King Fahd University of Petroleum and Minerals
Information and Computer Science Dept.

ICS 253 Discrete Structures I – Spring 2014

Handout #1

An Example of a proof by contradiction

Problem: Prove that the product of any irrational number and any nonzero rational number is irrational.

The statement to be proved has the form: $\forall x\ \forall y\ (\ P(x,y) \rightarrow\ Q(x,y)\ )$

Its negation is: $\exists x\ \exists y\ (\ P(x,y) \wedge \neg\ Q(x,y)\ )$.

Thus for the proof-by-contradiction, we assume (the negation of what is to be proved) that there exists an irrational number s and a rational number $r \neq 0$ and that (negation of Q) sr is rational.

Because r is a nonzero rational, there exist integers a, b ($b\neq 0$, $a\neq 0$) where $r = a/b$ (1)

Because sr is a rational, there exist integers c, d ($d\neq 0$) where $sr = c/d$ (2)

From (1) and (2), it follows that $s = (c/d) / (a/b) = (cb) / (da)$ (note that $da \neq 0$)

This means that s is rational which *contradicts* part of our starting assumption.

Pitfalls in Induction Proofs

Recall that the induction step is to prove $\forall n \geq n_0$: $P(n) \rightarrow P(n+1)$. A common fallacy occurs by not proving the induction step for *all* $n \geq n_0$, as illustrated by the next example. $P(n_0)$ is proved as a base step but then the induction step must prove $P(n_0+1)$.

Example: Prove that all horses are of the same color.

Proof (Incorrect): Let S be a set of n horses. We are to prove that for $n \geq 1$, all horses in S have the same color.

Base Step: Let n=1. Obviously, if S consists of just one horse, then all horses in S have the same color.

Induction Step: Let $n > 1$ and assume that for any set of n horses, all horses have the same color. Let S be a set of n+1 horses; i.e., $S = \{h_1, h_2, \ldots, h_{n+1}\}$. Then the sets

$$S' = S - \{h_1\} = \{h_2, h_3, \ldots, h_{n+1}\} \text{ and } S'' = S - \{h_2\} = \{h_1, h_3, \ldots, h_{n+1}\}$$

each contains exactly n horses, and so by the induction hypothesis, all horses in S' are of one color, and likewise for S''. Because horse h_3 is common to both S' and S'' and the fact that h_3 can have only one color, we conclude that the color of the horses in S' is identical to that of the horses in S''. (Note $n > 1 \Rightarrow n \geq 2 \Rightarrow n+1 \geq 3$, so there is, in fact, a third horse.) Because $S = S' \cup S''$, it follows that all horses in S are of the same color.

Obviously, the proposition being proved is false, so there is something wrong with the proof, but what? The base step is certainly correct, and the induction step, as stated, is also correct. The problem is that the induction step was not quantified properly. We should have proved $\forall n \geq 1$: $P(n) \rightarrow P(n+1)$. Instead, we proved (correctly) that $\forall n > 1$: $P(n) \rightarrow P(n+1)$. Indeed, it is true that, for instance, $P(2) \rightarrow P(3)$ (that is, whenever any two horses have the same color, then so do any three), but we never proved (and it is false that) $P(1) \rightarrow P(2)$.