King Fahd University of Petroleum and Minerals
Information and Computer Science Dept.

ICS 253 Discrete Structures I – Spring Term 2014 – Practice Problems on Generating Functions

---

You must know the following facts about GFs (you won't get any points for stating them):

- $\mathbf{1/(1-x)}$ is the GF for the *GP* infinite sequence **1, 1, 1, …** ⇔ $\mathbf{1 + x + x^2 + \dots}$
- $\mathbf{1/(1-x)^2}$ is the GF for the infinite sequence **1, 2, 3, …** ⇔ $\mathbf{1 + 2x + 3x^2 + \dots}$
- The *Binomial Theorem*, the exponent $\boldsymbol{n}$ is a positive integer

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- The *Extended Binomial Theorem*, the exponent $\boldsymbol{u}$ is a real number

$$(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$$

1. In the questions below write (*include proper reasoning*) a simple formula for the generating function associated with the given sequence.

a) 0, 1, 1, 1, 0, 1, 1, 1, 0, ….

The sequence S = 1,0,0,0,1,0,0,0,1, … corresponds to $1/(1-x^4)$ [by replacing x by x^4 in GP]
The sequence in question can be obtained as the sequence 1,1,1, … minus S
Thus, the sequence in question has its GF as $1/(1-x) - 1/(1-x^4)$

b) 0, 1, 2, 3, …

The sequence S = 1, 2, 3, … corresponds to $1/(1-x)^2$
The given sequence is the sequence is S minus the sequence 1,1,1, …
Thus, the GF for the given sequence is $1/(1-x)^2 - 1/(1-x) = [1 - (1-x)] / (1-x)^2 = x / (1-x)^2$

Another simpler solution:
The GF for the given sequence is $0 + 1x + 2x^2 + 3x^3 + \dots = 0 + x(1 + 2x + 3x^2 + \dots) = x / (1-x)^2$

2. Compute a non-recursive formula (in terms of $\boldsymbol{k}$) for $\boldsymbol{a_k}$ (the ***coefficient of*** $\boldsymbol{x^k}$) in the sequence determined by $1/(1+2x)^5$. Also give the values of the ***first three terms*** of the sequence.

Note: $1/(1+2x)^5 = (1+2x)^{-5}$. Thus, because of negative exponent, we use the Ext. Bin. Theorem.

By the Ext. Bin. Theorem, **the *k*-th term in the expansion of** $\mathbf{1/(1+2x)^5}$ **is** $\mathbf{C(-5, k)\,(2x)^k}$

By definition, $\boldsymbol{a_k}$ **is the coefficient of** x^k; thus, $\boldsymbol{a_k = C(-5, k)\, 2^k}$

Next, we compute the first three terms using the preceding formula.
$a_0 = C(-5, 0)\, 2^0 = 1$
$a_1 = C(-5, 1)\, 2^1 = -5*2 = -10$
$a_2 = C(-5, 2)\, 2^2 = [(-5*-6)/2]*4 = 60$ [Note: it is a mistake to compute $C(-5, 2)$ as $(-5*-4)/2$]