Error Probability of Coded Multi-Antenna Systems in Block Fading Environments

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***Abstract*— In this paper, a union bound on the error probability of coded multi-antenna systems over block fading channels is proposed. The bound is based on uniform interleaving of the coded sequence prior to transmission over the channel. Using the uniform interleaving argument the distribution of error bits over the fading blocks is computed. The pairwise error probability is derived for a specific distribution pattern of the error bits over the fading blocks. We consider coded systems that concatenate a binary code with a space-time block code (STBC). The tradeoff between channel diversity and channel estimation is investigated assuming pilot-aided channel estimation and the optimal channel memory is approximated analytically. Results show that the optimal channel memory increases with increasing the number of transmit antennas.**

I. INTRODUCTION

Multipath fading is a main problem in designing wireless communication systems. Coding techniques are effective in mitigating fading by providing the receiver with channel diversity. The *channel diversity* is defined as the number of fading realizations available at the decoder to decode a codeword. Alternatively, channel diversity can be provided at the transmitter as in space-time block codes (STBC)s [1, 2]. In the literature, infinite interleaving is assumed frequently to simplify the performance analysis of coded systems [3, 4], which is impossible in delay-sensitive applications. In practical communication systems the channel often exhibits memory. A popular model that captures the memory is the *block fading* channel [5], in which a frame undergoes a number of independent fading realizations, each affecting a number of consecutive signals. Recently, a union bound was derived in [6] to analyze the performance of binary coded systems over block fading channels. In this paper, the union bound in [6] is generalized to the case of coded STBCs over block fading channels is analyzed.

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If the receiver knows the *channel side information* (SI) perfectly, large channel diversity improves the system performance resulting in an optimal channel memory of unity. On the other hand, if the receiver estimates the channel, long channel memory provides more observations for each fading realization which permits a better channel estimation. Thus if the frame size is finite, there exists a fundamental tradeoff between the channel diversity and channel estimation [7]. It is well known [8] that channel estimation becomes more crucial to the performance of space-time codes as the number of transmit antennas increases. Moreover, increasing the number of transmit antennas provides more space diversity at the cost of more difficult channel estimation. In this paper, the tradeoff between channel diversity and estimation is investigated as well as the effect of the number of transmit antennas on the optimal channel memory.

The paper is organized as follows. The system model is described in Section II. Then, the union bound on the error probability of coded STBCs over block fading channels is discussed in Section III. In Section IV, the pairwise error probability is derived for the cases of coherent receivers with perfect and imperfect channel SI at the receiver. Conclusions are discussed in Section V.

II. SYSTEM MODEL

The coded system considered in the paper is shown in Figure 1. The transmitter consists of a binary encoder, random interleaver, a modulator and a multi-antenna transmission matrix. Time is divided into frames of duration NT, where T is the *transmission interval* of a bit. In each time interval of duration kT, a rate-R_c encoder maps k information bits into n coded bits, where $R_c = \frac{k}{n}$. Each coded bit is modulated to a signal using BPSK with coherent detection.

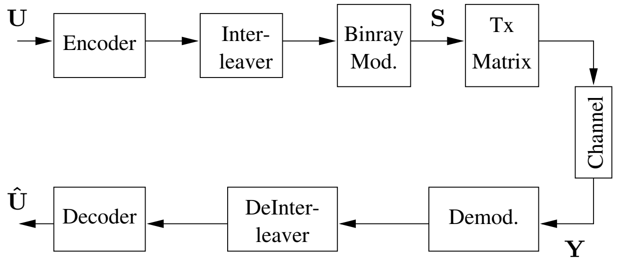

Fig. 1. The binary coded system with multi-antenna transmission.

The channel we adopt is a block fading channel, where each frame is subject to F independent fading realizations, and a block of length $m = \lceil \frac{N}{F} \rceil$ signals undergoes the same fading realization. The coded bits are interleaved prior to transmission over the channel in order to spread burst errors in the decoder.

After encoding and interleaving, each n_t signals are mapped into a $n_t \times n_t$ transmission matrix $\mathcal{G}_{n_t}$ as shown below for the case of $n_t = 2$ [1]

$$\mathcal{G}_2 = \begin{pmatrix} s_1 & s_2 \\ -s_2 & s_1 \end{pmatrix}. \tag{1}$$

The transmission of $\mathcal{G}_{n_t}$ takes place in a *time slot* of duration $n_t T$, where the i^{th} column of $\mathcal{G}_{n_t}$ is transmitted over the i^{th} transmit antenna during the time slot of duration $n_t T$. Thus the resulting STBC transmits one coded bit every T seconds. To be able to detect the STBC, the fading process from each antenna should remain constant for at least one time slot, i.e., n_t time intervals. This constrains the channel memory length to be a multiple of n_t, where each fading block contains $\frac{m}{n_t}$ time slots each of length n_t.

In the rest of the paper, the subscript of $\mathcal{G}_{n_t}$ is omitted and small letters in bold are used to denote column vectors. The received vector in time slot l of the f^{th} fading block is given by

$$\mathbf{y}_{f,l} = \sqrt{E_s}\mathcal{G}_{f,l}\mathbf{h}_f + \mathbf{z}_{f,l}, \tag{2}$$

where $\mathbf{z}_{f,l}$ is a length-n_t column random vector with a distribution $\mathcal{CN}(\mathbf{0}, N_0\mathbf{I})$ and $\mathbf{I}$ denotes the $n_t \times n_t$ identity matrix. The vector $\mathbf{h}_f$ contains the channel gains from the transmit antennas in fading block f and is modeled as $\mathcal{CN}(\mathbf{0}, \mathbf{I})$. The receiver employs sequence decoding based on the detection scheme of STBC in [1]. The decoder chooses the codeword $\mathbf{S}$ that maximizes the metric

$$\mathbf{m}(\mathbf{Y}, \mathbf{S}) = \sum_{f=1}^{F} \sum_{l=1}^{m/n_t} \mathrm{Re}\{\mathbf{y}_{f,l}^{*}\mathcal{G}_{f,l}\mathbf{h}_f\}, \tag{3}$$

where $(.)^*$ denotes the complex conjugate of a complex vector. The union bound for block fading channels is discussed in the following.

III. The Union Bound

Throughout the paper, the subscripts c, u and b are used to denote conditional, unconditional and bit error probabilities, respectively. For linear convolutional codes with k input bits, the bit error probability is upper bounded [3] as

$$P_b \leq \frac{1}{k} \sum_{d=d_{\min}}^{N} w_d P_u(d), \tag{4}$$

where $d_{\min}$ is the minimum distance of the code, w_d is the number of codewords with output weight d, and $P_u(d)$ is the unconditional pairwise error probability defined as the probability of decoding a received sequence as a weight-d codeword given that the all-zero codeword is transmitted.

In block fading channels, $P_u(d)$ is a function of the distribution of the d nonzero bits over the F fading blocks. This distribution is quantified assuming uniform interleaving of the coded bits over the fading blocks. Denote the number of fading blocks with weight v by f_v and define $w = \min(m, d)$, then the fading blocks are distributed according to the pattern $\mathbf{f} = \{f_v\}_{v=0}^{w}$ if the following conditions are satisfied

$$F = \sum_{v=0}^{w} f_v, \qquad d = \sum_{v=1}^{w} v f_v. \tag{5}$$

Denote by $L = F - f_0$ the number of fading blocks with nonzero weights. Then $P_u(d)$ averaged over all possible fading block patterns is given by

$$P_u(d) = \sum_{L=\lceil d/m \rceil}^{d} \sum_{f_1=0}^{L_1} \sum_{f_2=0}^{L_2} \cdots \sum_{f_w=0}^{L_w} P_u(d|\mathbf{f})p(\mathbf{f}), \tag{6}$$

where

$$L_v = \min\left\{ L - \sum_{r=1}^{v-1} f_r, \frac{d - \sum_{r=1}^{v-1} r f_r}{v} \right\}, \qquad 1 \leq v \leq w. \tag{7}$$

The probability of a fading block pattern $p(\mathbf{f})$ is

$$p(\mathbf{f}) = \frac{\binom{m}{1}^{f_1}\binom{m}{2}^{f_2}\cdots\binom{m}{w}^{f_w}}{\binom{mF}{d}} \cdot \frac{F!}{f_0! f_1! \ldots f_w!}. \tag{8}$$

Using (6)-(8) in (4), the union bound for convolutional codes over a block fading channels is computed. Since the number of summations involved in computing (6) increases as the channel memory length increases, a good approximation to the union bound is obtained by truncating the summation in (6) up to a small value of $d < N$. This results in an approximation to the error probability rather than an upper bound.

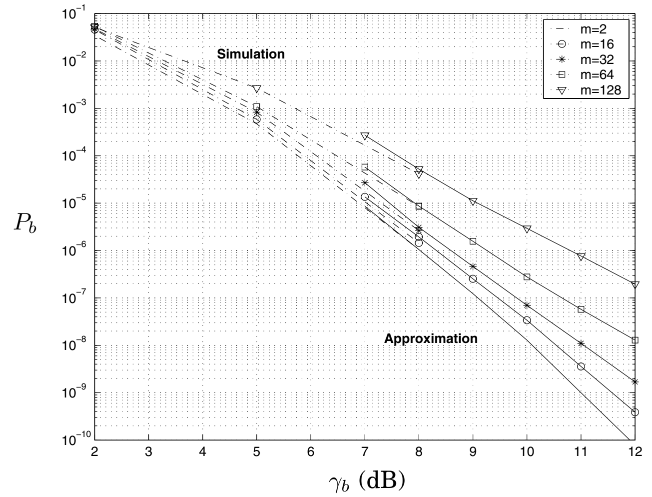

Fig. 2. Bit error probability of a convolutionally coded STBC with perfect SI and $n_t = 2$.

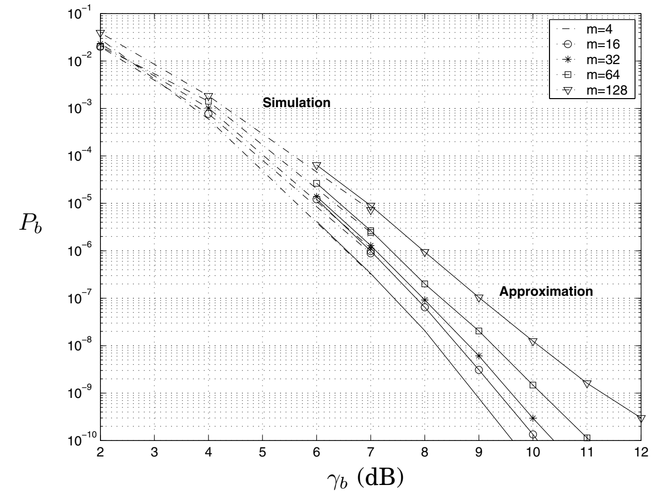

Fig. 3. Bit error probability of a convolutionally coded STBC with perfect SI and $n_t = 4$.

IV. Pairwise Error Probability

In this section, the pairwise error probabilities for coded STBC systems are derived for coherent detection with perfect and imperfect SI. The conditional pairwise error probability $P_c(d|\mathbf{f})$ is given by

$$P_c(d|\mathbf{f}) = \Pr\left(\mathbf{m}(\mathbf{Y},\mathbf{S}) - \mathbf{m}(\mathbf{Y},\hat{\mathbf{S}}) < 0 \middle| \mathbf{H},\mathbf{S}\right), \quad (9)$$

where $\mathbf{H} = \{h_f\}_{f=1}^{F}$, $\mathbf{Y} = \{\mathbf{y}_f\}_{f=1}^{F}$ and $\hat{\mathbf{S}}$ is the error codeword. Note that the d nonzero errors are distributed among the F fading blocks according to a pattern $\mathbf{f}$. The probability $P_u(d|\mathbf{f})$ is found by substituting the decoding metric in (9) and then averaging over the fading gains. The cases of perfect and imperfect SI are considered in the following.

A. Perfect SI

The conditional pairwise error probability results from substituting the metric (3) in (9) as

$$P_c(d|\mathbf{f}) = \Pr\left(\sum_{f=1}^{L}\sum_{l=1}^{w}\kappa_{f,l} < 0 \middle| \mathbf{H},\mathbf{S}\right), \quad (10)$$

where $\kappa_{f,l} = \mathrm{Re}\{\mathbf{y}_{f,l}^{*}\mathcal{E}_{f,l}\mathbf{h}_f\}$ and $\mathcal{E}_{f,l} = \mathcal{G}_{f,l} - \hat{\mathcal{G}}_{f,l}$. Here, $\mathcal{G}_{f,l}$ and $\hat{\mathcal{G}}_{f,l}$ are the transmission matrices in time slot l of the f^{th} fading block corresponding to the all-zero codeword and a weight-d error codeword, respectively. Here, we use $(.)^T$ to denote the transpose of a real matrix. In (10), $\kappa_{f,l}$ is a Gaussian random variable with conditional mean and variance given respectively by

$$\mathrm{E}\left[\kappa_{f,l}|\mathcal{G}_{f,l},\mathbf{h}_f\right] = \sqrt{E_s}d_{f,l}\sum_{i=1}^{n_t}|h_f^i|^2, \quad (11)$$

$$\mathrm{Var}\left[\kappa_{f,l}|\mathcal{G}_{f,l},\mathbf{h}_f\right] = d_{f,l}N_0\sum_{i=1}^{n_t}|h_f^i|^2, \quad (12)$$

where $d_{f,l}$ is the number of error bits in the time slot l in the f^{th} fading block. The error probability in (10) simplifies to

$$P_c(d|\mathbf{f}) = \mathrm{Q}\left(\sqrt{2R_c\gamma_b\sum_{v=1}^{w}v\sum_{l=1}^{f_v}\sum_{i=1}^{n_t}|h_l^i|^2}\right). \quad (13)$$

The unconditional pairwise error probability $P_u(d|\mathbf{f})$ is found by averaging over the fading gains and using the exact expression of the Q function, $\mathrm{Q}(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{(-x^2/2\sin^2\theta)}d\theta$ as

$$P_u(d|\mathbf{f}) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\prod_{v=1}^{w}\left(\frac{1}{1+vR_c\gamma_b/\sin^2\theta}\right)^{n_t f_v}d\theta. \quad (14)$$

As study cases, a rate-$\frac{1}{2}$ (23,35) convolutional code with 4 memory elements is concatenated with a STBC employing two and four transmit antennas with a frame size of $N = 2\times 512$ coded bits. The union bound was evaluated by substituting (14) in (4) and summing over codewords with distances $d \le 12$. The bound is compared to simulation results. In the simulations, the channel interleaver is chosen randomly and is changed every 10 frames to account for the uniform interleaving argument.

Figures 2 and 3, show the results for the cases of two and four transmit antennas, respectively for different channel memory lengths. It is observed that the performance degradation due to increasing the memory length reduces as the number of transmit antennas is increased. This is expected since there is more space diversity as the number of transmit anten-

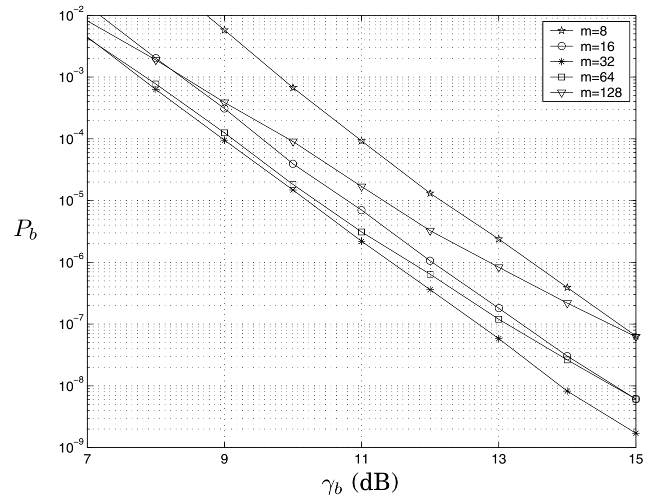

Fig. 4. Approximation of bit error probability of a convolutionally coded STBC with imperfect SI (OPE receiver with $E_p = E_s$) and $n_t = 2$.

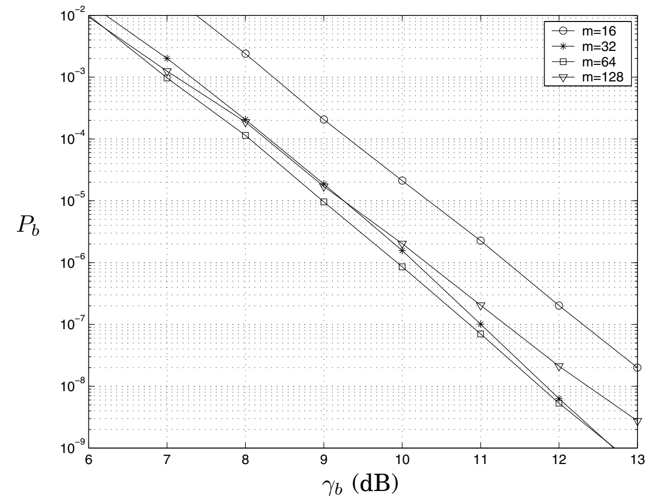

Fig. 5. Approximation of bit error probability of a convolutionally coded STBC with imperfect SI (OPE receiver with $E_p = E_s$) and $n_t = 4$.

nas increases, which reduces the sensitivity of the performance to the number of independent fading blocks.

B. Imperfect SI

Now we consider the case of coherent receivers with channel SI generated from pilot sequences. This is achieved by transmitting n_t orthogonal pilot sequences [8], each of length n_t, over the n_t antennas in each fading block. If pilot sequences from different transmit antennas are orthogonal, then the ML estimator of the channel gain is

$$\hat{h}_f^i = h_f^i + e_f^i, \tag{15}$$

where e_f^i is the estimation error associated with the channel from the i^{th} transmit branch in fading block f. The distribution of e_f^i follows $\mathcal{CN}(0, \sigma_e^2)$ where $\sigma_e^2 = \frac{N_0}{n_p E_p}$, where E_p is the energy spent on each pilot signal. The correlation coefficient between the true and estimated channel gains is defined as

$$\mu = \frac{\mathrm{E}\left[h_f^i \hat{h}_f^{i*}\right]}{\sqrt{\mathrm{Var}(h_f^i)\mathrm{Var}(\hat{h}_f^i)}} = \frac{1}{\sqrt{1+\sigma_e^2}}. \tag{16}$$

In [6] the ML decoding metric was shown to be difficult to implement in a Viterbi-like decoder as well as being difficult to analyze. Hence a suboptimal decoding rule that maximizes the likelihood function $p(\mathbf{Y}|\hat{\mathbf{H}}, \mathbf{S})$ is employed. It chooses the codeword $\mathbf{S}$ that maximizes the metric

$$\mathbf{m}(\mathbf{Y}, \mathbf{S}) = \sum_{f=1}^{F} \sum_{l=1}^{m/n_t} \mathrm{Re}\{\mathbf{y}_{f,l} G_{f,l} \hat{\mathbf{h}}_f\}. \tag{17}$$

The received signal vector $\mathbf{y}_{f,l}$ conditioned on the estimated channel gains is a complex Gaussian random vector with mean $\frac{\mu}{\sigma}\sqrt{E_s}\mathcal{G}_{f,l}\hat{\mathbf{h}}_f$ and covariance matrix $(N_0 + n_t E_s(1-\mu^2))\mathbf{I}$. Thus the pairwise error probability conditioned on the estimated fading gains is given by (10) with replacing $\mathbf{H}$ and $\kappa_{f,l}$ by $\hat{\mathbf{H}}$ and $\hat{\kappa}_{f,l} = \mathrm{Re}\{\mathbf{y}_{f,l}^* \mathcal{E}_{f,l} \hat{\mathbf{h}}_f\}$, respectively. Similar to the case of perfect SI, $\hat{\kappa}_{f,l}$ is a Gaussian random variable with mean and variance given respectively by

$$\mathrm{E}\left[\hat{\kappa}_{f,l} \middle| \mathcal{G}_{f,l}, \hat{\mathbf{h}}_f\right] = \frac{\mu}{\sigma}\sqrt{E_s} d_{f,l} \sum_{i=1}^{n_t} |\hat{h}_f^i|^2, \tag{18}$$

$$\mathrm{Var}\left[\hat{\kappa}_{f,l} \middle| \mathcal{G}_{f,l}, \hat{\mathbf{h}}_f\right] = \left(N_0 + n_t E_s(1-\mu^2)\right) d_{f,l} \sum_{i=1}^{n_t} |\hat{h}_f^i|^2. \tag{19}$$

Using the mean and variance of $\hat{\kappa}_{f,l}$, the pairwise error probability conditioned on the estimated fading gains is given by (13), with $\hat{\gamma}_b = \frac{\mu^2 \gamma_b}{1 + n_t R_c \gamma_b (1-\mu^2)}$ replacing γ_b. Thus the unconditional error probability is found by averaging over the estimated fading gains, resulting in (14) with the SNR value being $\hat{\gamma}_b$. Channel estimation using pilots with $E_p = E_s$ is referred to as only pilot estimation (OPE), with an estimation error variance of $\sigma_e^2 = \frac{N_0}{E_s}$.

In systems using pilot-aided channel estimation, n_p coded bits are punctured every m coded bits and replaced by a pilot sequence of length n_p. This reduces the error correcting capability of the code as the channel memory length becomes shorter which degrades the performance. The results for two transmit antennas with imperfect SI using an OPE receiver with

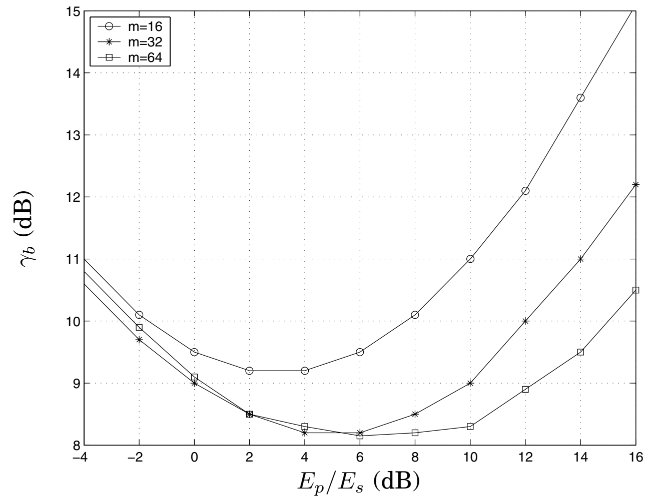

Fig. 6. SNR required for the convolutionally coded STBC with $n_t = 2$ to achieve $P_b = 10^{-4}$ versus E_p/E_s for the OPE receiver.

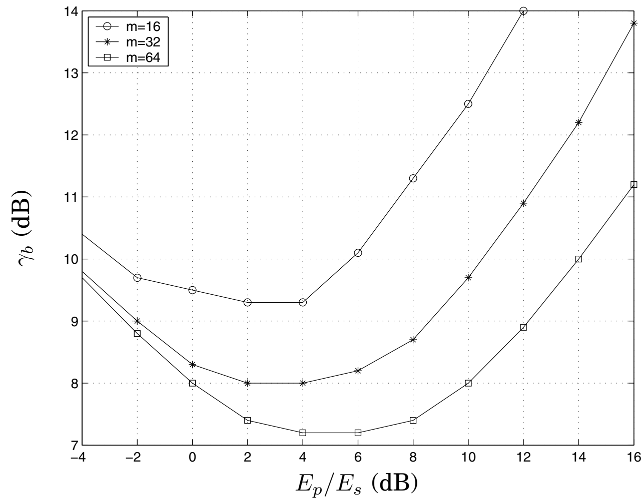

Fig. 7. SNR required for the convolutionally coded STBC with $n_t = 4$ to achieve $P_b = 10^{-4}$ versus E_p/E_s for the OPE receiver.

$n_p = n_t$ are shown in Figure 4. Note that the energy of the pilot sequences is taken into account in the SNR axis. In all cases the memory length $m = 8$ performs the worst among the shown curves because the resulting code is weak due to puncturing two coded bits every 8 coded bits. Using an OPE receiver, the case of $m = 64$ outperforms all other cases in the low SNR region, whereas the case of $m = 32$ starts to improve and becomes the best after an SNR value of 7 dB. Also, observe that the case of $m = 16$ outperforms the $m = 128$ case, which is reversed at low SNR.

When the energy allocated for the pilot signal is varied, the performance of an OPE receiver is expected to change as a function of the channel memory. The energy per information bit is written as a function of the energy allocated for transmitting signals and pilots as

$$E_b = \frac{(m - n_p)E_s + n_p E_p}{mR_c}. \tag{20}$$

Thus for a fixed channel memory, there exists an optimal value for pilot energy. Results for optimizing the pilot energy allocation are shown in Figures 6 and 7 for the cases of two and four transmit antennas, respectively. From the figures, we observe that the optimal pilot energy allocation is almost independent of the number of transmit antennas. This is mainly because the optimal pilot energy allocation is governed by the ratio of energy spent on estimating the channel, and this ratio is a function of the channel memory length only. As in single-antenna systems, optimizing the pilot energy results in an SNR gain as large as 1 dB over the case where $E_p = E_s$.

V. Conclusions

In this paper, a union bound on the performance of binary coded systems employing STBCs over block fading channels was derived. Results show that the SNR degradation due to channel memory reduces as the number of transmit antennas is increased. Moreover, under imperfect channel estimation conditions, the tradeoff between channel estimation and channel diversity was investigated and the optimal channel memory was approximated. It is observed that the optimal channel memory increases with increasing the number of transmit antennas.

VI. Acknowledgements

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References

[1] S. Alamouti, "A Simple Transmit Diversity Technique for Wireless Communications," *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 1451–1458, October 1998.

[2] V. Tarokh, H. Jafarkhani, and A. Calderbank, "Space-Time Block Codes from Orthogonal Designs," *IEEE Transactions on Information Theory*, vol. 45, pp. 1456–1467, July 1999.

[3] A. Viterbi, "Convolutional Codes and Their Performance in Communication Systems," *IEEE Transactions on Communications*, vol. COM-19, pp. 751–772, October 1971.

[4] D. Divsalar and M. Simon, "Trellis Coded Modulation for 4800-9600 bits/s Transmission Over a Fading Satellite Channel," *IEEE Journal on Selected Areas in Communications*, vol. SAC-5, pp. 162–175, February 1987.

[5] R. J. McEliece and W. E. Stark, "Channels with Block Interference," *IEEE Transactions on Information Theory*, vol. 30, pp. 44–53, January 1984.

[6] Salam Zummo, *Coding and Channel Estimation for Block Fading Channels*, Ph.D. thesis, University of Michigan, Ann Arbor, USA, June 2003.

[7] A. Worthen and W. Stark, *On the Channel Memory-Diversity Tradeoff in Communication Systems*, in Information, Coding and Mathematics. Kluwer Academic Publisher, Netherlands, 2002.

[8] V. Tarokh, A. Naguib, N. Seshadri, and A. Calderbank, "Space-Time Codes for High Data Rate Wireless Communication: Performance Criterion in the Presence of Channel Estimation Errors, Mobility, and Multiple Paths," *IEEE Transactions on Communications*, vol. 47, pp. 199–207, February 1999.