Spectral Efficiency of Maximum Ratio Combining: Slow Fading

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***Abstract*— In this paper we derive closed-form expressions for the single-user capacity of Maximum Ratio Combiner (MRC) system system, taking into account the effect of imperfect channel estimation at the receiver. The channel considered is a slowly varying spatially independent flat Rayleigh fading channel. The complex channel estimate and the actual channel are modelled as jointly Gaussian random variables with a correlation that depends on the estimation quality. Two adaptive transmission schemes are analyzed: 1) optimal power and rate adaptation; and 2) constant power with optimal rate adaptation. Our numerical results show the effect of Gaussian channel estimation error on the achievable spectral efficiency.**

I. INTRODUCTION

It is widely accepted that using diversity at the transmitter or at the receiver of a wireless communication system can improve significantly the performance of wireless links. Diversity combining, which skillfully combines multiple replicas of received signals has long been as one of the most efficient techniques to overcome the destructive effects of multipath fading in wireless communication systems. There are several diversity combining methods employed in communication receivers including maximal ratio combining (MRC), equal gain combining (EGC), selection combining (SC), and a combination of MRC and SC, called generalized selection combining (GSC). By definition, MRC combiner linearly combines the individually received branch signals so as to maximize the instantaneous output signal-to- noise ratio (SNR) [1–3].

Most system designs assume that perfect channel estimation is available at the receiver. In practise, however, the channel gains have to be estimated at the receiver for diversity combining which can be obtained either from a pilot signal or data signals (by applying a clairvoyant estimator). The work in [4] analyzed the performance of MRC with pilot tone-based weighting on frequency-selective Rayleigh fading channels. The pilot tone was assumed to be separated from the data signal and the resulting channel-estimation error was shown be to be Gaussian. Previous work on the analysis of imperfect channel estimation with no diversity can be found in [5] and [6]. In [7], Gans modelled the channel estimation errors as complex Gaussian and derived the distribution of the SNR statistics which has been used by Tomiuk in [8] to obtain the average probability of error for the MRC diversity schemes.

The pioneering work of Shannon [9] has established the significance of channel capacity as the maximum possible rate at which information can be transmitted over a channel. In [12], the capacity of a single user flat fading channel with perfect channel information at the transmitter and the receiver is derived for various adaptation policies, namely, 1) optimal rate and power adaptation (*opra*), 2) optimal rate adaptation and constant power (*ora*), and 3) channel inversion with fixed rate (*cifr*).

The first scheme requires channel information at the transmitter and receiver, whereas the second scheme is more practical since the transmission power remains constant. The last scheme is a suboptimal transmission adaptation scheme, in which the channel side information is used to maintain a constant received power by inverting the channel fading [12]. In [13], the general theory developed in [12] was applied to achieve closed-form expressions for the capacity of Rayleigh fading channel under different adaptive transmission and diversity combining techniques. In [16], the channel capacity of adaptive transmission with MRC in correlated fading was derived. Two kinds of correlations were considered including equal branch SNR and between any pair of branches, and unequal branch SNRs and arbitrary correlation between branches such that the eigenvalues of the branch covariance matrix are distinct. In [17], the capacity of MRC over generalized Rician fading channels was discussed.

In this paper, we extend the results in [13] to obtain closed-form expressions for the single-user capacity of MRC system, in the presence of Gaussian channel estimation errors. The contribution of this paper is to derive closed-form expressions for two adaptive transmission schemes including their asymptotic approximations and upper bounds and these schemes are:
(1) optimal simultaneous power and rate adaptation (*opra*).
(2) optimal rate adaptation with constant transmit power (*ora*).

The paper is organized as follows. In Section II, the system model used in this paper is discussed. In Section III, we derive closed-form expressions for the channel capacity under two adaptation schemes; *opra* and *ora* including their asymptotic approximations and upper bounds in sub-sections III-A and III-B, respectively. Results are presented and discussed in Section IV. The main outcomes of the paper are summarized in Section V.

II. SYSTEM MODEL

Consider an L-branch diversity receiver in slow fading channels. Assuming perfect timing and inter-symbol interference (ISI) free transmission, then, the received signal on the lth branch r_l at ith symbol interval can be expressed as

$$r_l = g_l s_i + n_l \quad l = 1 \ldots L \tag{1}$$

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where g_l is the zero mean complex Gaussian distributed channel gain,n_l is the complex additive white Gaussian noise with the variance $2N_0$, and s_i is the date symbol taken from normalized unit energy E_b set $\boldsymbol{S}$ at the ith symbol interval. The actual channel gains of L diversity branches are independent and identical distributed. We assume the noise per branch is identical for all branches. The pdf of the output of MRC combiner SNR γ with estimation errors has been derived by Gans [7] and re-arranged in a simple pdf form [8], and is given by

$$p_\gamma(\gamma) = \sum_{k=1}^{L} \binom{L-1}{k-1} (\rho^2)^{k-1}(1-\rho^2)^{L-k} \left(\frac{1}{\gamma_t}\right)^k \gamma^{k-1} e^{\frac{-\gamma}{\gamma_t}} = \sum_{k=1}^{L} \frac{W_{k-1}^{L-1}(\rho^2)}{\Gamma(k)} \gamma^{k-1} e^{\frac{-\gamma}{\gamma_t}} \quad (2)$$

where $W_{k-1}^{L-1}(\rho) = \binom{L-1}{k-1}(\rho^2)^{k-1}(1-\rho^2)^{L-k}$ is representing the weighting coefficients in the sum, and γ_t is the average SNR per bit per branch ($\gamma_t = \frac{E_b}{N_0}$) and ρ denotes the correlation between the actual channel coefficients g_l and their estimates $\widehat{g}_l$ (i.i.d). The actual channel gain can be related to the channel estimate by

$$g_l = \rho \widehat{g}_l + z_l \quad (3)$$

where ρ is a complex number representing the normalized correlation between g_l and $\widehat{g}_l$, and z_l is a complex Gaussian Random Variable (R.V) independent of $\widehat{g}_l$ with zero mean and variance σ_z^2, it defined as :

$$\rho^2 = \left|\mathbb{E}[g_l \widehat{g}_l^*]\right|^2 \quad (4)$$

where $\mathbb{E}[.]$ stands for mathematical expectation. In a system with no estimation errors $\rho = 1$, (the summation of $W_{k-1}^{L-1}(\rho^2) = 1$), the distribution of γ in (2) reduces to

$$f_\gamma(\gamma) = \left(\frac{1}{\gamma_t}\right)^L \frac{\gamma^{L-1}}{\Gamma(L)} e^{\frac{-\gamma}{\gamma_t}} \quad (5)$$

III. ADAPTIVE CAPACITY POLICIES

In this section, we derive close-form expressions for different adaptive schemes with MRC over Rayleigh fading channels. In the derivation, we will rely on the main results from [13].

A. Optimal Adaptation At The Transmitter

Given an average transmit power constraint, the channel capacity C_{opra} in (bits/seconds) of a fading channel is given by [12, 13]

$$C_{opra} = \frac{B}{\ln 2} \int_{\gamma_0}^{\infty} \ln\left(\frac{\gamma}{\gamma_0}\right) p_\gamma(\gamma) d\gamma \quad (6)$$

where B (in hertz) is the channel bandwidth and γ_0 is the optimum cutoff SNR satisfying [12]

$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p_\gamma(\gamma) d\gamma = 1 \quad (7)$$

However, C_{opra} in (6) can be expressed in terms of cumulative distribution function (CDF) by applying integration by-parts resulting in

$$\frac{C_{opra}\ln(2)}{B} = -\int_{\gamma_0}^{\infty} \frac{1}{\gamma} F(\gamma) d\gamma \quad (8)$$

Substituting (2) into (7) yields the equlity

$$\sum_{k=1}^{L} \frac{W_{k-1}^{L-1}(\rho^2)}{\Gamma(k)} \left(\frac{1}{\gamma_t}\right)^k \left[\int_{\gamma_0}^{\infty} \frac{\gamma^{k-1}}{\gamma_0} e^{\frac{-\gamma}{\gamma_t}} - \int_{\gamma_0}^{\infty} \gamma^{k-2} e^{\frac{-\gamma}{\gamma_t}}\right] d\gamma = 1 \quad (9)$$

We evaluate the integrals in (9) by making the use of the following equality [18]

$$\Gamma(n, x) = \int_x^{\infty} s^{n-1} e^{-s} ds \quad (10)$$

where $\Gamma(.,.)$ is the incomplete gamma function, Upon substitution of (10) into (9), it is found that the optimal cutoff SNR, γ_0 has satisfy the following equality

$$\sum_{k=1}^{L} W_{k-1}^{L-1}(\rho^2) \left[\frac{\gamma_t}{\gamma_0} \Gamma\left(k, \frac{\gamma_0}{\gamma_t}\right) - \Gamma\left(k-1, \frac{\gamma_0}{\gamma_t}\right)\right] = \Gamma(k)\gamma_t^k$$

To obtain the optimal cutoff SNR γ_0 in (9), we follow the following procedure. Let $x = \frac{\gamma_0}{\gamma_t}$ and define the $f_{MRC}(x)$ as

$$f_{MRC} = W_{k-1}^{L-1}(\rho^2) \frac{\Gamma\left(k, x\right)}{x} - W_{k-1}^{L-1}(\rho^2) \Gamma\left(k-1, x\right) - \Gamma(k)\gamma_t^k \quad (11)$$

By differentiating f_{MRC} with respect to x over the interval $]0, +\infty[$, gives

$$f'_{MRC} = -W_{k-1}^{L-1}(\rho^2)\Gamma\left(k, x\right)/x^2 \quad (12)$$

Hence, $f'_{MRC}(x) < 0$ for $\forall\ x > 0$, meaning that f'_{MRC} is strictly decreasing function of x. Observe that, from (11) it can be observed that $\lim_{x\to 0} f_{MRC} = +\infty$ and $\lim_{x\to 0} f_{MRC} = -\Gamma(k)\gamma_t^k$ Note that, f_{MRC} is continues function of x, which leads to a unique positive γ_0 such that $f_{MRC}(x) = 0$. We thereby conclude that for *each* $\gamma_t > 0$ *there* unique γ_0 satisfying (11). Numerical results using MATLAB shows that $\gamma_o \in [0, 1]$ as γ_t increases, and $\gamma_o \to 1$ as $\gamma_t \to \infty$.

Now, inserting (2) into (6) yields the channel capacity with *opra* scheme as follows

$$\frac{C_{opra}}{B} = \int_{\gamma_0}^{\infty} \sum_{k=1}^{L} W_{k-1}^{L-1}(\rho^2) \ln\left(\frac{\gamma}{\gamma_0}\right) \frac{\gamma^{k-1}}{\Gamma(k)\gamma_t^k} e^{-\frac{\gamma}{\gamma_t}} d\gamma \quad (13)$$

The summation in (13) is of finite order, thus, the order of summation and integral can be inverted, it yields

$$\frac{C_{opera}}{B} = \sum_{k=1}^{L} W_{k-1}^{L-1}(\rho^2) \underbrace{\int_{\gamma_0}^{\infty} \ln\left(\frac{\gamma}{\gamma_0}\right) \gamma_t \frac{\left(\gamma_t\gamma\right)^{k-1}}{\Gamma(k)} e^{-\frac{\gamma}{\gamma_t}} d\gamma}_{I_1} \quad (14)$$

The integral I_1 can be evaluated using the fact from [13] which states the following

$$J_s(\mu) = \int_1^{\infty} t^{s-1}\ln(t)e^{-\mu t}dt = \frac{\Gamma(s)}{\mu^s}\left\{E_1(\mu)+\sum_{k=1}^{s-1}\frac{1}{k}P_k(\mu)\right\} \quad (15)$$

where E_1 denotes *exponential integral* of the first order which is given by [18]

$$E_1(x) = \int_1^{\infty}\frac{e^{x\gamma}}{\gamma}d\gamma \quad x \geq 0 \quad (16)$$

and $P_k(\mu)$ denotes the *poisson distribution* [18]

$$P_k(x) = \frac{\Gamma(k,x)}{\Gamma(k)} = e^{-x}\sum_{i=0}^{k-1}\frac{x^n}{n!} \quad (17)$$

Upon substituting (15) into (14) implies that the capacity C_{opra} per unit bandwidth (in bits/seconds/hertz) can be written as:

$$\frac{C_{opra}}{B} = \left[E_1\left(\frac{\gamma_0}{\gamma_t}\right)+\sum_{k=1}^{L}\left\{\binom{L-1}{k-1}(\rho^2)^{k-1}(1-\rho^2)^{L-k} \times\sum_{i=1}^{k-1}\frac{P_i\left(\frac{\gamma_0}{\gamma_t}\right)}{i}\right]\right\} \quad (18)$$

1) Asymptotic Approximation: : We can obtain asymptotic approximation C_{opra} using the series representation of E_1 in [18] which is given by

$$E_1(x) = -E-\ln(x)-\sum_{i=1}^{+\infty}\frac{(-x)^i}{i.i!} \quad (19)$$

where $E = 0.5772156659$ is Euler-Mascheroni constant. Thus, the asymptotic approximation C_{opra}^{∞} per unit bandwidth (in bits/seconds/hertz) is expressed as:

$$\frac{C_{opra}^{\infty}}{B} \simeq \left[\left(-E-\ln\left(\frac{1}{\gamma_t}\right)+\frac{\gamma_0}{\gamma_t}\right)+ \sum_{k=1}^{L}\left\{\binom{L-1}{k-1}(\rho^2)^{k-1}(1-\rho^2)^{L-k} \times\sum_{i=1}^{k-1}\frac{P_i\left(\frac{\gamma_0}{\gamma_t}\right)}{i}\right]\right\} \quad (20)$$

2) Opra Upper Bound: The capacity expression C_{opra} can be upper bounded by applying Jensen's inequality to (6) as follows:

$$C_{OPRA}^{UP}\ln\left(\frac{E\{\gamma\}}{\gamma_0}\right) \quad (21)$$

The expression in (21) can be evaluated by averaging it over the PDF in (2) and making the help the identity [18]

$$\int_0^{\infty} x^n e^{-\mu x}dx = n!\mu^{-n-1} \quad (22)$$

for Re$[\mu] > 0$.The resulting expression can be further simplified to obtain the upper bound for C_{opera} as follows

$$\frac{C_{opra}^{UB}}{B} = \ln\left(\sum_{k=0}^{L-1}\frac{1}{\gamma_t}\binom{L-1}{k}(\rho^2)^{k-1}(1-\rho^2)^{L-k}\right) \quad (23)$$

B. Constant Transmit Power

By adapting the code rate to channel fading with a constant power, the channel capacity C_{ora} is given by [9, 10]

$$C_{ora} = \frac{B}{\ln 2} = \int_0^{\infty}\ln\big(1+\gamma\big)p_{\gamma}(\gamma)d\gamma \quad (24)$$

Inserting (2) into (24), it yields

$$C_{ora} = \sum_{k=1}^{L}W_{k-1}^{L-1}(\rho^2)\underbrace{\int_0^{\infty}\ln\big(1+\gamma\big)\frac{\gamma^{k-1}}{\Gamma(k)\gamma_t^k}e^{\frac{-\gamma}{\gamma_t}}d\gamma}_{I_2} \quad (25)$$

The integral I_2 in (25) has been evaluated in terms of *Poisson* distribution in closed form as stated in [13]- [14]

$$I_2 = P_k\left(\frac{-1}{\gamma_t}\right)E_1\left(\frac{1}{\gamma_t}\right)+\sum_{i=1}^{k-1}\frac{P_i\left(\frac{-1}{\gamma_t}\right)P_{k-i}\left(\frac{-1}{\gamma_t}\right)}{i} \quad (26)$$

Substituting (26) into (25) implies that the capacity C_{ora} per unit bandwidth (in bits/seconds/hertz) can be expressed as:

$$\frac{C_{ora}}{B} = \sum_{k=1}^{L}\left\{\binom{L-1}{k-1}(\rho^2)^{k-1}(1-\rho^2)^{L-k} \times\left[P_k\left(\frac{-1}{\gamma_t}\right)E_1\left(\frac{1}{\gamma_t}\right)+\sum_{i=1}^{k-1}\frac{P_i\left(\frac{-1}{\gamma_t}\right)P_{k-i}\left(\frac{-1}{\gamma_t}\right)}{i}\right]\right\} \quad (27)$$

1) Asymptotic Approximation: : Following the same procedure in III-A, the approximated C_{ora}^{∞} per unit bandwidth (in bits/seconds/hertz) can be obtained as

$$\frac{C_{ora}^{\infty}}{B} = \sum_{k=1}^{L}\left\{\binom{L-1}{k-1}(\rho^2)^{k-1}(1-\rho^2)^{L-k} \times\left[P_k\left(\frac{-1}{\gamma_t}\right)\left[-E-\ln\left(\frac{1}{\gamma_t}\right)+\frac{1}{\gamma_t}\right] +\sum_{i=1}^{k-1}\frac{P_i\left(\frac{-1}{\gamma_t}\right)P_{k-i}\left(\frac{-1}{\gamma_t}\right)}{i}\right]\right\} \quad (28)$$

2) Ora Upper Bound: The capacity expression C_{opra} can be upper bounded by applying Jensen's inequality to (6) as follows:

$$\frac{C_{opra}^{UB}}{B} = \ln\left(1+\sum_{k=0}^{L-1}\frac{1}{\gamma_t}\binom{L-1}{k}(\rho^2)^{k-1}(1-\rho^2)^{L-k}\right) \quad (29)$$

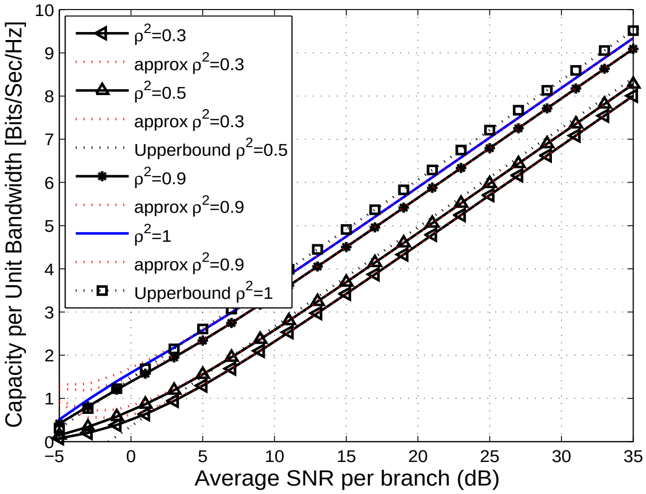

Fig. 1. Capacity per unit bandwidth for a Rayleigh fading with SCD diversity (L=3) and various values of different ρ^2 under power and rate adaptation

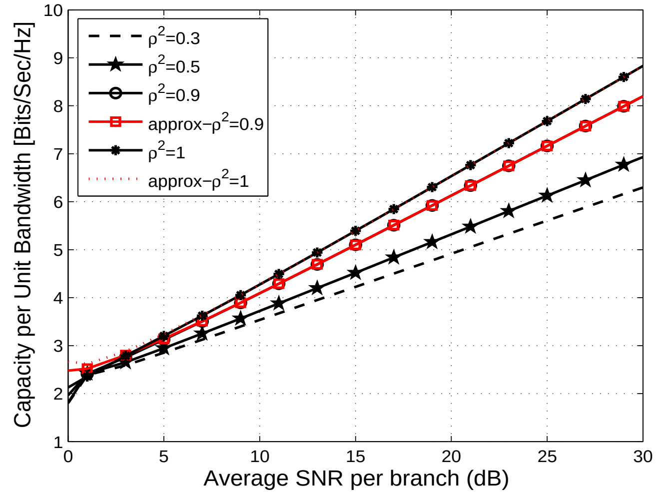

Fig. 2. Capacity per unit bandwidth for a Rayleigh fading with MRC diversity (L=3) and various values of different ρ^2 under constant power and rate adaptation

IV. Numerical Results

In this section we provide some numerical results that illustrate the mathematical derivation of the channel capacity per unit bandwidth as a function of average receiver SNR (γ_t) in dB for two different adaptation policies with MRC over slow Rayleigh fading with weight estimation errors. All curves provided are obtained using the closed-form expressions (18), (20), (23) ,(27), (28), and (29).

From the expression in (18), it can be observed that the capacity increases along with increase of the diversity order L at the receiver and increase of the average of received SNR per branch γ_t. Figure (1) compares C_{opra} for different values of correlation between the channel and its estimate; namely, $\rho^2 = 0.3$, $\rho^2 = 0.5$, $\rho^2 = 0.7$, $\rho^2 = 0.9$ and $\rho^2 = 1$. It can be noticed that the highest C_{opra} that can be achieved when $\rho^2 = 1$. Furthermore, C_{opra} decreases when the value of ρ^2 decreases where in this case the weight error increases. It can be observed from Figure (18) that there is almost a 7 dB difference in C_{opra} between $\rho^2 = 1$ and $\rho^2 = 0.3$.

Furthermore, the same figure depicts both the asymptotic approximated capacity expressed in (20) and the upper bound expressed in (23). As can be shown that the upper bound gives a tight approximation of the exact average capacity C_{opra}.
In Figure 2, the exact, asymptotic, and upper bound of the average capacity C_{ora} are plotted against γ_t for different values of ρ^2 $\{0.3, 0.5, 0.9 \text{ and } 1\}$ when $L = 3$. As it can be observed from Figure 2 that the difference in the capacity of ora between $\rho^2 = 1$ and $\rho^2 = 0.3$ is increasing along with increase of the average of received SNR per branch γ_t which makes it more sensitive to the estimation errors than $opra$ policy.

V. Conclusion

The channel capacity for unit bandwidth for two adaptive schemes including their approximations and upper bounds over a slow Rayleigh fading channel for MRC with estimation error is discussed. Closed-form expressions including the exact, asymptotic, and upper bound of the average capacity for $opra$ and ora policies are derived for L-selection combiner. Our numerical results showed that for the same bandwidth, the capacity increases with increase of the diversity order L and increase of of the average γ_t per branch. Also, simulation showed that *opra* outperforms *ora*, and less sensitive to the estimation error.

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