Symmetries of static, spherically symmetric space-times

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In this paper it is shown that reduction from maximal to minimal static, spherical symmetry of a space-time occurs in only one step reducing the number of independent Killing vector fields from 10 to 4. Maximal symmetry corresponds only to the de Sitter, anti-de Sitter, and Minkowski metrics, without reference to the Einstein field equations.

I. INTRODUCTION

By Noether's theorem[1] the symmetries of a Lagrangian imply the existence of conserved quantities. These symmetries have been used[2] to obtain the constants of motion for the trajectories of freely falling particles in the field of a gravitating source, e.g., in the Schwarzschild, Reissner–Nordstrom, and Kerr–Newmann geometries. In general relativity, symmetries are expressed in terms of Killing vector fields (or Killing tensor fields, as in the case of the Kerr–Newmann geometry[3]). The number of independent Killing vector fields (KV's) is related to the number of generators of the corresponding symmetry group. Rather than trying to work out the symmetries of some particular space-time by group theoretic methods, we work out all possible KV's for a static, spherically symmetric space-time by the process of elimination.

A Killing vector field is a vector field $\mathbf{k}$ relative to which the Lie derivative of the metric tensor g is zero, i.e.,

$$\mathscr{L}_{\mathbf{k}} g = 0. \tag{1}$$

In a torsion-free space, in a coordinate basis, the Killing equation reduces to[4]

$$g_{ab,c} k^c + g_{ac} k^c_{\ ,b} + g_{bc} k^c_{\ ,a} = 0 \quad (a,b,c = 0,...,3). \tag{2}$$

The number of KV's for the de Sitter, anti-de Sitter, and Minkowski geometries are known to be maximal (10) and for the Schwarzchild geometry to be minimal (4). A point that needs to be determined is whether the gaps in the number of KV's from the maximal to the minimal symmetry for a static, spherically symmetric space-time can be filled or not. In this paper we examine this point. We start by considering the most general static, spherically symmetric line element,

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{3}$$

The Killing equations are solved for all possible cases. It is found that there can be either ten or four KV's for the metric given by Eq. (3), in general.

The authors have not found any work in recent literature exactly along the lines followed here. However, there are two major lines followed that are fairly close to the approach taken in this paper. One follows the standard work of Petrov,[5] where he considers Einstein spaces, and the other is the work on exact solutions of Einstein's field equations, given by Kramer, Stephani, MacCallum, and Herlt,[6] for example.

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Since we are not dealing with Einstein spaces only, the work on Einstein spaces does not apply to our considerations. We have replaced the requirement by the conditions of spherical symmetry and staticity. Thus ours is, in many ways, a more restrictive assumption. Nevertheless, there are many examples of spherically symmetric, static metrics that do not belong to Einstein spaces.

Of course, all cases considered by us are exact solutions of *some* Einstein field equations. However, the procedure generally adopted is to deal with given Einstein equations and determine the symmetry of their exact solutions. We have reversed the order to deal with a given symmetry and determine, where possible, the stress-energy tensor for such a symmetry. This procedure may seem to provide a pointless approach at first sight. However, our point of view was to look only at the symmetries obtaining in a space-time, provided that it is static and spherically symmetric.

It is instructive to put the work in group theoretic terms. What we show in our paper is that the maximal symmetry group of a spherically symmetric static four-dimensional space-time is one of the three: (a) SO(1,4), (b) SO(2,3), or (c) SO(1,3) $\otimes \mathbf{R}^4$. Here the $\mathbf{R}^4$ give the four space-time translations. Thus the groups are either the de Sitter, anti-de Sitter, or Poincaré groups. The minimal allowed symmetry group is SO(3) $\otimes$ $\mathbf{R}$, where the $\mathbf{R}$ gives time translation and SO(3) the spatial rotations only. The remarkable result is that there does not exist any group properly containing the minimal group and properly contained in one of the minimal groups.

In the next section we explain the procedure adopted for finding KV's. This procedure is applied, in full, to one case in Sec. III while mentioning the results for all other cases without giving details. Finally, we state our main result in the form of a theorem in the concluding section.

II. PROCEDURE ADOPTED

To find the KV's for the metric given by Eq. (3) we write the complete set of first-order coupled partial differential equations obtained by inserting Eq. (3) into Eq. (2). Now, by differentiating these equations, we can obtain identities between pairs of equations, leading to first- or second-order partial differential equations that are decoupled. We then solve these differential equations by using the separation of variables. The separation and integration constants

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