A Tight Bound on the Error Probability of Space-Time Codes for Rapid Fading Channels

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Abstract- **This paper presents the performance of Space-Time (ST) codes over rapid fading channels. A tight upper bound on the pairwise error probability (PWEP) of ST codes over rapid fading channels is derived. Also, an upper bound on the bit error probability (BEP) is evaluated using the derived PWEP. The existing and new bounds are evaluated for different QPSK ST codes and compared to the simulation results. The new bound is shown to be tighter than the existing bound by almost 2 dBs and is very tight to the simulation results.**

I. INTRODUCTION

As technology advances, wireless systems are required to provide higher data rates with improved quality of service and better support for multimedia applications. All these requirements have led to new developments in wireless communications. In general, diversity and error control codes are known to improve the link quality of wireless systems [1,2]. In particular, transmit diversity can be used to increase the transmission rate. Clearly, systems combining transmit diversity and error control codes are promising to provide higher transmission rate at good quality via providing diversity in time and space.

The concept of space-time (ST) codes had appeared first in [3] as the delay diversity system, where different symbols are simultaneously transmitted via different transmit antennas. In [4], this concept was extended to transmit N encoded symbols from a trellis encoder simultaneously using N transmit antennas. The performance of ST coded systems was analyzed in [4] for rapid and quasi-static fading channels and upper bounds on the pairwise error probability (PWEP) were derived. From the derived bounds, code design criteria were established for rapid and quasi-static fading channels. A similar analysis was carried out for the case of correlated transmit branches. In [5], the performance of ST coded systems under different mobility conditions was investigated. The design criteria of ST codes for quasi-static fading channels in [4] were examined for frequency-selective fading channels. In [6], the existence of non-ideal channel state information (CSI) at the receiver was examined. The performance of ST codes in this case was also analyzed.

All the mentioned bounds appear to be loose since they are based on the Chernoff bound of the PWEP. In this paper, a tight upper bound on the PWEP of ST coded systems over rapid fading channels is derived. In addition, an expression for the bit error probability (BEP) is presented. It is based on the transfer function of the trellis encoder.

The paper starts with a general description of a typical ST coded system and the existing bound on the PWEP over rapid fading channels. Then, the tight bound is presented. The BEP is then evaluated for two QPSK ST schemes. Finally, some conclusions and suggestions for future work are presented.

II. SYSTEM MODEL

A typical system that employs ST coding using N transmit and M receive antennas is shown in Figure 1. The transmitter consists of a ST trellis encoder, serial-to-parallel converter, N modulators and a vector block interleaver. The ST encoder encodes the input bits into N symbols drawn from a signal constellation. The ST signals are interleaved using a vector block interleaver. Each element in the interleaver is a vector containing the N signals to be transmitted via the N transmit antennas. The depth and span of the interleaver depend on the channel's fading rate ($f_D T$) and the encoder's constraint length, respectively. The interleaver is used in order to break the memory of the channel so that it approaches the behavior of independent fading channels, and hence the diversity provided by the coded system is fully utilized. The ST siganls are distributed over the N transmit antennas via the serial-to-parallel converter and then modulated.

The received signal at the j^{th} receive antenna is a noisy superposition of all transmitted symbols over all transmit antennas and is given by:

$$d_t^j = \sum_{i=1}^{N} \alpha_{ij,t} c_t^i + \eta_t^j \,, \tag{1}$$

where η_t^j is an AWGN modeled as independent samples of a zero-mean complex Gaussian random process with variance $N_o/2$ per dimension. The coefficient $\alpha_{ij,t}$ is the path gain from the i^{th} transmit antenna to the j^{th} receive antenna at time t

which is a sample of a complex Gaussian random process with a variance of 1. Also, c_t^i is the transmitted symbol from the i^{th} transmit antenna at time t. At the receiver, Maximal Ratio Combining (MRC) is used to combine signals at different receive antennas and the Viterbi algorithm is employed at the decoder.

The performance of ST coded systems employing N transmit and M receive antennas is derived in [4] for rapid fading channels. Define the codeword C_l as:

$$C_l = \underline{c}_1\underline{c}_2...\underline{c}_l = c_1^1c_1^2....c_1^N c_2^1c_2^2....c_2^N....c_l^1c_l^2....c_l^N$$

Consider that it has been transmitted over l time intervals and was erroneously decoded as $\hat{C}_l$. The conditional probability of deciding $\hat{C}_l$ in favor of C_l using maximum liklihood decoding is upper bounded using the Chernoff bound [4] as:

$$P_c \leq \exp[-d_E^2(C_l,\hat{C}_l)/4N_o], \tag{2}$$

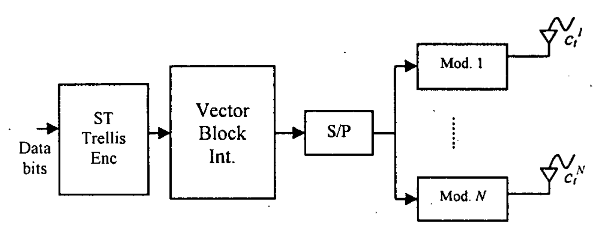
where

$$d_E^2(C_l,\hat{C}_l) = \sum_{t=1}^{l}\sum_{j=1}^{M} E_s \,|\sum_{i=1}^{N}\alpha_{ij,t}(c_t^i - \hat{c}_t^i)|^2 \tag{3}$$

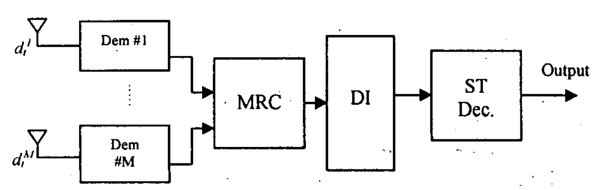
After going through the derivation in [4], the conditional probability in (2) yields the unconditional probability as:

$$P \leq \prod_{t\in\eta}[1+\sum_{i=1}^{N}|c_t^i - \hat{c}_t^i|^2 \,(E_s/4N_o)]^{-M} \tag{4}$$

Where $\eta = \{t : \underline{c}_t \neq \hat{\underline{c}}_t\}$ and $\underline{c}_t = (c_t^1c_t^2....c_t^N)$ is the codeword of ST symbols transmitted simultaneously over all transmit antennas at time t.



(a)



(b)

Figure 1: General ST system (a) Encoder, (b) Decoder.

Define the cardinality of the set η to be $L_\eta = |\eta|$. Then, $L = min\{L_\eta\}$ is the length of the shortest error path with L time intervals and can be referred to as the Space-Time Minimum Time Diversity (ST-MTD) of the code. In other words, ST-MTD is the "branch-wise" Hamming distance (HD) in conventional trellis codes, by considering the whole codeword $\underline{c}_t$ as one symbol. Since the above bound is derived from the Chernoff bound, it is expected to be loose. The proposed tight bound is derived in the following.

III. TIGHTER UPPER BOUND

Tight bounds on the BEP of trellis coded systems employing receive diversity were derived in [7] for fading environments. In this paper, the same methodology is used to derive a tight bound on the PWEP of ST coded systems. This bound is based on the exact expression of the conditional PWEP given by:

$$P_c = \frac{1}{2}erfc\left(\sqrt{\frac{d_E^2(C_l,\hat{C}_l)}{4N_o}}\right) \tag{5}$$

Define an NxN matrix, $A_t = \underline{C}_t \cdot \underline{C}_t^{T*}$, where $\underline{C}_t = [(c_t^1 - \hat{c}_t^1)(c_t^2 - \hat{c}_t^2)...(c_t^N - \hat{c}_t^N)]^T$ and $\underline{C}_t^{T*}$ is the conjugate transpose of $\underline{C}_t$. The A_t matrix contains the difference term between the correct and error codewords in (3). Then, the distance expression in (3) can be represented in matrix form [4] as:

$$d_E^2(C_l,\hat{C}_l) = E_s\sum_{t=1}^{l}\sum_{j=1}^{M} \Omega_{j,t}\, A_t\, \Omega_{j,t}^{\,T} \tag{6}$$

Where $\Omega_{j,t} = [\alpha_{1j,t}\ \alpha_{2j,t}\ ....\ \alpha_{Nj,t}]$ represents the fading coefficients at the j^{th} receive antenna. The expression in (6) can be further simplified using similarity transformation of the matrix A_t. It is expressed as a product of a diagonal matrix D_t, whose diagonals are the eigenvalues of A_t, and a unitary matrix containing the set of orthonormal eigenvectors of the matrix A_t: i.e.,

$$V_t^{T*}A_tV_t = D_t, \tag{7}$$

where V_t is a matrix whose columns are the N eigenvectors of the matrix A_t. Defining $\lambda_{i,t}$ as the i^{th} eigenvalue of A_t, and

$$B_{j,t} = [\beta_{1j,t}\ \beta_{2j,t}\ ......\beta_{Nj,t}] = \Omega_{j,t}\cdot V_t$$
$$= [\alpha_{1j,t}\ \alpha_{2j,t}\ ......\ \alpha_{Nj,t}] \times [v_{1,t}\ v_{2,t}\ .....\ v_{N,t}], \tag{8}$$

then, the conditional PWEP in (5) is simplified to:

$$P_c = \frac{1}{2}erfc\left(\sqrt{\frac{E_s}{4N_o}\sum_{t=1}^{l}\sum_{j=1}^{M}\sum_{i=1}^{N}\lambda_{i,t}\,|\beta_{ij,t}|^2}\right) \tag{9}$$

Define the variables $d_{i,t}$ and $\gamma_{i,t}$ as:

$$d_{i,t} = \lambda_{i,t}(E_s/4N_o), \quad \text{and} \quad \gamma_{i,t} = \sum_{j=1}^{M}|\beta_{ij,t}|^2,$$

Then the expression in (9) simplifies to:

$$P_c = \frac{1}{2} erfc\left(\sqrt{\sum_{t=1}^{l}\sum_{i=1}^{N} d_{i,t}\gamma_{i,t}}\right) \quad (10)$$

Since all the eigenvalues $\lambda_{i,t}$'s are zero except one [4], then the nonzero $d_{i,t}$ is replaced by:

$$|\underline{c}_t - \hat{\underline{c}}_t|^2 (E_s/4N_o) = \sum_{i=1}^{N} |c_t^i - \hat{c}_t^i|^2 (E_s/4N_o), \quad (11)$$

So, the inner summation in (10) reduces to $d_t\Gamma_t$, where the quantity $d_t = |\underline{c}_t - \hat{\underline{c}}_t|^2 (E_s/4N_o)$ and $\Gamma_t=\{\gamma_{i,t}: \lambda_{i,t} \neq 0\}$. Hence, the PWEP becomes:

$$P_c = \frac{1}{2} erfc\left(\sqrt{\sum_{t=1}^{l} d_t\Gamma_t}\right) \quad (12)$$

Since $|\beta_{ij,t}|$'s constitute a form of sum of the N $\alpha_{ij,t}$'s at the j^{th} receive antenna as appears in (8), then they have Rayleigh distribution with mean square $E[|\beta_{ij,t}|^2]=1$. Hence, $|\beta_{ij,t}|^2$'s are exponentially distributed with unity mean. Therefore, the variables Γ_t's, which are sums of exponentially distributed random variables, will have an M-Erlang distribution with parameter one [8]. The probability density function (*pdf*) of this distribution is written [8] as:

$$f_\Gamma(\Gamma_t) = \frac{1}{(M-1)!}\Gamma_t^{(M-1)} e^{-\Gamma_t}, \ \Gamma_t \geq 0. \quad (13)$$

The PWEP in (12) is conditional on the fading coefficients. To find the unconditional probability, it is averaged with respect to these coefficients, resulting in:

$$P = \frac{1}{2}\int_0^\infty \cdots \int_0^\infty erfc\left(\sqrt{\sum_{t=1}^{l}\Gamma_t d_t}\right) f_\Gamma(\Gamma_1)\ldots f_\Gamma(\Gamma_l)\, d_{\Gamma_1}\ldots d_{\Gamma_l} \quad (14)$$

Defining $\delta_t=d_t/(1+d_t)$ and $\omega_t=\Gamma_t(1+d_t)$, and using *pdf* transformation formula [8], then the unconditional PWEP is written as:

$$P = \prod_{t\in\eta}\frac{1}{(1+d_t)^M}\int_0^\infty \cdots \int_0^\infty erfc\left(\sqrt{\sum_{t=1}^{l}\delta_t\omega_t}\right)$$
$$\times \exp\left(\sum_{t=1}^{l}\delta_t\omega_t\right) f_\omega(\omega_1)\ldots f_\omega(\omega_l)\, d_{\omega_1}\ldots d_{\omega_l} \quad (15)$$

Where $\eta = \{t : \underline{c}_t \neq \hat{\underline{c}}_t\}$. Defining $L_\eta = |\eta|$ and $\delta_m = \min\{\delta_t: t\in\eta\}$, then it is clear that:

$$\sum_{t=1}^{l}\delta_t\omega_t \geq \delta_m \sum_{t=1}^{l}\omega_t \quad (16)$$

Since the function $erfc(x)\exp(x^2)$ is a monotonically decreasing function for $x \geq 0$, then the PWEP is upper bounded by:

$$P \leq \frac{1}{2}\prod_{t\in\eta}\frac{1}{(1+d_t)^M}\int_0^\infty erfc\left(\sqrt{\delta_m\Omega}\right) e^{(\delta_m\Omega)} f_\Omega(\Omega)\, d\Omega \quad (17)$$

Where $\Omega = \sum_{t\in\eta}\omega_t$. Since each of the ω_t's. The variable Ω is a sum of L_η M-Erlang random variables with parameter one, and hence its distribution is an ML_η-Erlang distribution with parameter equal to one. Substituting in the PWEP expression:

$$P \leq \frac{1}{2(ML_\eta - 1)!}\prod_{t\in\eta}\frac{1}{(1+d_t)^M}$$
$$\times \int_0^\infty erfc\left(\sqrt{\delta_m\Omega}\right) e^{(\delta_m\Omega)}\Omega^{(ML_\eta-1)} e^{-\Omega} d\Omega \quad (18)$$

The integral in (18) is evaluated using the following equality [9]:

$$\frac{1}{2(K-1)!}\int_0^\infty erfc\left(\sqrt{xy}\right) e^{-y(1-x)} y^{(K-1)} dy$$
$$= \frac{1}{2^{2K}}\sum_{j=1}^{K}\binom{2K-j-1}{K-1}\left(\frac{2}{1+\sqrt{x}}\right)^j \quad (19)$$

Then, the PWEP is evaluated to be:

$$P \leq \frac{1}{2^{2ML_\eta}}\sum_{j=1}^{ML_\eta}\binom{2ML_\eta - j - 1}{ML_\eta - 1}\left(\frac{2}{1+\sqrt{\delta_m}}\right)^j \times \prod_{t\in\eta}\frac{1}{(1+d_t)^M} \quad (20)$$

Since L is the length of the shortest error path, and since $L \leq L_\eta$, then the PWEP can be written as:

$$P \leq \underbrace{\frac{1}{2^{2ML}}\sum_{j=1}^{ML}\binom{2ML - j - 1}{ML - 1}\left(\frac{2}{1+\sqrt{\delta_m}}\right)^j}\times\underbrace{\prod_{t\in\eta}\frac{1}{(1+d_t)^M}} \quad (21)$$

The new bound consists of two terms: the first one is the tightening constant while the second one is the same as in Equation (4). The tightening constant is a function of the number of receive antennas, the ST-MTD of the ST code and the SNR of the channel. It can be easily shown that the constant is always less than one, proving the tightness of the new bound.

The BEP is derived using the modified transfer function approach [9] as:

$$P_b \leq \frac{1}{2k}\frac{\partial T(I,D)}{\partial I}\Big|_{I=1} \quad (22)$$

Where $T(I,D)$ is the modified transfer function of the ST code and k is the number of input bits to the ST encoder. Finally, the BEP is evaluated as:

$$P_b \leq \left[\frac{1}{k.2^{2ML}}\sum_{j=1}^{ML}\binom{2ML - j - 1}{ML - 1}\left(\frac{2}{1+\sqrt{\delta_m}}\right)^j\right] \times \frac{\partial T(I,\bar{D})}{\partial I}\Big|_{\substack{D=e^{(-E_s/4N_o)} \\ I=1}} \quad (23)$$

Two examples are considered to test the tightness of the bounds. In Figure 2, the existing and the new bounds along with simulation results of the 4-state QPSK ST code in [4] are plotted. The same information regarding the QPSK ST code in [10] is shown in Figure 3. It can be seen that the existing bound is loose compared to the simulation results while the new bound is very tight. Also, the new bound is tighter than the existing one by almost 2 dB. The bound has also been evaluated for other codes and has consistently shown tightness to simulation results.

IV. Conclusions & Future Work

A tight bound on the PWEP of ST coded systems over rapid fading channels was derived. The corresponding BEP was evaluated from the existing and new PWEP bounds for different ST codes in the literature. Results showed that the new bound is very tight to the simulation curves. Future work concentrates on deriving tight bounds for the cases of correlated transmit branches as well as for correlated fading channels.

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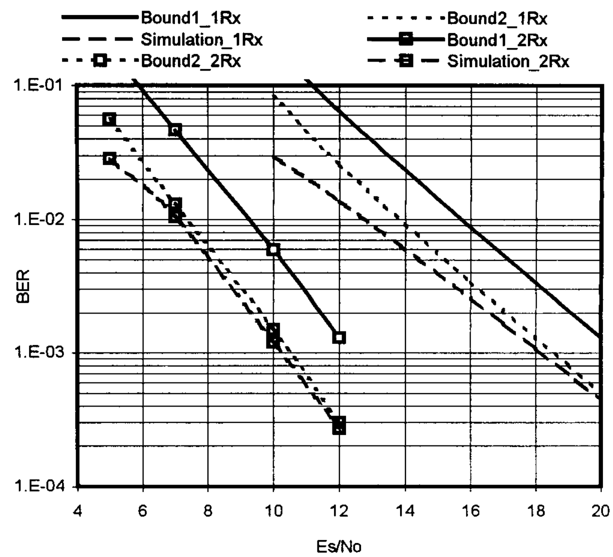

Figure 2: Performance of the 4-state QPSK ST code designed in [4]. Bound1: existing, Bound2: new, Rx: # of receive antennas.

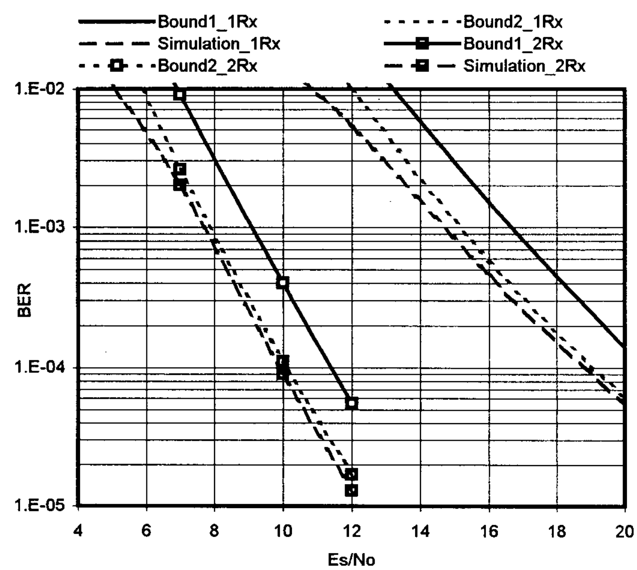

Figure 3: Performance of the 4-state QPSK ST code in [10] for one and two receive antennas.