Performance Analysis of Coded Systems over Block Fading Channels

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***Abstract*— This paper considers the performance analysis of binary coded systems over block fading (BF) channels. In the transmitter, the coded bits are interleaved prior to transmission to spread burst errors resulting from deeply faded blocks. The union bound on the bit error probability (BEP) is derived assuming uniform interleaving and using the weight enumerator of the code. The union bound is a function of the pairwise error probability (PEP). Hence, the PEP is derived for coherent detection with different assumptions on the availability of the channel side information (SI) at the receiver. The proposed bound is evaluated for convolutional and turbo codes. For the case of imperfect SI, the tradeoff between channel estimation and the code's diversity is investigated. Results show that the bound provides insight on the gain loss due to channel memory.**

I. Introduction

Radio communications suffer mainly from fading of the received signal, and diversity is powerful to mitigate this problem. Error control codes provide a form of time diversity. If perfect SI is assumed, ideal interleaving provides the best performance, which is not possible for slow fading channels and limited delay applications. In practice, each frame is transmitted over a number of independent fading channels, as in frequency hopping and orthogonal frequency division multiplexing (OFDM) systems. In this case, a block of bits undergo the same channel, resulting in a BF channel [1, 2]. If iterative channel estimation and decoding is performed, there exists a tradeoff between channel estimation and the code's effective diversity [3].

The performance of coded systems over ideally interleaved channels is commonly analyzed using the union bound, which was computed in [4] for specific convolutional codes using the weight enumerator of the code. Similar bounds were derived in [5] for trellis codes, and in [6] for turbo codes with perfect and no SI assumptions. Multifrequency trellis codes [7] are special codes for BF channels, in which the output signals from the encoder are transmitted over different fading blocks. In [8], upper bounds on the performance of the multifrequency convolutional codes were derived. Several block and trellis codes for the BF channels were presented in [9].

In this paper, the union bound of binary coded systems over BF channels is derived. Expressions for PEP are derived for different assumptions about the channel SI at the receiver. Moreover, the tradeoff between channel estimation and the code's diversity is investigated. The paper starts with the system model in Section II. The proposed bound is derived in Section III and the PEP is considered in Section IV. Results are discussed in Section V and conclusions are presented in Section VI.

This work was supported in part by DoD and managed by ARO under the MURI program under grant 96-1-0377, and also by NSF under grant ECS-9979347.

II. System Model

The block diagram of the system is shown in Figure 1. In each transmission interval, the encoder maps k information bits into n coded bits, resulting in a code rate of $R_c = \frac{k}{n}$. Each coded bit is modulated to a signal from a binary unit-energy constellation. The frame of signals $\{s_l\}_{l=1}^{N}$ is transmitted over a BF channel, where a block of length $m = \lfloor \frac{N}{F} \rfloor$ signals undergoes the same fading realization. Here, F is the number of fading blocks. This occurs if the channel's coherence time is longer than the transmission duration of each block. Due to deeply faded blocks, bursts of low instantaneous signal-to-noise ratio (SNR) occur at the demodulator output, and therefore the coded bits are interleaved to spread the bursts in the decoder. The received signal at time l in the f^{th} fading block is

$$y_{f,l} = \sqrt{E_s}\alpha_f s_{f,l} + \eta_{f,l}, \tag{1}$$

where E_s is the average received energy and $\eta_{f,l}$ is an additive white noise modeled as independent zero-mean complex Gaussian random variables with variance σ_η^2, $\mathcal{CN}(0, \sigma_\eta^2)$. The variable α_f is the channel gain in the block f and is modeled as $\mathcal{CN}(0,1)$. Hence, it can be written as $\alpha_f = a_f \exp(j\theta_f)$, where $j = \sqrt{-1}$, the amplitude a_f has a Rayleigh distribution and the phase θ_f is uniformly distributed on $[0, 2\pi]$.

The receiver employs maximum likelihood (ML) decoding. In this paper, coherent reception is considered, where the channel phase is assumed to be known. Thus, SI refers solely to the fading amplitudes $\{a_f\}_{f=1}^{F}$. Therefore, the receiver may have either perfect, imperfect or no SI. The decoder maximizes

$$\mathbf{m}(\mathbf{y}, \mathbf{s}) = -\sum_{f=1}^{F}\sum_{l=1}^{m}\left|y_{f,l} - \sqrt{E_s} a_f s_{f,l}\right|^2, \tag{2}$$

if perfect SI is available, and maximizes

$$\mathbf{m}(\mathbf{y}, \mathbf{s}) = -\sum_{f=1}^{F}\sum_{l=1}^{m}\left|y_{f,l} - \sqrt{E_s} s_{f,l}\right|^2, \tag{3}$$

for no SI as in [5, 10]. If a pilot signal is transmitted with energy E_p in each block, the received signal is

$$y_{f,p} = \sqrt{E_p}\alpha_f + \eta_{f,p} \tag{4}$$

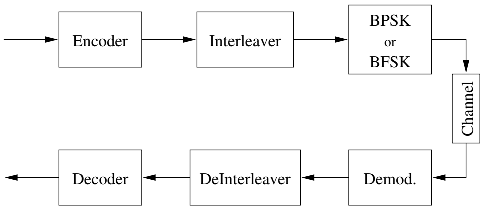

Fig. 1. The structure of the binary coded system.

The ML estimator for α_f is $\hat{\alpha}_f = \alpha_f + e_f$, where $e_f = \frac{\eta_{f,p}}{\sqrt{E_p}}$ is the estimation error, which is modeled as $\mathcal{CN}(0, \sigma_e^2)$ with $\sigma_e^2 = \sigma_\eta^2 / E_p$. The estimation quality is measured by

$$\mu = \frac{E[\alpha_f \hat{\alpha}_f^*]}{\sqrt{\mathrm{Var}(\alpha_f)\mathrm{Var}(\hat{\alpha}_f)}} = \frac{1}{\sqrt{1+\sigma_e^2}}. \quad (5)$$

The received signal $y_{f,l}$ conditioned on $\hat{\alpha}_f$ is complex Gaussian with mean $\frac{\mu}{\sigma}\sqrt{E_s}\hat{\alpha}_f s_{f,l}$ and variance $\sigma_\eta^2 + (1-\mu^2)E_s$. Thus, the ML decoding metric for imperfect SI is

$$\mathbf{m}(\mathbf{y}, \mathbf{s}) = -\sum_{f=1}^{F}\sum_{l=1}^{m}\left| y_{f,l} - \frac{\mu}{\sigma}\sqrt{E_s}\hat{\alpha}_f s_{f,l}\right|^2. \quad (6)$$

The proposed union bound is derived in the following.

III. The Union Bound

In this section, the union bound for convolutional and turbo codes over BF channels is derived. For linear convolutional codes with k input bits, the BEP is upper bounded [11] as

$$P_b \leq \frac{1}{k}\sum_{d=d_{free}}^{N} w_d P_u(d), \quad (7)$$

where d_{free} is the free distance of the code, $P_u(d)$ is the unconditional PEP, which is defined as the probability of decoding in favor of a codeword with weight d when the all-zero codeword is transmitted. In (7), $w_d = \sum_{i=1}^{N} i A_{i,d}$ where $A_{i,d}$ is the number of codewords with output weight d and input weight i. The weight distribution $\{w_d\}_{d=d_{free}}^{N}$ is obtained directly from the weight enumerator of the code [11]. For turbo codes with interleaver size $\tilde{N} = R_c N$, the union bound is found by averaging the BEP over all possible interleavers [12]

$$P_b \leq \sum_{i=1}^{\tilde{N}} \frac{i}{\tilde{N}}\binom{\tilde{N}}{i}\sum_{d=d_{free}}^{\tilde{N}} p(i,d)P_u(d), \quad (8)$$

where $p(i,d)$ is the probability of having an input sequence with weight i and an output codeword with weight d. For a turbo code with two component codes, $p(i,d)$ is given by

$$p(i,d) = \sum_{d_0,d_1,d_2 : d_0+d_1+d_2=d} p_0(i,d_0)p_1(i,d_1)p_2(i,d_2), \quad (9)$$

Where $p_0(i,d_0) = \delta(i,d_0)$ represents the systematic bit; and $p_j(i,d_j) = A_{i,d_j}/\binom{\tilde{N}}{i}$ for $j = 1,2$, accounts for the interleaver.

For BF channels, $P_u(d)$ is a function of the distribution of the d nonzero bits among the F fading blocks, which is quantified assuming uniform interleaving. Denote the number of fading blocks with weight v by f_v and $w = \min(m,d)$, then the pattern $\mathbf{f} = \{f_v\}_{v=0}^{w}$ occurs if

$$F = \sum_{v=0}^{w} f_v, \qquad d = \sum_{v=1}^{w} v f_v. \quad (10)$$

By averaging over possible patterns, $P_u(d)$ is found as

$$E_{\mathbf{f}|d}\left[P_u(d)\right] = \sum_{f_1=1}^{F}\sum_{f_2=1}^{F/2}\dots\sum_{f_w=1}^{F/w} P_u(d|\mathbf{f})p(\mathbf{f}). \quad (11)$$

Here, $p(\mathbf{f}) = \frac{n(\mathbf{f})}{\sum_{\mathbf{f}} n(\mathbf{f})}$, where $n(\mathbf{f})$ is the number of occurrence of the pattern $\mathbf{f}$, which depends on the number of combinations of $\{f_v\}_{v=0}^{w}$ among the F blocks, and the number of ways the bits can be ordered in the pattern. Using combinatorics,

$$n(\mathbf{f}) = \frac{F!}{f_0! f_1! \dots f_w!}\cdot\frac{d!}{(2!)^{f_2}(3!)^{f_3}\dots(w!)^{f_w}}. \quad (12)$$

Computing (11) requires w summations for each value of d, which is difficult for $m > 5$. However, $P_u(d|\mathbf{f})p(\mathbf{f})$ becomes small when $f_v > 2$, for $v \geq 3$. Also, the slope of the bound is determined mainly by the terms containing $f_v = 0, 1$ for $v \geq 3$. Hence, (11) can be approximated as

$$E_{\mathbf{f}|d}\left[P_u(d)\right] \approx \sum_{f_1=1}^{F}\sum_{f_2=1}^{F/2}\sum_{f_3=0}^{1}\dots\sum_{f_w=0}^{1} P_u(d|\mathbf{f})p(\mathbf{f}), \quad (13)$$

Thus, the union bounds for convolutional and turbo codes over BF channel are found by substituting $E_{\mathbf{f}|d}[P_u(d)]$ from (13) for $P_u(d)$ in (7) and (8), respectively. Expressions for $P_u(d|\mathbf{f})$ for different SI assumptions are derived in the next section.

IV. Pairwise Error Probability (PEP)

The conditional PEP, $P_c(d|\mathbf{f})$ is given by

$$P_c(d|\mathbf{f}) = \Pr\left(\mathbf{m}(\mathbf{y},\mathbf{s^0}) - \mathbf{m}(\mathbf{y},\mathbf{s}) < 0 \middle| \{\alpha_f\}_{f=1}^{F}, \mathbf{s^0}\right), \quad (14)$$

where $\mathbf{s}$ is a codeword with weight d. The cases of perfect, imperfect and no SI are considered in the following.

A. Perfect SI

For the case of perfect SI, the conditional PEP is

$$P_c(d|\mathbf{f}) = \mathrm{Q}\left(\sqrt{\rho R_c \gamma_b \sum_{v=1}^{w} v \sum_{i=1}^{f_v} a_i^2}\right), \quad (15)$$

where $\gamma_b = \frac{E_b}{\sigma_\eta^2}$ is the SNR per information bit, $\rho = 2$ for antipodal signals and $\rho = 1$ for orthogonal signals. To find $P_u(d|\mathbf{f})$, (15) is averaged over the fading amplitudes as

$$P_u(d|\mathbf{f}) = E_a\left[\mathrm{Q}\left(\sqrt{\rho R_c \gamma_b \sum_{v=1}^{w} v \sum_{i=1}^{f_v} a_i^2}\right)\right]. \quad (16)$$

Using $Q(x) \le \frac{1}{2}e^{-x^2/2}$, the Chernoff bound on (16) is

$$P_u(d|\mathbf{f}) \le \frac{1}{2}\prod_{v=1}^{w}\left(\frac{1}{1+v\rho R_c\gamma_b/2}\right)^{f_v}, \tag{17}$$

where the product results from the independence of the channels in different fading blocks. An exact expression of the PEP is found by using the integral expression of the Q-function [13]

$$P_u(d|\mathbf{f}) = \frac{1}{\pi}E_a\left[\int_0^{\frac{\pi}{2}}\exp\left(\frac{\rho R_c\gamma_b}{2\sin^2\theta}\sum_{v=1}^{w}v\sum_{i=1}^{f_v}a_i^2\right)d\theta\right]$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\prod_{v=1}^{w}\left(\frac{1}{1+v\rho R_c\gamma_b/2\sin^2\theta}\right)^{f_v}d\theta. \tag{18}$$

B. Imperfect SI

The PEP for the case of imperfect SI is found by plugging (6) in (14). After some algebra, the PEP conditioned on the estimated amplitudes $\{\hat{a}_f\}_{f=1}^{F}$ is given by

$$P_c(d|\mathbf{f}) = \Pr\Big(\eta > \frac{\mu^2}{\sigma^2}\rho E_s\sum_{v=1}^{w}v\sum_{i=1}^{f_v}\hat{a}_i^2\Big|\{\hat{a}\}\Big), \tag{19}$$

where η is complex Gaussian with zero mean and variance of $\rho\left(\sigma_\eta^2+(1-\mu^2)E_s\right)\frac{\mu^2}{\sigma^2}E_s\sum_{v=1}^{w}v\sum_{i=1}^{f_v}\hat{a}_i^2$. Let $\zeta = \frac{\hat{a}}{\sigma}$, and define $\hat{\gamma}_b = \frac{\mu^2\gamma_b}{1+(1-\mu^2)R_c\gamma_b}$ to be the effective SNR that takes into account the additional noise in the channel estimation, then the conditional PEP simplifies to

$$P_c(d|\mathbf{f}) = Q\left(\sqrt{\rho R_c\hat{\gamma}_b\sum_{v=1}^{w}v\sum_{i=1}^{f_v}\zeta_i^2}\right). \tag{20}$$

Since (20) is similar to (15), then the PEP of imperfect SI is the same as that of perfect SI, with replacing γ_b by $\hat{\gamma}_b$. Two cases are considered for channel estimation using pilots with $E_p = E_s$. In the first case, only pilot estimation (OPE) is performed, resulting in $\sigma_e^2 = \sigma_\eta^2/E_s$. The second case considers iterative channel estimation and decoding [3]. In this case, a lower bound on σ_e^2 is obtained by assuming the signals in each block are known, i.e., $\sigma_e^2 = \sigma_\eta^2/(mE_s)$. This case is referred to as correct data estimation (CDE).

C. No SI

When no SI is available at the receiver, (14) is upper bounded using the Chernoff bound [5] as

$$P_c(d|\mathbf{f}) \le \prod_{f=1}^{F}E\Big[\exp\Big(\lambda\sum_{l=1}^{m}|y_{f,l}-s_{f,l}^0|^2-|y_{f,l}-s_{f,l}|^2\Big)\Big], \tag{21}$$

where $\lambda > 0$ is the Chernoff parameter. Expanding,

$$P_c(d|\mathbf{f}) \le \prod_{f=1}^{F}\exp\Big(-\lambda\sum_{l=1}^{m}|s_{f,l}^0-s_{f,l}|^2-2\lambda(a_f-1)$$

$$\times\mathrm{Re}\{s_{f,l}^0(s_{f,l}^0-s_{f,l})^*\}\Big)$$

$$\times E\Big[\exp\Big(-2\lambda\sum_{l=1}^{m}\mathrm{Re}\{\eta_{f,l}(s_{f,l}^0-s_{f,l})^*\}\Big)\Big]. \tag{22}$$

For constant-envelope constellations, $2\mathrm{Re}\{s_{f,l}^0(s_{f,l}^0-s_{f,l})^*\} = |s_{f,l}^0-s_{f,l}|^2$. As in [11], it can be shown that

$$E\Big[e^{\left(-2\lambda\sum_{l=1}^{m}\mathrm{Re}\{\eta_l(s_l^0-s_l)^*\}\right)}\Big] = e^{\left(\lambda^2\sigma_\eta^2\sum_{l=1}^{m}|s_l^0-s_l|^2\right)}. \tag{23}$$

Substituting back in (22), the conditional PEP simplifies to

$$P_c(d|\mathbf{f}) \le \prod_{f=1}^{F}\exp\Big(-\lambda a_f\sum_{l=1}^{m}|s_{f,l}^0-s_{f,l}|^2$$

$$+\lambda^2\sigma_\eta^2\sum_{l=1}^{m}|s_{f,l}^0-s_{f,l}|^2\Big). \tag{24}$$

Since $\sum_{l=1}^{m}|s_{f,l}^0-s_{f,l}|^2 = 2\rho E_s d_f$, where d_f is the number of nonzero locations in block f, (24) simplifies to

$$P_c(d|\mathbf{f}) \le \prod_{f=1}^{F}\exp\left(-2\rho E_s\lambda a_f d_f+2\rho E_s\lambda^2\sigma_\eta^2 d_f\right). \tag{25}$$

Substituting $\tilde{\lambda} = \lambda\sigma_\eta^2$ and $\gamma_b = \frac{E_s}{R_c\sigma_\eta^2}$, then

$$P_c(d|\mathbf{f}) \le \prod_{f=1}^{F}\exp\left(-2\tilde{\lambda}d_f a_f\rho R_c\gamma_b+2\tilde{\lambda}^2 d_f\rho R_c\gamma_b\right). \tag{26}$$

By averaging over the fading gains $\{a_f\}_{f=1}^{F}$,

$$P_u(d|\mathbf{f}) \le \prod_{f=1}^{F}\exp\left(2\tilde{\lambda}^2 d_f\rho R_c\gamma_b\right)E\left[\exp\left(-2\tilde{\lambda}d_f a_f\rho R_c\gamma_b\right)\right]$$

$$= \prod_{f=1}^{F}\exp\left(2\tilde{\lambda}^2\rho R_c\gamma_b\right)\left[1-2\sqrt{\pi}\beta_f\exp(\beta_f^2)Q(\sqrt{2}\beta_f)\right], \tag{27}$$

where $\beta_f = \tilde{\lambda}d_f\rho R_c\gamma_b$. To find $\tilde{\lambda}$ that minimizes the bound, the approximation $Q(x) \approx \frac{1}{2\sqrt{\pi}}\exp(-\frac{x^2}{2})(1-\frac{1}{x^2})$ is used, and (27) becomes

$$P_u(d|\mathbf{f}) \le \prod_{v=1}^{w}\left(\frac{1}{2\beta_v^2}\right)^{f_v}\exp\left(2\tilde{\lambda}^2 f_v\rho R_c\gamma_b\right), \tag{28}$$

where in this case $\beta_v = \tilde{\lambda}v\rho R_c\gamma_b$. Let $L = F - f_0$ denote the number of fading blocks with nonzero weights, and $\bar{d} = \sum_{v=1}^{w}vf_v$ represent the average weight in a fading block. Then, the PEP simplifies to

$$P_u(d|\mathbf{f}) \le \frac{\exp\left(2\tilde{\lambda}^2\bar{d}\rho R_c\gamma_b\right)}{2^L(\tilde{\lambda}\rho R_c\gamma_b)^{2L}\prod_{v=1}^{w}v^{2f_v}}. \tag{29}$$

This can be minimized over $\tilde{\lambda}$, resulting in an optimum value $\lambda_{opt}^2 = L/(2\bar{d}\rho R_c \gamma_b)$. Substituting for λ_{opt}, the final expression of the PEP for no SI is given by

$$P_u(d|\mathbf{f}) \leq \left(\frac{e\bar{d}}{L\rho R_c \gamma_b}\right)^L \left(\prod_{v=1}^{w} v^{2f_v}\right)^{-1}. \quad (30)$$

V. Results

The proposed union bound was evaluated for the (5,7) convolutional code with a frame size $N = 200$, and the (1,5/7,5/7) turbo code with a frame size $N = 300$, both with BPSK signaling. In computing the union bound, (18) is used for perfect and imperfect SI with the appropriate SNR, where (30) is used for the case of no SI. Figure 2 shows the results for the convolutional code with perfect SI information. The bound is tight to the simulation up to $m = 10$, where it is less tight for $m = 20$. This is due to the approximation used in (13), whose effect increases as the memory length increases since more combinations of $\mathbf{f}$ are excluded by the approximation. In Figure 3, the case of no SI is shown. The bound is less tight because the PEP in (30) is the Chernoff bound, not the exact expression as (18).

Figures 4 and 5 show the results of imperfect SI with POE and CDE, respectively. The energy of the pilot is taken into account in the x-axis. For the POE, the gain loss in SNR due to the channel memory is reduced compared to the case of perfect SI. This is expected since shorter memory length reduces the number of transmitted pilot signals, and hence the system becomes more energy efficient. This phenomenon becomes more clear in the CDE case because the estimation quality improves with increasing memory length, since there is more information used in the estimation. Note that the cases of $m = 10$ and $m = 5$ are very close in the low and moderate SNR region, and they become more distinguished as the SNR increases. This is because at high SNR, the channel estimation is better since σ_e^2 is small. This suggests that the optimal memory length is between $m = 10$ and $m = 5$, but closer to $m = 5$.

The results of the turbo code are shown in Figures 6, 7 and 8 for the cases of perfect SI, imperfect SI with OPE and CDE, respectively. The case of no SI is not plotted for the space lack, but it bears no additional information than Figure 3. Again, the union bound is tight to simulation. From Figure 7, it can be seen that the gain losses due to the channel memory is less than the convolutional codes. Moreover, the cases of $m = 10$ and $m = 5$ cross at 4 dB, which is also dictated by the bound. This is expected since turbo codes is more connected than convolutional codes due to the interleaving, and hence more sensitive to channel memory. In the case of CDE, this becomes clearer, where the case of $m = 10$ outperforms $m = 5$ up to SNR value of 5 dB. Also, the bounds of these cases cross at 15 dB, indicating that the optimal memory length is somewhere between $m = 10$ and $m = 5$, depending on the SNR value.

VI. Conclusions

In this paper, the union bound of coherently detected binary coded systems over BF channels was derived. Several assumptions on the availability of SI at the receiver were considered, and the corresponding PEP were derived. The proposed bounds were evaluated for convolutional and turbo codes. Results show that the proposed bound is tight to the simulation and gives the trend of the code performance with channel memory, in the case of iterative decoding and channel estimation. Turbo codes were shown to be more sensitive to channel memory, and the optimal memory length was investigated.

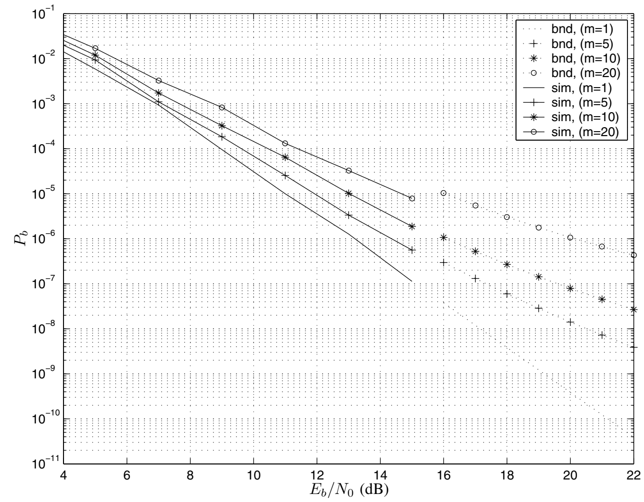

Fig. 2. Performance of the (5,7) convolutional code with perfect SI.

References

[1] R. J. McEliece and W. E. Stark, "Channels with Block Iinterference," *IEEE Transactions on Information Theory*, vol. 30, pp. 44 –53, January 1984.

[2] E. Biglieri, J. Proakis, and S. Shamai, "Fading Channels: Information-Theoretic and Communications Aspects," *IEEE Transactions on Information Theory*, vol. 44, pp. 2619 –2692, October 1998.

[3] A. P. Worthen, *Codes and Iterative Receivers for Wireless Communication Systems*, Ph.D. thesis, University of Michigan, Ann Arbor, May 2001.

[4] A. Viterbi, "Convolutional Codes and Their Performance in Communication Systems," *IEEE Transactions on Communications*, vol. COM-19, pp. 751–772, Oct. 1971.

[5] D. Divsalar and M. Simon, "Trellis Coded Modulation for 4800-9600 bits/s Transmission Over a Fading Satellite Channel," *IEEE Journal on Selected Areas in Communications*, vol. SAC-5, pp. 162–175, Feb. 1987.

[6] E. Hall and S. Wilson, "Design and Analysis of Turbo Codes on Rayleigh Fading Channels," *IEEE Journal on Selected Areas in Communications*, vol. 16, pp. 160 –174, Feb. 1998.

[7] Y. Leung, S. Wilson, and J. Ketchum, "Multifrequency Trellis Coding with Low Delay for Fading Channels," *IEEE Transactions on Communications*, vol. 41, pp. 1450 –1459, October 1993.

[8] E. Malkamaki and H. Leib, "Coded Diversity on Block-Fading Channels," *IEEE Transactions on Information Theory*, vol. 45, pp. 771 –781, March 1999.

[9] R. Knopp and P. Humblet, "On Coding for Block-Fading Channels," *IEEE Transactions on Information Theory*, vol. 46, pp. 189 –205, January 2000.

[10] D. Divsalar and M. Simon, "The Design of Trellis Coded MPSK for Fading Channels: Set Partitioning for Optimum Code Design," *IEEE Transactions on Communications*, vol. 36, pp. 1013 –1021, September 1988.

[11] J. G. Proakis, *Digital Communications*, McGraw-Hill, New York, USA, 4th edition, 2000.

[12] D. Divsalar and R. Pollara, "Performance Analysis of Turbo Codes," *IEEE Military Communications Conference, MILCOM*, 1995.

[13] M. Simon and D. Divsalar, "Some New Twists to Problems Involving the Gaussian Probability Integral," *IEEE Transactions on Communications*, vol. 46, pp. 200–210, February 1998.

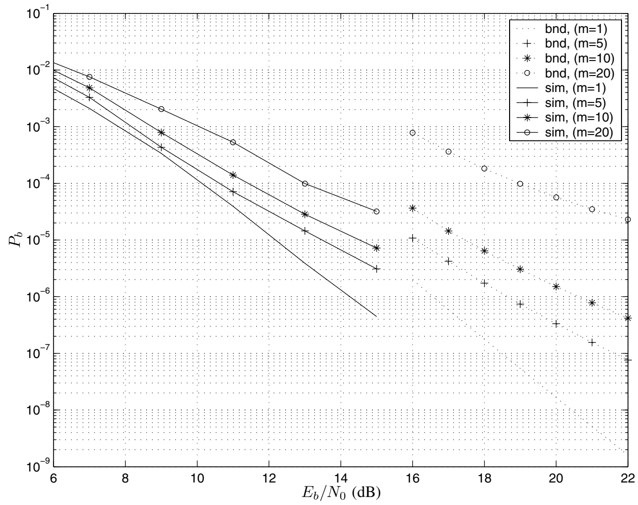

Fig. 3. Performance of the (5,7) convolutional code with no SI.

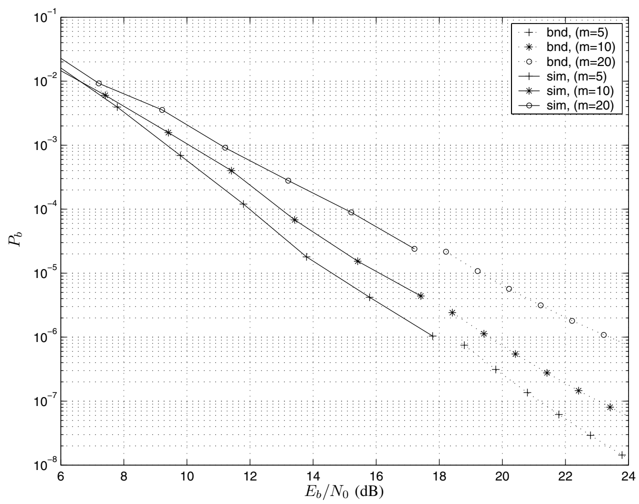

Fig. 4. Performance of the (5,7) convolutional code with OPE.

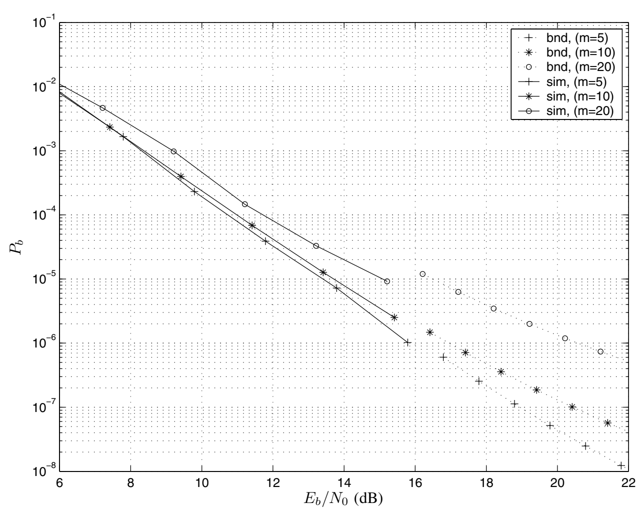

Fig. 5. Performance of the (5,7) convolutional code with CDE.

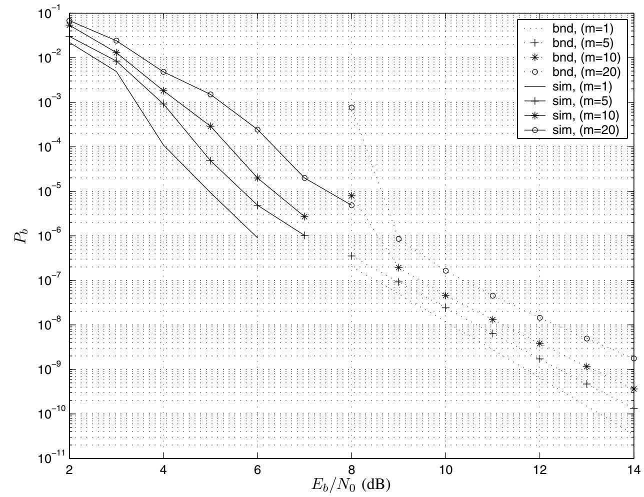

Fig. 6. Performance of the (1,5/7,5/7) turbo code with $\tilde{N} = 100$ and perfect SI.

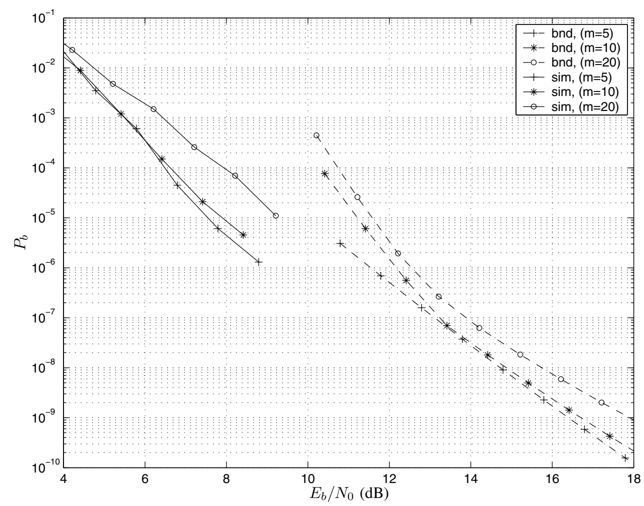

Fig. 7. Performance of the (1,5/7,5/7) turbo code with $\tilde{N} = 100$ and OPE.

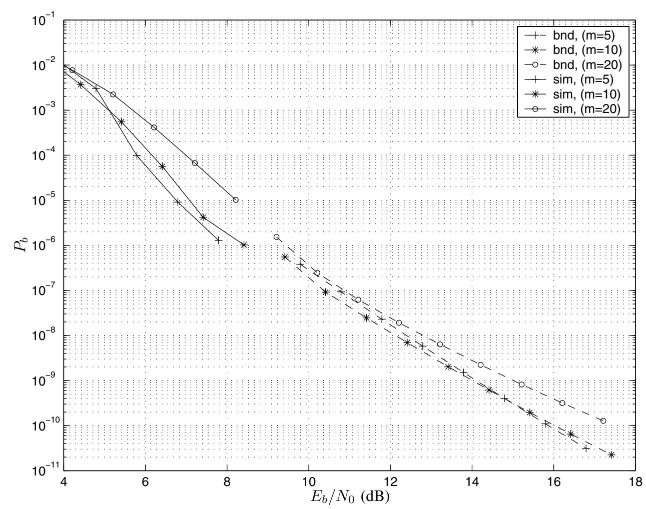

Fig. 8. Performance of the (1,5/7,5/7) turbo code with $\tilde{N} = 100$ and CDE.