Throughput of ARQ Protocols Over Nakagami and MIMO Block Fading Channels

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Abstract

Automatic-repeat request (ARQ) protocols are used to provide reliable communication in wireless networks. In this paper the throughput of the basic selective-repeat (SR) ARQ in block fading environments is derived. Single-antenna ARQ systems employing coherent BPSK and non-coherent BFSK are analyzed over block fading channels characterized by Nakagami fading distribution. Moreover, the performance of multi-input multi-output (MIMO) ARQ systems employing space-time block codes (STBCs) is derived. The effect of antenna correlation in MIMO ARQ systems is investigated analytically. Results show that longer channel memory lengths improve the performance of basic ARQ protocols. As the fading severity increases, the performance improvement resulting from increasing the channel memory length increases.

1. Introduction

A serious challenge to having good communication quality in wireless networks is the time-varying multipath fading environments, which causes the received signal-to-noise ratio (SNR) to vary randomly. The fading distribution varies according to the environment. A popular and general model for the fading process is the Nakagami distribution [8], which provides a family of distributions that fit measurements under different propagation environments [1].

Error-free communication is often accomplished using ARQ techniques. Basic ARQ protocols are based on error detection and retransmission [6]. If a received packet is detected in error at the receiver, a retransmission of the same packet is initiated through a feedback channel [6]. The performance of an ARQ protocol is characterized by its *throughput*, which is defined as the number of packets successfully transmitted per a channel use. The selective repeat (SR) ARQ is known as the best among the basic ARQ protocols [6]. In SR ARQ the sender retransmits only the negatively acknowledged packets.

The throughput of an ARQ protocol is a function of the fading statistics affecting transmitted packets. In practice, channel models that exhibit memory are often used to model wireless systems. Of a particular interest, the *block fading* channel [7] provides an acceptable model for many wireless communication systems such as frequency-hopped spread-spectrum (FH-SS) and time-division multiple access (TDMA). In this model a packet undergoes several statistically independent propagation channels, where each block of symbols undergo the same fading.

Most research efforts have focused on the analysis of hybrid ARQ protocols where channel coding is used [3, 4, 6]. Particularly, in [4] the throughput of basic ARQ protocols over slow Rayleigh fading was derived. However, basic ARQ protocols are used to insure correct delivery of data packets in many systems such a Bluetooth. In some packet formats used in Bluetooth, data is transmitted without using channel coding [12]. Thus it is of interest to analyze the throughput of uncoded transmission (i.e., basic ARQ) in wireless networks. In this paper, we analyze the throughput of basic SR ARQ in block fading environments with Nakagami fading distribution. Moreover, multiple-antenna ARQ systems employing space-time block codes (STBCs) [2] are considered. Furthermore, the effect of antenna correlation on the performance of SR ARQ is studied analytically.

The outline of the paper is as follows. The SR ARQ system model is described in the next section. In Section 3, the throughput of basic SR ARQ employing single and multiple antennas is derived for single and multiple antenna systems, and results are discussed therein. Conclusions are presented in Section 4.

2. System Model

2.1. Single-Antenna System

In the basic SR ARQ, the sender retransmits the packet if it contains some errors. A packet is composed of kN bits, where each k bits are mapped onto one symbol of an M-ary signal constellation. Thus each packet contains N symbols. In this paper we consider coherent BPSK and noncoherent BFSK. The channel affecting each packet is a block fading. In this model each packet undergoes F independent fading realizations, where each block of $m = \lceil \frac{N}{F} \rceil$ symbols are affected by the same fading realization.

In coherent receivers the matched filter sampled output at time l in the f^{th} fading block is given by

$$y_{f,l} = \sqrt{E_s} h_f s_{f,l} + z_{f,l}, \qquad (1)$$

where E_s is the average received energy, $s_{f,l}$ is the transmitted signal and $z_{f,l}$ is a noise modeled as an AWGN with a $\mathcal{CN}(0, N_0)$ distribution. The coefficient h_f is the channel gain in fading block f given by $h_f = a_f \exp(j\theta_f)$, where θ_f is uniformly distributed phase and a_f is the amplitude with a Nakagami distribution. The coherent receiver chooses the signal s that maximizes the metric $\{y^s_{f,l}\}$.

In a noncoherent BFSK receiver, a square-law combining [14] is employed, whose outputs are represented by

$$r^{(I,c)}_{f,l} = \sqrt{E_s} a_f \delta(c_{f,l}, c) \cos(\theta_f) + \eta^{(I,c)}_{f,l}$$
$$r^{(Q,c)}_{f,l} = \sqrt{E_s} a_f \delta(c_{f,l}, c) \sin(\theta_f) + \eta^{(Q,c)}_{f,l}, \qquad (2)$$

where $r^{(I,c)}_{f,l}$ and $r^{(Q,c)}_{f,l}$ are defined respectively as the correlation of the received signal with the inphase and quadrature dimensions of the signal corresponding to the bit $c = 0, 1$. In (2), $\delta(x, y) = 1$ if $x = y$ and $\delta(x, y) = 0$ otherwise; and $\eta^{(I,0)}_{f,l}$, $\eta^{(Q,0)}_{f,l}$ $\eta^{(I,1)}_{f,l}$ and $\eta^{(Q,1)}_{f,l}$ are independent random variables with a $\mathcal{N}(0, \frac{N_0}{2})$ distribution. The receiver chooses the bit c with the maximum $|r^{(I,c)}_{f,l}|^2 + |r^{(Q,c)}_{f,l}|^2$.

2.2. MIMO System

In transmitters employing STBCs [2,13], every n_t QPSK signals are mapped into a $n_t \times n_t$ transmission matrix $\mathcal{G}$. For $n_t = 2$ the transmission matrix is

$$\mathcal{G} = \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}, \qquad (3)$$

where $(.)^*$ denotes the complex conjugate operation. For $n_t = 2$, antennas 1 and 2 transmit the signals $s_1, -s_2^*$ and s_2, s_1^*, respectively during two symbol durations. In general, the i^{th} column of $\mathcal{G}$ is transmitted over the i^{th} transmit antenna during a *time slot*, which consists of n_t symbol durations.

Let $\mathcal{G}_{f,l}$ be the transmission matrix in the l^{th} time slot of fading block f, the corresponding received vector is given by

$$\mathbf{y}_{f,l} = \sqrt{E_s} \mathcal{G}_{f,l} \mathbf{h}_f + \mathbf{z}_{f,l}, \qquad (4)$$

where $\mathbf{z}_{f,l}$ is a length-n_t column random vector with a distribution $\mathcal{CN}(\mathbf{0}, N_0\mathbf{I})$ and $\mathbf{I}$ denotes the $n_t \times n_t$ identity matrix. The vector $\mathbf{h}_f$ is the channel gain in fading block f and is modeled as $\mathcal{CN}(\mathbf{0}, C_\mathbf{h})$, where $C_\mathbf{h}$ is the covariance matrix of the channel. In order to enable simple detection, the fading process should remain constant for at least one time slot. This constrains the channel memory length to be a multiple of n_t. A simple receiver for STBC was proposed in [2], whose outputs for $n_t = 2$ have the form

$$\tilde{s}_1 = (|h_1|^2 + |h_2|^2) y_1 + \tilde{z}_1$$
$$\tilde{s}_2 = (|h_1|^2 + |h_2|^2) y_2 + \tilde{z}_2, \qquad (5)$$

which makes a n_t-branch STBC equivalent to a n_t-branch maximal-ratio combiner (MRC), where $\tilde{z}_1$ and $\tilde{z}_2$ are the noise samples at the outputs of the detector. Note that the rows of $\mathcal{G}$ should be orthogonal [2].

3. Throughput Analysis

The performance of an ARQ protocol is characterized by its throughput η, which is defined as the ratio of the average number of packets accepted as error-free by the receiver to the total number of transmitted packets [3]. It is assumed that the probability of undetected errors is negligible. Under the assumptions of infinite buffer size at the receiver and noiseless feedback channel, the throughput of SR ARQ protocols [6] is given by

$$\eta = 1 - P_p, \qquad (6)$$

where P_p is the probability of packet error, which is a function of the fading process affecting a packet.

Under the block fading model a packet of length N symbols undergoes F independent fading realizations, where $m = \lceil \frac{N}{F} \rceil$ symbols is the size of each fading block, which represents the channel memory length. The packet error probability conditioned on the fading channel gains $\mathbf{H} = \{\mathbf{h}_f\}_{f=1}^F$ is the probability that at least one fading block is in error, i.e.,

$$P_{p|\mathbf{H}} = 1 - \prod_{f=1}^{F} (1 - P_{B|\mathbf{h}_f}), \qquad (7)$$

where $P_{B|\mathbf{h}_f}$ is the probability that the f^{th} fading block is in error conditioned on the fading gains $\mathbf{h}_f$. It is the probability that at least one symbol in the fading block is in error, i.e.,

$$P_{B|\mathbf{h}_f} = 1 - (1 - P_{s|\mathbf{h}_f})^m, \quad (8)$$

where $P_{s|\mathbf{h}_f}$ is the conditional symbol error probability, which is a function of the modulation scheme and the receiver employed. Let $\gamma_f = a_f^2 \gamma_s$ denote the received SNR value for a symbol in the f^{th} fading block, where $\gamma_s = \frac{E_s}{N_0}$ is the average SNR. Substituting (8) in (7), the conditional packet error probability becomes

$$P_{p|\Gamma} = 1 - \prod_{f=1}^{F} (1 - P_{s|\gamma_f})^m, \quad (9)$$

where $\Gamma = \{\gamma_f\}_{f=1}^{F}$. The unconditional packet error probability is obtained by averaging (7) over the probability density function (pdf) of the fading statistics Γ. Since the fading gains affecting different fading blocks are independent, the unconditional packet error probability becomes

$$P_p = 1 - \prod_{f=1}^{F} \mathrm{E}_{\gamma_f}\left[(1 - P_{s|\gamma_f})^m\right], \quad (10)$$

where the product results from the independence of fading random variables affecting different fading blocks. Since all fading blocks are affected by identical fading processes the packet error probability is

$$P_p = 1 - \left\{\mathrm{E}_{\gamma}\left[(1 - P_{s|\gamma})^m\right]\right\}^F. \quad (11)$$

Using the binomial expansion

$$\mathrm{E}_{\gamma}\left[(1 - P_{b|\gamma})^m\right] = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \mathrm{E}_{\gamma}[P_{s|\gamma}^i], \quad (12)$$

where we have used the linearity of the expectation operation. The expression in (12) can be simplified for modulation schemes with a conditional symbol error probability given by an exponential function such as noncoherent BFSK, whose symbol error probability is given by

$$P_{s|\gamma_f} = \frac{1}{2} \mathrm{e}^{-\frac{1}{2}\gamma_f}. \quad (13)$$

Substituting (13) in (12) and (11) yields the packet error probability for noncoherent BFSK as

$$P_p = 1 - \left\{\sum_{i=0}^{m} (-1)^i \binom{m}{i} \mathrm{E}_{\gamma}\left[\frac{1}{2}\mathrm{e}^{-\frac{1}{2}i\gamma}\right]\right\}^F. \quad (14)$$

Let $P_{si} = \mathrm{E}_{\gamma}[P_{s|\gamma}^i]$ and using multi-nomial expansion to

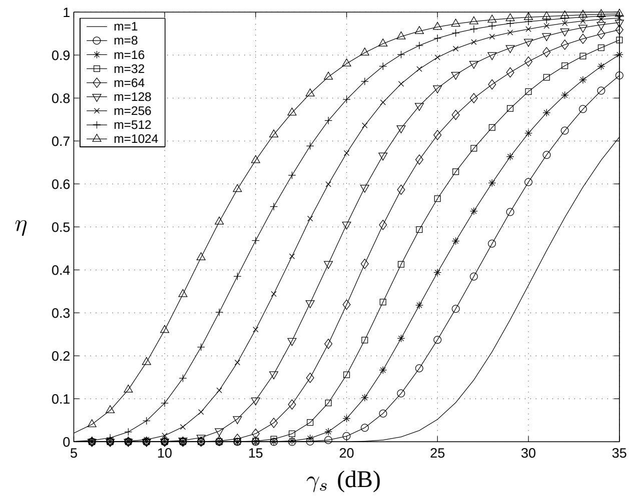

Figure 1. Analytical throughput of SR ARQ system employing BFSK over Rayleigh block fading channels with memory length m.

simplify (14) the packet error probability becomes

$$P_p = 1 - \sum_{j_1, j_2, \ldots, j_m \in J} \frac{F!}{j_1! j_2! \ldots j_m!} \prod_{i=0}^{m} (-1)^{j_i} \binom{m}{j_i}^{j_i} P_{sj_i}^{j_i}, \quad (15)$$

where $J = \{j_1, j_2, \ldots, j_m : j_1 + j_2 + \ldots + j_m = F\}$. The throughput of SR ARQ is obtained by substituting (15) in (6) and computing (15) for the modulation scheme and the channel model under study. Equation (15) is hard to be evaluated in a closed form for coherent QPSK, whose conditional symbol error probability is given by

$$P_{s|\gamma_f} = \mathrm{Q}\left(\sqrt{2\gamma_f}\right), \quad (16)$$

Therefore, numerical integration is used to evaluate (15) using (16) and averaging over the fading statistics. The throughput of SR ARQ employing single and multiple transmit antennas is derived below.

3.1. Nakagami Channels

Nakagami distribution was shown to fit a large variaty of channel measurements [1]. Under Nakagami distribution, the pdf of the received SNR [8] is given by

$$f_{\gamma}(\gamma) = \left(\frac{M}{\Omega}\right)^M \frac{\gamma^{M-1}}{\Gamma(M)} \exp\left(-\frac{M\gamma}{\Omega}\right), \quad \gamma > 0, M > 0.5, \quad (17)$$

where $\Gamma(.)$ is the Gamma function and $M = \frac{\Omega^2}{\mathrm{Var}[\sqrt{\gamma}]}$ is the Nakagami parameter that indicates the fading severity.

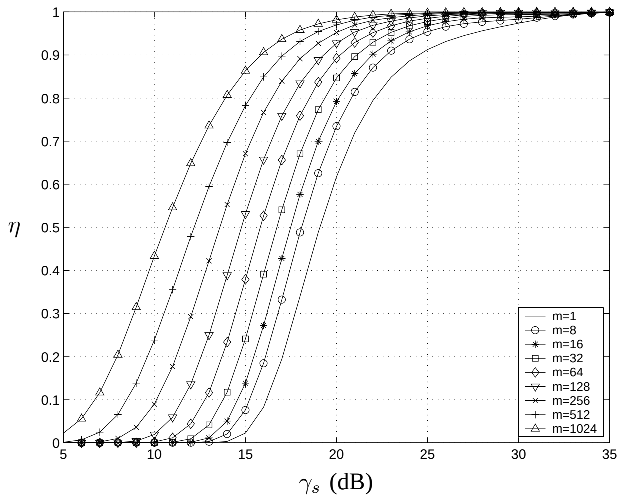

Figure 2. Analytical throughput of SR ARQ system employing noncoherent BFSK over Nakagami block fading channels with fading parameter $M = 2$.

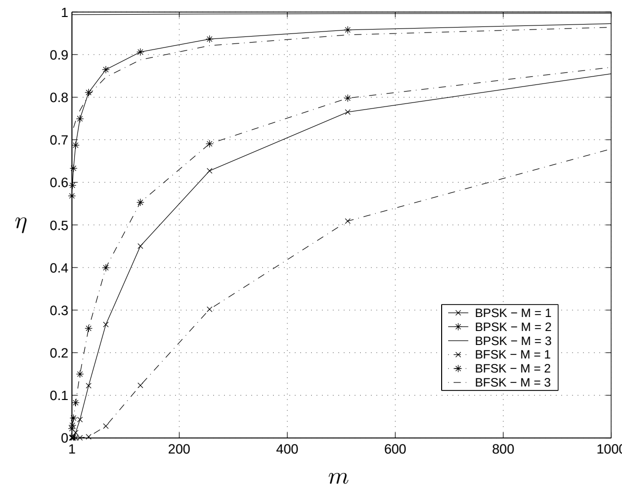

Figure 3. Analytical throughput of SR ARQ system employing coherent BPSK and non-coherent BFSK modulation schemes over Nakagami block fading channels at $\gamma_s = 15$ dB versus the channel memory length m.

As M increases, the fading becomes less severe and approaches the AWGN channel when $M \to \infty$. The Nakagami distribution covers a wide range of fading scenarios including Rayleigh fading when $M = 1$. The bit error probability for BFSK over Nakagami fading is found by averaging (13) over the statistics in (17) and is given by

$$P_{si} = \frac{1}{2}\left(\frac{1}{1 + i\gamma_s/2M}\right)^M . \quad (18)$$

Substituting (18) in (15) and (6) yields the throughput of SR ARQ employing BFSK over Nakagami block fading channels.

In the following analytical results are shown and discussed. Figure 1 shows the throughput of the SR ARQ employing BFSK over Nakagami block fading channels with $M = 1$, i.e., Rayleigh fading. We observe that the throughput improves with longer channel memory because the probability of a packet error is lower. This is because a packet is considered in error if it includes at least one symbol error. Shorter channel memory results in larger number of independent fading realizations affecting a packet, which increases the probability that a symbol falls in a deep fade. Hence, the probability of error increases with shorter memory lengths and therefore the throughput degrades. Also, the every double in the channel memory length results in a 2 dB gain in the SNR.

Figure 2 shows the throughput of SR ARQ employing BFSK over Nakagami block fading channels with $M = 2$. Comparing with Figure 1, which is the Rayleigh case ($M = 1$), we observe that the SNR gain due to longer channel memory decreases as M increases, i.e., as the channel becomes less random. In Figure 3 the throughput of SR ARQ achieved at SNR of 15 dB is shown for BPSK and noncoherent BFSK with different Nakagami parameters. For the same operating SNR, the throughput improves significantly in the short memory range and a little improvement is achieved in the long memory range. Moreover, as the channel becomes less random (larger M values), the memory length above which a little performance improvement is achieved becomes smaller.

3.2. MIMO Channels

In the following the throughput of SR ARQ is derived for MIMO channels with one receive antenna. Note that the presented results can be easily extended to multiple receive antennas. The conditional symbol error probability corresponding to the decision variables in (5) is given by

$$P_{s|\mathbf{h}} = \mathrm{Q}\left(\sqrt{2\gamma_s \sum_{i=1}^{n_t} |h_i|^2}\right) . \quad (19)$$

In the MIMO case we assume that the channel amplitude $|h_i|$ is Rayleigh distributed. If the fading processes from different transmit antennas are uncorrelated, then the variable $q = \gamma_s \sum_{i=1}^{n_t} |h_i|^2$ has a Chi-square distribution with

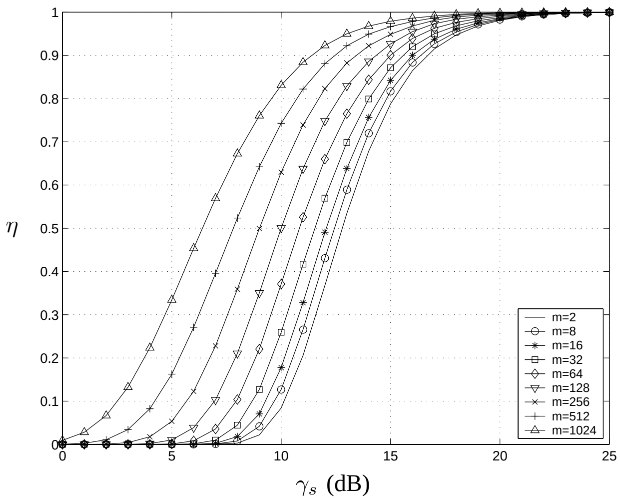

Figure 4. Analytical throughput of SR ARQ system employing BPSK STBC with $n_t = 2$ over Rayleigh block fading channels.

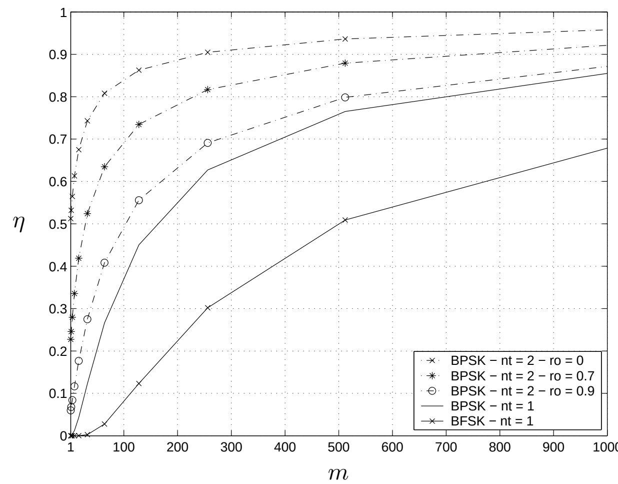

Figure 5. Analytical comparison of the throughput of SR ARQ system employing different modulation schemes over Rayleigh block fading channels at $\gamma_s = 15$ dB versus the channel memory length m.

$2n_t$ degrees of freedom [9] given by

$$f_q(q) = \frac{1}{(n_t-1)!}\frac{q^{n_t-1}}{\Omega_q^{n_t}} e^{-q/\Omega_q}, \quad q > 0, \tag{20}$$

where $\Omega_q = \mathrm{E}[q] = n_t\gamma_s$.

When antennas are placed relatively close to each others, the fading processes from different transmit antennas will be correlated [5, 11]. In this case, the channel vector, $\mathbf{h}_f$ in (4) is a correlated complex Gaussian random vector with a covariance matrix $C_{\mathbf{h}}$ whose $(i,j)^{th}$ element is $\mathrm{E}[h_i^* h_j] = \rho_{ij}$, where ρ_{ij} is the correlation coefficient between channels from the i^{th} and j^{th} transmit antennas. Note that $\mathrm{E}[h_i^* h_i] = 1$. In order to derive the unconditional error probability from (19), we need to average over the pdf of the inner product $\mathbf{h}^*\mathbf{h} = \sum_{i=1}^{n_t} |h^i|^2$ is needed, which is difficult to perform.

Recall the eigenvalue decomposition of the covariance matrix $C_{\mathbf{h}} = U\Lambda U^T$, where Λ is a diagonal matrix containing the eigenvalues of $C_{\mathbf{h}}$, i.e., $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{n_t}\}$, and U is a unitary matrix that contains the eigenvectors of $C_{\mathbf{h}}$ in its rows. Thus an uncorrelated Gaussian random vector $\mathbf{g}$ with a covariance matrix $C_{\mathbf{g}} = \Lambda$ is generated by applying the linear transformation $\mathbf{g} = U^T\mathbf{h}$, and the conditional symbol error probability becomes

$$P_{s|\mathbf{h}} = \mathrm{Q}\left(\sqrt{2\gamma_s \sum_{i=1}^{n_t} \lambda_i |g_i|^2}\right). \tag{21}$$

Let $q = \gamma_s \sum_{i=1}^{n_t} \lambda_i |g_i|^2$ and assume distinct eigenvalues $\{\lambda_i\}_{i=1}^{n_t}$, the pdf of q is found using the inverse Fourier transform [10] to be

$$f_q(q) = \sum_{i=1}^{n_t} \prod_{j\neq i} \frac{1}{\lambda_i - \lambda_j} e^{-q/(\lambda_i\Omega_q)}, \quad q > 0. \tag{22}$$

Substituting (20) or (22) in (15) with γ is replaced by q results in the packet error probability of STBC over Rayleigh block fading channels with uncorrelated and correlated transmit branches, respectively. The throughput follows directly from substituting (15) in (6). Since the conditional symbol error probability is given by the Q-function, numerical integration is used to compute (15).

Figure 4 shows the throughput of the SR ARQ employing STBC with BPSK and two transmit antennas over Rayleigh block fading channels. Note the the throughput of STBC employing QPSK is double that of the BPSK system. In Figure 5 the throughput of SR ARQ achieved at SNR of 15 dB is shown for BFSK and BPSK with single and two transmit antennas. We observe that the SNR gain due to longer channel memory decreases with increasing the number of transmit antennas. This is because having more antennas reduces the probability of packet error, which improves the throughput. As the order of space diversity increases, the memory length above which a little performance improvement is achieved becomes larger. The figure also shows the effect of antenna correlation on the throughput. We observe that a correlation coefficient of 0.7 results in small degradation in the performance and a significant SNR gain compared to the single-antenna scenario.

4. Conclusions

In this paper we derived the throughput of the SR ARQ over Nakagami block fading channels employing single and multiple transmit antennas. It was found that longer channel memory improves the performance of basic SR ARQ. Furthermore, the performance improvement gained by increasing the channel memory increases as the fading becomes more severe (random). Results show that space diversity reduces the need for long channel memory used in single-antenna systems to improve the performance. Moreover, a little degradation in the performance is observed when a correlation of 0.7 exists between the transmit antenna in a STBC system.

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