and $N \in \sigma[L_{\mathcal{S}}]$:

$$\begin{aligned} \text{Hom}_{-\mathcal{S}}(M \otimes_A B, N) &\simeq \text{Hom}_{-\mathcal{S}}((\iota(M) \otimes_{\mathcal{T}} \mathcal{T}) \otimes_A B, N) \\ &\simeq \text{Hom}_{-\mathcal{S}}(\iota(M) \otimes_{\mathcal{T}} (\mathcal{T} \otimes_A B), N) \\ &\simeq \text{Hom}_{-\mathcal{T}}(\iota(M), \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, N)). \\ &\simeq \text{Hom}_{-\mathcal{T}}(M, \text{Sp}(\sigma[K_{\mathcal{T}}], \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, N))). \\ &\simeq \text{Hom}_{-\mathcal{T}}(M, \text{Coind}_L^K(N)). \blacksquare \end{aligned}$$

4 Coinduction Functors between Categories of Comodules for Measuring α -Pairings

In this section we apply our results in the previous section to the special case of coinduction functors between categories of comodules induced by a morphism of measuring left α -pairing. It turns out that our results in this setting are direct generalization of the previous results on coinduction functors between categories of comodules for coalgebras obtained by the author in his dissertation [Abu01] (see also [Abu]).

The (Ad-)Induction Functor

4.1. Let A be an R -algebra and $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an A -coring, $(M, \varrho_M) \in \mathbb{M}^{\mathcal{C}}$ and $(N, \varrho_N) \in {}^{\mathcal{C}}\mathbb{M}$. The *cotensor product* $M \square_{\mathcal{C}} N$ is defined through the exactness of the following sequence of R -modules

$$0 \longrightarrow M \square_{\mathcal{C}} N \longrightarrow M \otimes_A N \xrightarrow{\varpi} M \otimes_A \mathcal{C} \otimes_A N,$$

where $\varpi := \varrho_M \otimes_A id_N - id_M \otimes_A \varrho_N$. For more information about the cotensor product of comodules the reader may consult [BW03, Sections 21].

4.2. The induction functor. ([G-T02, Proposition 4.4]) Let $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a morphism of corings. Then we have a covariant functor, the so called *induction functor*

$$- \otimes_A B : \mathbb{M}^{\mathcal{C}} \rightarrow \mathbb{M}^{\mathcal{D}}, \quad M \mapsto M \otimes_A B,$$

where the canonical right B -module $M \otimes_A B$ has a structure of a right \mathcal{D} -comodule through

$$M \otimes_A B \rightarrow (M \otimes_A B) \otimes_B \mathcal{D} \simeq M \otimes_A \mathcal{D}, \quad m \otimes_A b \mapsto \sum m_{\langle 0 \rangle} \otimes_A \theta(m_{\langle 1 \rangle})b. \quad (10)$$

4.3. The ad-induction functor. ([G-T02, 4.5], [BW03, 24.7-24.9]) Let $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D}, B)$ be a morphism of corings. Then the canonical (B, A) -bimodule $B \otimes_A \mathcal{C}$ has a structure of a $(\mathcal{D}, \mathcal{C})$ -bicomodule with the canonical right \mathcal{C} -comodule structure and the left \mathcal{D} -comodule structure given by

$$\varrho : B \otimes_A \mathcal{C} \rightarrow \mathcal{D} \otimes_B (B \otimes_A \mathcal{C}) \simeq \mathcal{D} \otimes_A \mathcal{C}, \quad b \otimes_A c \mapsto \sum b\theta(c_1) \otimes_A c_2.$$

Assume the morphism of corings $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D}, B)$ to be pure. For every right \mathcal{D} -comodule N , the canonical right A -submodule $N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \subseteq N \otimes_B (B \otimes_A \mathcal{C}) \simeq N \otimes_A \mathcal{C}$ given by

$$N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) := \left\{ \sum n^i \otimes_A c^i \mid \sum n^i \otimes_B \theta(c^i_1) \otimes_A c^i_2 = \sum n^i_{\langle 0 \rangle} \otimes_B n^i_{\langle 1 \rangle} \otimes_A c^i \right\}$$

is a right \mathcal{C} -comodule by

$$\tilde{\varrho} : N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \rightarrow N \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \otimes_A \mathcal{C} : \sum n^i \otimes_A c^i \mapsto \sum n^i \otimes_A c^i_1 \otimes_A c^i_2.$$

Moreover we have a functor, the so called *ad-induction functor*

$$- \square_{\mathcal{D}}(B \otimes_A \mathcal{C}) : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}.$$

Proposition 4.4. ([G-T02, Proposition 4.5]) *Let $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a pure morphism of corings. Then $(- \otimes_A B, - \square_{\mathcal{D}}(B \otimes_A \mathcal{C}))$ is an adjoint pair of covariant functors.*

Theorem 4.5. *Let $P = (\mathcal{T}, \mathcal{C} : A)$ and $Q = (\mathcal{S}, \mathcal{D} : B)$ be measuring left α -pairings with a morphism in $\mathcal{P}_{ml}^{\alpha}$:*

$$(\xi, \theta : \beta) : (\mathcal{T}, \mathcal{C} : A) \rightarrow (\mathcal{S}, \mathcal{D} : B).$$

Then we have a covariant functor

$$\text{Hom}_{-\mathcal{S}}^{(\mathcal{C})}(\mathcal{T} \otimes_A B, -) := \text{Rat}^{\mathcal{C}}(-) \circ \text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, -) : \mathbb{M}_{\mathcal{S}} \rightarrow \mathbb{M}^{\mathcal{C}}, \quad (11)$$

which restricts to a functor

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}, \quad N \mapsto \text{Rat}^{\mathcal{C}}(\text{Hom}_{-\mathcal{S}}(\mathcal{T} \otimes_A B, N)) \quad (12)$$

defined by the commutativity of the following diagram

$$\begin{array}{ccc}
\mathbb{M}_{\mathcal{D}^*} & \xrightarrow{\text{Hom}_{-S}(\mathcal{T} \otimes_A B, -)} & \mathbb{M}_{\mathcal{C}^*} \\
\uparrow & \searrow^{\text{HOM}_{-S}^{(\mathcal{C})}(\mathcal{T} \otimes_A B, -)} & \downarrow \text{Rat}^{\mathcal{C}}(-) \\
\mathbb{M}^{\mathcal{D}} & \xrightarrow{\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-)} & \mathbb{M}^{\mathcal{C}}
\end{array}$$

Moreover $(- \otimes_A B, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-))$ is an adjoint pair of covariant functors.

Proof. Since P, Q satisfy the α -condition, Theorem 2.10 implies that $\mathbb{M}^{\mathcal{C}} \simeq \sigma[\mathcal{C}_{\mathcal{T}}]$ and $\mathbb{M}^{\mathcal{D}} \simeq \sigma[\mathcal{D}_{\mathcal{S}}]$. Moreover the canonical right B -module $\mathcal{C} \otimes_A B$ is a right \mathcal{D} -comodule by

$$\mathcal{C} \otimes_A B \mapsto \mathcal{C} \otimes_A B \otimes_B \mathcal{D} \simeq \mathcal{C} \otimes_A \mathcal{D}, \quad c \otimes_A b \mapsto \sum c_1 \otimes_A \theta(c_2)b,$$

and is hence \mathcal{D} -subgenerated as a right \mathcal{T} -module. The result follows now by Theorem 3.6. ■

In the case of a base field, the cotensor functor is equivalent to a suitable Hom-functor (e.g. [AW02, Proposition 3.1]). Analogous to the corresponding result for comodules of coalgebras over commutative base rings ([Abu01, Proposition 2.3.12], [Abu, Proposition 2.8]) we have

Proposition 4.6. *Let B be an R -algebra, \mathcal{D} a B -coring, \mathcal{S} a B -ring with a morphism of B -rings $\kappa : \mathcal{S} \rightarrow {}^*\mathcal{D}^*$ and assume the left B -pairing $Q := (\mathcal{D}, \mathcal{S} : B)$ to satisfy the α -condition.*

1. Let $(M, \varrho_M) \in \mathbb{M}^{\mathcal{D}}$, $(N, \varrho_N) \in {}^{\mathcal{D}}\mathbb{M}$ and consider $M \otimes_B N$ with the canonical \mathcal{S}^{op} -bimodule structure. Then we have for $\sum m_i \otimes_B n_i \in M \otimes_B N$:

$$\sum m_i \otimes_B n_i \in M \square_{\mathcal{D}} N \Leftrightarrow \sum s^{op} m_i \otimes_B n_i = \sum m_i \otimes_B n_i s^{op} \text{ for all } s^{op} \in \mathcal{S}^{op}.$$

2. We have an isomorphism of functors for every right \mathcal{D} -comodule M and every left \mathcal{D} -comodule N :

$$M \square_{\mathcal{D}} N \simeq {}_{\mathcal{S}^{op}}\text{Hom}_{\mathcal{S}^{op}}(\mathcal{S}^{op}, M \otimes_B N).$$

Proof. 1. Let $M \in \mathbb{M}^{\mathcal{D}}$, $N \in {}^{\mathcal{D}}\mathbb{M}$ and set $\psi := \alpha_{M \otimes_B N}^Q \circ \tau_{(23)}$. Then we have:

$$\begin{aligned}
& \sum m_i \otimes_B n_i \in M \square_{\mathcal{D}} N \\
\Leftrightarrow & \sum m_{i \langle 0 \rangle} \otimes_B m_{i \langle 1 \rangle} \otimes_B n_i = \sum m_i \otimes_B n_{i \langle -1 \rangle} \otimes_B n_{i \langle 0 \rangle}, \\
\Leftrightarrow & \psi(\sum m_{i \langle 0 \rangle} \otimes_B m_{i \langle 1 \rangle} \otimes_B n_i)(s) = \psi(\sum m_i \otimes_B n_{i \langle -1 \rangle} \otimes_B n_{i \langle 0 \rangle})(s), \forall s \in \mathcal{S}. \\
\Leftrightarrow & \sum m_{i \langle 0 \rangle} \langle s, m_{i \langle 1 \rangle} \rangle \otimes_B n_i = \sum m_i \otimes_B \langle s, n_{i \langle -1 \rangle} \rangle n_{i \langle 0 \rangle}, \forall s \in \mathcal{S}. \\
\Leftrightarrow & \sum s^{op} m_i \otimes_B n_i = \sum m_i \otimes_B n_i s^{op}, \forall s^{op} \in \mathcal{S}^{op}.
\end{aligned}$$

2. The isomorphism is given by the morphism

$$\gamma_{M,N} : M \square_{\mathcal{D}} N \rightarrow {}_{\mathcal{S}^{op}} \text{Hom}_{\mathcal{S}^{op}}(\mathcal{S}^{op}, M \otimes_B N), \quad m \otimes_B n \mapsto [s^{op} \mapsto s^{op} m \otimes_B n (= m \otimes_B n s^{op})]$$

with inverse $\beta_{M,N} : f \mapsto f(1_{\mathcal{S}^{op}})$. It is easy to see that $\gamma_{M,N}$ and $\beta_{M,N}$ are functorial in M and N . ■

As a consequence of Proposition 4.4, Theorem 4.5 and Proposition 4.6 we get

Corollary 4.7. *Let $P = (\mathcal{T}, \mathcal{C} : A)$, $Q = (\mathcal{S}, \mathcal{D} : B)$ be measuring left α -pairings with a morphism in \mathcal{P}_{ml}^α :*

$$(\xi, \theta : \beta) : (\mathcal{T}, \mathcal{C} : A) \rightarrow (\mathcal{S}, \mathcal{D} : B).$$

Then we have an isomorphism of functors

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) \simeq -\square_{\mathcal{D}}(B \otimes_A \mathcal{C}).$$

If moreover $\kappa_Q(\mathcal{S}) \subseteq {}^ \mathcal{D}^*$ then we have an isomorphisms of functors*

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) \simeq -\square_{\mathcal{D}}(B \otimes_A \mathcal{C}) \simeq \text{Hom}_{\mathcal{S}^{op} - \mathcal{S}^{op}}(\mathcal{S}^{op}, - \otimes_B (B \otimes_A \mathcal{C})).$$

4.8. Corings over the Same Base Ring. Let A be an R -algebra, \mathcal{C}, \mathcal{D} be A -corings and $\theta : \mathcal{C} \rightarrow \mathcal{D}$ be an A -coring morphism. Then we have the *corestriction functor*

$$(-)^\theta : \mathbb{M}^{\mathcal{C}} \rightarrow \mathbb{M}^{\mathcal{D}}, \quad (M, \varrho_M) \mapsto (M, (id_M \otimes_A \theta) \circ \varrho_M). \quad (13)$$

In particular, \mathcal{C} is a $(\mathcal{D}, \mathcal{C})$ -bicomodule with the canonical right \mathcal{C} -comodule structure and the left \mathcal{D} -comodule structure by

$${}^{\mathcal{D}} \varrho_{\mathcal{C}} : \mathcal{C} \xrightarrow{\Delta_{\mathcal{C}}} \mathcal{C} \otimes_A \mathcal{C} \xrightarrow{\theta \otimes_A id_{\mathcal{C}}} \mathcal{D} \otimes_A \mathcal{C}.$$

If $(N, \varrho_N^{\mathcal{D}})$ is a right \mathcal{D} -comodule and

$$\omega_{N,\mathcal{C}} := \varrho_N^{\mathcal{D}} \otimes_A id_{\mathcal{C}} - id_N \otimes_A {}^{\mathcal{D}} \varrho_{\mathcal{C}} : N \otimes_A \mathcal{C} \rightarrow N \otimes_A \mathcal{D} \otimes_A \mathcal{C}$$

is \mathcal{C} -pure in \mathbb{M}_A (e.g. ${}_A \mathcal{C}$ flat), then the cotensor product $N \square_{\mathcal{D}} \mathcal{C}$ has a structure of a right \mathcal{C} -comodule by

$$N \square_{\mathcal{D}} \mathcal{C} \xrightarrow{id_N \square_{\mathcal{D}} \Delta_{\mathcal{C}}} N \square_{\mathcal{D}} (\mathcal{C} \otimes_A \mathcal{C}) \simeq (N \square_{\mathcal{D}} \mathcal{C}) \otimes_A \mathcal{C}$$

and we get the *ad-induction functor*

$$-\square_{\mathcal{D}} \mathcal{C} : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}, \quad N \mapsto N \square_{\mathcal{D}} \mathcal{C}.$$

Corollary 4.9. *Let A be an R -algebra, \mathcal{C}, \mathcal{D} be A -corings and $\theta : \mathcal{C} \rightarrow \mathcal{D}$ be a morphism of A -corings. If ${}_A \mathcal{C}$ is locally projective, then the morphism of A -rings ${}^* \theta : {}^* \mathcal{D} \rightarrow {}^* \mathcal{C}$ induces a covariant functor*

$$\text{HOM}_{-{}^* \mathcal{D}}^{(\mathcal{C})}({}^* \mathcal{C}, -) := \text{Rat}^{\mathcal{C}}(-) \circ \text{Hom}_{-{}^* \mathcal{D}}({}^* \mathcal{C}, -) : \mathbb{M}_{-{}^* \mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}},$$

which restricts to the coinduction functor

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}, \quad N \mapsto \text{Rat}^{\mathcal{C}}(\text{Hom}_{-{}^* \mathcal{D}}({}^* \mathcal{C}, N)).$$

Moreover $((-)^{\theta}, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-))$ is an adjoint pair of covariant functors and we have isomorphisms of functors

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) \simeq -\square_{\mathcal{D}} \mathcal{C} \simeq \text{Hom}_{({}^* \mathcal{D}^*)^{op} - ({}^* \mathcal{D}^*)^{op}}(({}^* \mathcal{D}^*)^{op}, - \otimes_A \mathcal{C}).$$

5 The general case

In this section we consider the case of coinduction functors induced by a morphism of corings. Throughout the section, fix R -algebras A, B with a morphism of R -algebras $\beta : A \rightarrow B$, an A -coring \mathcal{C} , a B -coring \mathcal{D} with a morphism of corings $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ and set $\# \mathcal{C} := \text{Hom}_{A-}(\mathcal{C}, B)$.

Lemma 5.1. 1. $\# \mathcal{C}$ is a $({}^* \mathcal{C}, {}^* \mathcal{D})$ -bimodule through

$$(\phi \rightharpoonup h)(c) = \sum h(c_1 \phi(c_2)) \text{ and } (h \leftarrow g)(c) = \sum g(\theta(c_1)h(c_2)). \quad (14)$$

2. If N is a right ${}^* \mathcal{D}$ -module, then $\text{Hom}_{-{}^* \mathcal{D}}(\# \mathcal{C}, N)$ is a right ${}^* \mathcal{C}$ -module by

$$(\varphi \leftarrow \psi)(h) = \varphi(\psi \rightharpoonup h) \text{ for all } \psi \in {}^* \mathcal{C}, \varphi \in \text{Hom}_{-{}^* \mathcal{D}}(\# \mathcal{C}, N) \text{ and } h \in \# \mathcal{C}.$$

If N is a left ${}^* \mathcal{C}$ -module, then $\text{Hom}_{{}^* \mathcal{C}-}(\# \mathcal{C}, N)$ is a left ${}^* \mathcal{D}$ -module by

$$(g \rightharpoonup \varphi)(h) = \varphi(h \leftarrow g) \text{ for all } g \in {}^* \mathcal{D}, \varphi \in \text{Hom}_{{}^* \mathcal{C}-}(\# \mathcal{C}, N) \text{ and } h \in \# \mathcal{C}.$$

Proof. 1. For arbitrary $h \in \# \mathcal{C}$, $\phi, \psi \in {}^* \mathcal{C}$, $a \in A$ and $c \in \mathcal{C}$ we have

$$(\phi \rightharpoonup h)(ac) = \sum h(ac_1 \phi(c_2)) = \sum ah(c_1 \phi(c_2)) = a((\phi \rightharpoonup h)(c))$$

and

$$\begin{aligned} ((\phi *_l \psi) \rightharpoonup h)(c) &= \sum h(c_1(\phi *_l \psi)(c_2)) = \sum h(c_1 \psi(c_{21} \phi(c_{22}))) \\ &= \sum h(c_{11} \psi(c_{12} \phi(c_2))) = (\psi \rightharpoonup h)(\sum c_1 \phi(c_2)) \\ &= (\phi \rightharpoonup (\psi \rightharpoonup h))(c), \end{aligned}$$

i.e. $\# \mathcal{C}$ is a left ${}^* \mathcal{C}$ -module.

For arbitrary $h \in \# \mathcal{C}$, $f, g \in {}^* \mathcal{D}$, $a \in A$ and $c \in \mathcal{C}$ we have

$$\begin{aligned} (h \leftarrow g)(ac) &= \sum g(\theta(ac_1)h(c_2)) = \sum g(\beta(a)\theta(c_1)h(c_2)) \\ &= \sum \beta(a)g(\theta(c_1)h(c_2)) = \beta(a) \sum g(\theta(c_1)h(c_2)) \\ &= \beta(a)[(h \leftarrow g)(c)] = a[(h \leftarrow g)(c)] \end{aligned}$$

and

$$\begin{aligned} (h \leftarrow f *_l g)(c) &= \sum (f *_l g)(\theta(c_1)h(c_2)) = \sum g(\theta(c_1)_1 f(\theta(c_1)_2 h(c_2))) \\ &= \sum g(\theta(c_{11})f(\theta(c_{12})h(c_2))) = \sum g(\theta(c_1)f(\theta(c_{21})h(c_{22}))) \\ &= \sum g(\theta(c_1)(h \leftarrow f)(c_2)) = ((h \leftarrow f) \leftarrow g)(c), \end{aligned}$$

i.e. $\# \mathcal{C}$ is a right ${}^* \mathcal{D}$ -module.

To show the compatibility between the left ${}^* \mathcal{C}$ -action and the right ${}^* \mathcal{D}$ -action on $\# \mathcal{C}$ pick arbitrary $h \in \# \mathcal{C}$, $\psi \in {}^* \mathcal{C}$, $g \in {}^* \mathcal{D}$ and $c \in \mathcal{C}$. Then we have

$$\begin{aligned} (\psi \rightharpoonup (h \leftarrow g))(c) &= (h \leftarrow g)(\sum c_1 \psi(c_2)) = g(\sum \theta(c_{11})h(c_{12} \psi(c_2))) \\ &= g(\sum \theta(c_1)h(c_{21} \psi(c_{22}))) = g(\sum \theta(c_1)(\psi \rightharpoonup h)(c_2)) \\ &= ((\psi \rightharpoonup h) \leftarrow g)(c). \end{aligned}$$

2. Straightforward. ■

5.2. Assuming ${}_A\mathcal{C}$ to be locally projective we have the functor

$$\text{HOM}_{-^*\mathcal{D}}(\# \mathcal{C}, -) := \text{Rat}^{\mathcal{C}}(-) \circ \text{Hom}_{-^*\mathcal{D}}(\# \mathcal{C}, -) : \mathbb{M}_{^*\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}},$$

which restricts to a covariant coinduction functor

$$\mathcal{G} : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}, \quad N \mapsto \text{Rat}^{\mathcal{C}}(\text{Hom}_{-^*\mathcal{D}}(\# \mathcal{C}, N))$$

defined through the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{M}_{^*\mathcal{D}} & \xrightarrow{\text{Hom}_{-^*\mathcal{D}}(\# \mathcal{C}, -)} & \mathbb{M}_{^*\mathcal{C}} \\ \uparrow & \searrow \text{HOM}_{-^*\mathcal{D}}^{(\mathcal{C})}(\# \mathcal{C}, -) & \downarrow \text{Rat}^{\mathcal{C}}(-) \\ \mathbb{M}^{\mathcal{D}} & \xrightarrow{\mathcal{G}} & \mathbb{M}^{\mathcal{C}} \end{array}$$

Remark 5.3. Without further assumptions, it is not evident that $(- \otimes_A B, \mathcal{G})$ is an adjoint pair of covariant functors.

To get a right adjoint to $- \otimes_A B : \mathbb{M}^{\mathcal{C}} \rightarrow \mathbb{M}^{\mathcal{D}}$ we modify the definition of \mathcal{G} . Before we introduce the new version of the coinduction functor, some technical results are to be proved.

Lemma 5.4. *Consider the maps*

$$\beta \circ - : {}^*\mathcal{C} \rightarrow \# \mathcal{C}, \quad - \circ \theta : {}^*\mathcal{D} \rightarrow \# \mathcal{C}$$

and set $\# \mathcal{C} := \text{Im}(\beta \circ -)$ and $\# \mathcal{D} := \text{Im}(- \circ \theta)$.

1. $\# \mathcal{C}$ is an A -ring with multiplication

$$(\beta \circ \phi) * (\beta \circ \psi) := \beta \circ (\phi *_l \psi), \quad (15)$$

unit $\beta \circ \varepsilon_{\mathcal{C}}$ and

$$\beta \circ - : {}^*\mathcal{C} \rightarrow \# \mathcal{C}$$

is a morphisms of A -rings. Moreover $\# \mathcal{C} = {}^*\mathcal{C} \rightarrow (\beta \circ \varepsilon_{\mathcal{C}})$, and hence is a cyclic ${}^*\mathcal{C}$ -submodule of $\# \mathcal{C}$.

2. $\# \mathcal{D}$ is an A -ring with multiplication

$$(f \circ \theta) * (g \circ \theta) := (f *_l g) \circ \theta, \quad (16)$$

unit $\varepsilon_{\mathcal{D}} \circ \theta$ and

$$- \circ \theta : {}^*\mathcal{D} \rightarrow \# \mathcal{D}$$

is morphisms of A -rings. Moreover $\# \mathcal{D} = (\varepsilon_{\mathcal{D}} \circ \theta) \leftarrow {}^*\mathcal{D}$, and hence is a cyclic ${}^*\mathcal{D}$ -submodule of $\# \mathcal{C}$.

3. If $\# \mathcal{D} \subseteq \# \mathcal{C}$, then $\# \mathcal{C} \subseteq \# \mathcal{C}$ is a $({}^* \mathcal{C}, {}^* \mathcal{D})$ -subbimodule. Analogously, if $\# \mathcal{C} \subseteq \# \mathcal{D}$ then $\# \mathcal{D} \subseteq \# \mathcal{C}$ is a $({}^* \mathcal{C}, {}^* \mathcal{D})$ -subbimodule.
4. If $\# \mathcal{D} \subseteq \# \mathcal{C}$ and N is a right ${}^* \mathcal{D}$ -module, then $\text{Hom}_{-{}^* \mathcal{D}}(\# \mathcal{C}, N)$ has a right ${}^* \mathcal{C}$ -module structure by

$$(\varphi \leftarrow \psi)(h) = \varphi(\psi \rightarrow h) \text{ for all } \psi \in {}^* \mathcal{C}, \varphi \in \text{Hom}_{-{}^* \mathcal{D}}(\# \mathcal{C}, N) \text{ and } h \in \# \mathcal{C}.$$

If $\# \mathcal{C} \subseteq \# \mathcal{D}$ and N is a left ${}^* \mathcal{C}$ -module, then $\text{Hom}_{-{}^* \mathcal{C}}(\# \mathcal{D}, N)$ has a left ${}^* \mathcal{D}$ -module structure by

$$(g \rightarrow \varphi)(h) = \varphi(h \leftarrow g) \text{ for all } g \in {}^* \mathcal{D}, \varphi \in \text{Hom}_{-{}^* \mathcal{C}}(\# \mathcal{D}, N) \text{ and } h \in \# \mathcal{D}.$$

Proof. 1. It's obvious, that $(\# \mathcal{C}, *, \beta \circ \varepsilon_{\mathcal{C}})$ is an A -ring and that $\beta \circ -$ is a morphism of A -rings. For all $\phi \in {}^* \mathcal{C}$ and $c \in \mathcal{C}$ we have

$$\begin{aligned} (\beta \circ \phi)(c) &= (\beta \circ \phi)(\sum \varepsilon_{\mathcal{C}}(c_1)c_2) = \sum [(\beta \circ \varepsilon_{\mathcal{C}})(c_1)][(\beta \circ \phi)(c_2)] \\ &= \sum (\beta \circ \varepsilon_{\mathcal{C}})(c_1\phi(c_2)) = (\phi \rightarrow \beta \circ \varepsilon_{\mathcal{C}})(c), \end{aligned}$$

hence $\# \mathcal{C} = {}^* \mathcal{C} \rightarrow (\beta \circ \varepsilon_{\mathcal{C}}) \subseteq \# \mathcal{C}$ is a cyclic left ${}^* \mathcal{C}$ -submodule.

2. It's obvious that $(\# \mathcal{D}, *, \varepsilon_{\mathcal{D}} \circ \theta)$ is an A -ring and $- \circ \theta$ is a morphism of A -rings. Moreover, for every $g \in {}^* \mathcal{D}$ we have

$$\begin{aligned} (g \circ \theta)(c) &= (g \circ \theta)(\sum c_1 \varepsilon_{\mathcal{C}}(c_2)) = g(\sum \theta(c_1)(\beta \circ \varepsilon_{\mathcal{C}})(c_2)) \\ &= g(\sum \theta(c_1)(\varepsilon_{\mathcal{D}} \circ \theta)(c_2)) = ((\varepsilon_{\mathcal{D}} \circ \theta) \leftarrow g)(c), \end{aligned}$$

hence $\# \mathcal{D} = (\varepsilon_{\mathcal{D}} \circ \theta) \leftarrow {}^* \mathcal{D} \subseteq \# \mathcal{C}$ is a cyclic right ${}^* \mathcal{D}$ -submodule.

3. Assume that $\# \mathcal{D} \subseteq \# \mathcal{C}$. Then we have

$$\begin{aligned} \# \mathcal{C} \leftarrow {}^* \mathcal{D} &= ({}^* \mathcal{C} \rightarrow \beta \circ \varepsilon_{\mathcal{C}}) \leftarrow {}^* \mathcal{D} = {}^* \mathcal{C} \rightarrow (\beta \circ \varepsilon_{\mathcal{C}} \leftarrow {}^* \mathcal{D}) \\ &= {}^* \mathcal{C} \rightarrow (\varepsilon_{\mathcal{D}} \circ \theta \leftarrow {}^* \mathcal{D}) = {}^* \mathcal{C} \rightarrow \# \mathcal{D} \\ &\subseteq {}^* \mathcal{C} \rightarrow \# \mathcal{C} = \# \mathcal{C}, \end{aligned}$$

hence $\# \mathcal{C} \subseteq \# \mathcal{C}$ is closed under the right ${}^* \mathcal{D}$ -action (i.e. a right ${}^* \mathcal{D}$ -submodule).

Analogously, if $\# \mathcal{C} \subseteq \# \mathcal{D}$, then

$$\begin{aligned} {}^* \mathcal{C} \rightarrow \# \mathcal{D} &= {}^* \mathcal{C} \rightarrow (\varepsilon_{\mathcal{D}} \circ \theta \leftarrow {}^* \mathcal{D}) = ({}^* \mathcal{C} \rightarrow \varepsilon_{\mathcal{D}} \circ \theta) \leftarrow {}^* \mathcal{D} \\ &= ({}^* \mathcal{C} \rightarrow \beta \circ \varepsilon_{\mathcal{C}}) \leftarrow {}^* \mathcal{D} = \# \mathcal{C} \leftarrow {}^* \mathcal{D} \\ &= \# \mathcal{D} \leftarrow {}^* \mathcal{D} = \# \mathcal{D}, \end{aligned}$$

hence $\# \mathcal{D} \subseteq \# \mathcal{C}$ is closed under the left ${}^* \mathcal{C}$ -action (i.e. a left ${}^* \mathcal{C}$ -submodule).

4. Straightforward. ■

Lemma 5.5. 1. $\# \mathcal{C} \cap \# \mathcal{D}$ is an A -ring with either multiplications (15) or (16) and unit $\beta \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{D}} \circ \theta$. If $\# \mathcal{D} \subseteq \# \mathcal{C}$ (respectively $\# \mathcal{C} \subseteq \# \mathcal{D}$), then $-\circ \theta : {}^* \mathcal{D} \rightarrow \# \mathcal{C}$ (respectively $\beta \circ - : {}^* \mathcal{C} \rightarrow \# \mathcal{D}$) is a morphism of A -rings.

2. For all $\phi \in {}^* \mathcal{C}$ and $g \in {}^* \mathcal{D}$ we have

$$(\beta \circ \phi) \leftarrow g = \phi \rightarrow (g \circ \theta).$$

Hence $\# \mathcal{C} \leftarrow {}^* \mathcal{D} = {}^* \mathcal{C} \rightarrow \# \mathcal{D}$ is a $({}^* \mathcal{C}, {}^* \mathcal{D})$ -subbimodule of $\# \mathcal{C}$.

Proof. 1. For arbitrary $f \circ \theta = \beta \circ \phi$ and $g \circ \theta = \beta \circ \psi$ in $\# \mathcal{C} \cap \# \mathcal{D}$ we have

$$\begin{aligned} (f \circ \theta * g \circ \theta)(c) &= (f *_l g)(\theta(c)) &= \sum g(\theta(c)_1 f(\theta(c)_2)) \\ &= \sum g(\theta(c_1) f(\theta(c_2))) &= \sum g(\theta(c_1) (\beta \circ \phi)(c_2)) \\ &= (g \circ \theta)(\sum c_1 \phi(c_2)) &= \beta(\sum \psi(c_1 \phi(c_2))) \\ &= (\beta \circ (\phi *_l \psi))(c) &= ((\beta \circ \phi) * (\beta \circ \psi))(c), \end{aligned}$$

i.e.

$$(f \circ \theta) * (g \circ \theta) = (\beta \circ \phi) * (\beta \circ \psi). \quad (17)$$

The last statement follows immediately from (17).

2. For all $\phi \in {}^* \mathcal{C}$, $g \in {}^* \mathcal{D}$ and $c \in \mathcal{C}$ we have

$$((\beta \circ \phi) \leftarrow g)(c) = \sum g(\theta(c_1) (\beta \circ \phi)(c_2)) = \sum g(\theta(c_1 \phi(c_2))) = (\phi \rightarrow (g \circ \theta))(c). \blacksquare$$

Definition 5.6. Let $(\mathcal{C} : A)$, $(\mathcal{D} : B)$ be corings, $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ a morphism of corings, and consider the maps

$$\beta \circ - : {}^* \mathcal{C} \rightarrow \# \mathcal{C}, \quad - \circ \theta : {}^* \mathcal{D} \rightarrow \# \mathcal{C}.$$

Set $\# \mathcal{C} := \text{Im}(\beta \circ -)$ and $\# \mathcal{D} := \text{Im}(- \circ \theta)$. We say $(\theta : \beta)$ is a *compatible morphism of corings*, provided $\# \mathcal{D} \subseteq_{\#} \mathcal{C}$ (i.e. for every $f \in {}^* \mathcal{C}$, there exists some $g \in {}^* \mathcal{D}$ with $\beta \circ f = g \circ \theta$).

5.7. The coinduction functor. Assume ${}_A \mathcal{C}$ to be locally projective and the morphism of corings $(\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ to be compatible. Then we have a covariant functor

$$\text{HOM}_{-{}^* \mathcal{D}}^{(\mathcal{C})}(\# \mathcal{C}, -) := \text{Rat}^{\mathcal{C}}(-) \circ \text{Hom}_{-{}^* \mathcal{D}}(\# \mathcal{C}, -) : \mathbb{M}_{-{}^* \mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}},$$

which restricts to the covariant functor

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) : \mathbb{M}^{\mathcal{D}} \rightarrow \mathbb{M}^{\mathcal{C}}, \quad N \mapsto \text{Rat}^{\mathcal{C}}(\text{Hom}_{-{}^* \mathcal{D}}(\# \mathcal{C}, N))$$

defined through the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{M}_{*\mathcal{D}} & \xrightarrow{\text{Hom}_{-*\mathcal{D}}(\#\mathcal{C}, -)} & \mathcal{M}_{*\mathcal{C}} \\
\uparrow & \searrow^{\text{Hom}_{-*\mathcal{D}}^{(\mathcal{C})}(\#\mathcal{C}, -)} & \downarrow \text{Rat}^{\mathcal{C}}(-) \\
\mathcal{M}^{\mathcal{D}} & \xrightarrow{\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-)} & \mathcal{M}^{\mathcal{C}}
\end{array}$$

Lemma 5.8. 1. If ${}_{A}\mathcal{C}$ is locally projective then we have functorial morphisms in $M \in \mathbb{M}^{\mathcal{C}}$ and $N \in \mathbb{M}^{\mathcal{D}}$:

$$\begin{array}{ccc}
\Phi_{M,N} : \text{Hom}^{\mathcal{D}}(M \otimes_A B, N) & \rightarrow & \text{Hom}^{\mathcal{C}}(M, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(N)) \\
\mathfrak{x} & \mapsto & m \mapsto [h \mapsto \mathfrak{x}(\sum m_{\langle 0 \rangle} \otimes_A h(m_{\langle 1 \rangle}))].
\end{array} \quad (18)$$

2. If ${}_{A}\mathcal{C}, {}_{B}\mathcal{D}$ are locally projective and $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ is compatible, then we have functorial morphisms

$$\begin{array}{ccc}
\Psi_{M,N} : \text{Hom}^{\mathcal{C}}(M, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(N)) & \rightarrow & \text{Hom}^{\mathcal{D}}(M \otimes_A B, N) \\
\zeta & \mapsto & [m \otimes_A b \mapsto (\zeta(m)(\beta \circ \varepsilon_{\mathcal{C}}))b]
\end{array} \quad (19)$$

Proof. 1. First of all we prove that $\Phi_{M,N}$ is well-defined for all $M \in \mathbb{M}^{\mathcal{C}}$ and $N \in \mathbb{M}^{\mathcal{D}}$. For all $\mathfrak{x} \in \text{Hom}^{\mathcal{D}}(M \otimes_A B, N)$, $m \in M$, $\phi \in {}^*\mathcal{C}$ and $h \in \#\mathcal{C}$ we have

$$\begin{aligned}
[(\Phi_{M,N}(\mathfrak{x})(m)) \leftarrow \phi](h) &= [\Phi_{M,N}(\mathfrak{x})(m)](\phi \rightarrow h) \\
&= \mathfrak{x}(\sum m_{\langle 0 \rangle} \otimes_A (\phi \rightarrow h)(m_{\langle 1 \rangle})) \\
&= \mathfrak{x}(\sum m_{\langle 0 \rangle} \otimes_A h(m_{\langle 1 \rangle} \phi(m_{\langle 1 \rangle 2}))) \\
&= \mathfrak{x}(\sum m_{\langle 0 \rangle \langle 0 \rangle} \otimes_A h(m_{\langle 0 \rangle \langle 1 \rangle} \phi(m_{\langle 1 \rangle}))) \\
&= [\Phi_{M,N}(\mathfrak{x})(\sum m_{\langle 0 \rangle} \phi(m_{\langle 1 \rangle}))](h) \\
&= [\Phi_{M,N}(\mathfrak{x})(m \leftarrow \phi)](h).
\end{aligned}$$

So $\Phi_{M,N}(\mathfrak{x}) \in \text{Hom}_{-*\mathcal{C}}(M, \text{Hom}_{-*\mathcal{D}}(\#\mathcal{C}, N)) = \text{Hom}^{\mathcal{C}}(M, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(N))$, i.e. $\Phi_{M,N}$ is well-defined. It's easy to see that $\Phi_{M,N}$ is functorial in M and N .

2. For all $\zeta \in \text{Hom}^{\mathcal{C}}(M, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(N))$, $m \in M$, $b \in B$ and $g \in {}^*\mathcal{D}$ with $bg \circ \theta = \beta \circ \phi \in$

$\#\mathcal{C}$ we have

$$\begin{aligned}
\Psi_{M,N}(\zeta)[(m \otimes_A b) \leftarrow g] &= \Psi_{M,N}(\zeta)[\sum(m_{<0>} \otimes_A g(\theta(m_{<1>}b))] \\
&= \sum[\zeta(m_{<0>})(\beta \circ \varepsilon_{\mathcal{C}})][(bg)(\theta(m_{<1>}))] \\
&= \sum[\zeta(m)_{<0>}(\beta \circ \varepsilon_{\mathcal{C}})][(bg \circ \theta)(\zeta(m)_{<1>})] \\
&= \sum[\zeta(m)_{<0>}(\beta \circ \varepsilon_{\mathcal{C}})][(\beta \circ \phi)(\zeta(m)_{<1>})] \\
&= \sum[\zeta(m)_{<0>}(\beta \circ \varepsilon_{\mathcal{C}})]\phi(\zeta(m)_{<1>}) \\
&= (\zeta(m) \leftarrow \phi)(\beta \circ \varepsilon_{\mathcal{C}}) \\
&= \zeta(m)(\phi \rightarrow (\beta \circ \varepsilon_{\mathcal{C}})) \\
&= \zeta(m)(\beta \circ \phi) \\
&= \zeta(m)(bg \circ \theta) \\
&= \zeta(m)((\varepsilon_{\mathcal{D}} \circ \theta) \leftarrow bg) \\
&= [\zeta(m)(\varepsilon_{\mathcal{D}} \circ \theta)b] \leftarrow g \\
&= [\Psi_{M,N}(\zeta)(m \otimes_A b)] \leftarrow g
\end{aligned}$$

So $\Psi_{M,N}(\zeta) \in \text{Hom}_{-\ast_{\mathcal{D}}}(M \otimes_A B, N) = \text{Hom}^{\mathcal{D}}(M \otimes_A B, N)$. It's easy to see that $\Psi_{M,N}$ is functorial in M and N . ■

We are now in a position that allows us to introduce the main theorem in this section

Theorem 5.9. *Let $(\mathcal{C} : A)$, $(\mathcal{D} : B)$ be corings with ${}_A\mathcal{C}$, ${}_B\mathcal{D}$ locally projective and $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a compatible morphism of corings. Then $(- \otimes_A B, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-))$ is a pair of adjoint functors.*

Proof. By Lemma 5.8 it remains to prove that $\Phi_{M,N}$ and $\Psi_{M,N}$ are inverse isomorphisms for all $M \in \mathbb{M}^{\mathcal{C}}$ and $N \in \mathbb{M}^{\mathcal{D}}$. For all $\varkappa \in \text{Hom}^{\mathcal{D}}(M \otimes_A B, N)$, $m \in M$ and $b \in B$ we have

$$\begin{aligned}
[(\Psi_{M,N} \circ \Phi_{M,N})(\varkappa)](m \otimes_A b) &= [(\Phi_{M,N}(\varkappa)(m))(\beta \circ \varepsilon_{\mathcal{C}})]b \\
&= [\varkappa(\sum m_{<0>} \otimes_A (\beta \circ \varepsilon_{\mathcal{C}})(m_{<1>}))]b \\
&= [\varkappa(\sum m_{<0>} \varepsilon_{\mathcal{C}}(m_{<1>}) \otimes_A 1_B)]b \\
&= [\varkappa(m \otimes_A 1_B)]b \\
&= \varkappa(m \otimes_A b),
\end{aligned}$$

i.e. $\Psi_{M,N} \circ \Phi_{M,N} = \text{id}_{\text{Hom}^{\mathcal{D}}(M \otimes_A B, N)}$.

On the other hand, for all $\zeta \in \text{Hom}^{\mathcal{C}}(M, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(N))$, $m \in M$ and $h = \beta \circ \phi \in \#\mathcal{C}$ we have

$$\begin{aligned}
[((\Phi_{M,N} \circ \Psi_{M,N})(\zeta))(m)](\beta \circ \phi) &= \Psi_{M,N}(\zeta)(\sum m_{<0>} \otimes_A (\beta \circ \phi)(m_{<1>})) \\
&= \sum[\zeta(m_{<0>})(\beta \circ \varepsilon_{\mathcal{C}})][(\beta \circ \phi)(m_{<1>})] \\
&= \sum[\zeta(m)_{<0>}(\beta \circ \varepsilon_{\mathcal{C}})]\phi(\zeta(m)_{<1>}) \\
&= (\zeta(m) \leftarrow \phi)(\beta \circ \varepsilon_{\mathcal{C}}) \\
&= \zeta(m)(\phi \rightarrow \beta \circ \varepsilon_{\mathcal{C}}) \\
&= \zeta(m)(\beta \circ \phi),
\end{aligned}$$

i.e. $\Phi_{M,N} \circ \Psi_{M,N} = \text{id}_{\text{Hom}^{\mathcal{C}}(M, \text{Coind}_{\mathcal{D}}^{\mathcal{C}}(N))}$. ■

Corollary 5.10. *Let $(\mathcal{C} : A)$, $(\mathcal{D} : B)$ be corings with ${}_A\mathcal{C}$, ${}_B\mathcal{D}$ locally projective and $(\theta : \beta) : (\mathcal{C} : A) \rightarrow (\mathcal{D} : B)$ be a compatible morphism of corings. Then we have an isomorphism of covariant functors*

$$\text{Coind}_{\mathcal{D}}^{\mathcal{C}}(-) \simeq -\square_{\mathcal{D}}(B \otimes_A \mathcal{C}).$$

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