Performance Analysis of Coded Multi-Carrier Wideband Systems over Fading Channels

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ABSTRACT

In this paper, we propose an efficient method to analyze the performance of coded multi-carrier systems over multipath fading channels. The computation complexity is low and moreover, the analysis is able to capture the system performance with a specific interleaver which is not possible under the common uniform interleaving assumption. The framework in this paper can also be generalized to coded multi-channel systems that suffer from correlated fading process.

I. INTRODUCTION

An approach of achieving high data rates is the use of ultra-wideband (UWB) [1]–[3]. Although the majority of the initially proposed UWB systems were single-carrier (SC) systems employing bandwidths in excess of 500 MHz [4]–[8], there has been recent interest in employing multi-carrier (MC) signaling. Among the primary motivations for this development is the flexibility afforded by a MC scheme with regards to adapting to heterogeneous spectrum allocations across various countries. In addition, MC systems can avoid the non-intentional interference generated by WLAN systems operating at 5 GHz, whereas SC schemes require interference mitigation techniques such as nonlinear processing or notch filtering.

In order to optimize the system configuration for a wideband MC system, we need to have an efficient way to obtain the system performance over practical channels, multipath fading channels. Since the number of carriers is usually large in a wideband MC system, simulation of such system would consume a huge amount of time. In the literature, several papers analyze the performance of coded MC systems under multipath fading. Due to the difficulty of analyzing the system performance, various approximations are generally made to facilitate the performance analysis. In [9], [10], it is assumed that the fading level experienced by each coded bit is independent from the others due to the bit interleaving prior to the transmission. This is not true in reality since finite interleaving does not completely eliminate the correlation among the fading levels. In [11], [12], the pairwise error probability is expressed in terms of the eigenvalues of some matrix that depends on the error pattern. The disadvantage of the method is that the eigen-decomposition corresponding to each error pattern should be computed, which requires high computation complexity. As a result, the system performance can only be approximated by the pairwise error probabilities of a few error patterns, which reduces the accuracy. In [13], the performance of a Reed-Solomon coded MC system is analyzed, yet the work can not be generalized to coded systems that apply soft decision decoding.

In this paper, an efficient method is proposed to analyze the performance of coded MC systems over multipath fading channels. In particular, the performance analysis can be efficiently performed for most well-behaved coded MC systems such as turbo coded MC systems due to one of their important features. That is, the number of codewords with small weights is usually small. By deriving a simple expression for the pairwise error probability, we can consider all small weight codewords and sum up the corresponding pairwise error probabilities to obtain an approximation to the packet error rate (PER) for the system with a specific interleaver prior to the transmission. There are several advantages of our method over the existing methods. First, the computation complexity is low. Secondly, the system performance with a specific interleaver can be captured while the common uniform interleaving assumption can only give the average performance over all possible interleavers, which could be far away from the actual performance. Finally, the framework in this paper is not restricted to the correlated fading process of different frequencies. It can be generalized to other coded multi-channel systems that suffers from correlated fading process in time or space. For example, it can be generalized to a coded TDMA system with slow time variation of the fading

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process.

This paper is organized as follows. In Section II, the system and the channel models are described. In Section III, the PER approximation and the pairwise error probability are derived. In Section IV, various numerical examples are demonstrated and discussed. Finally, the conclusions are addressed in Section V.

II. Model Description

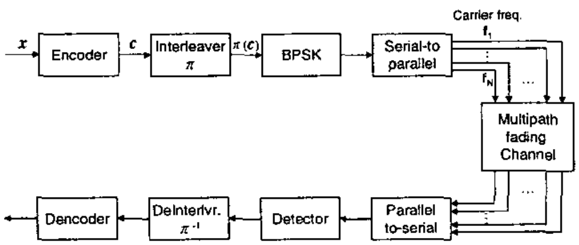

Fig. 1. System model.

In this paper, we consider a multi-carrier system as shown in Fig. 1. The data bit sequence x of length K is first encoded to c, and then interleaved to $\pi(\mathbf{c})$. After BPSK modulation and serial-to-parallel operation, the modulated signals are transmitted through N different carriers with carrier frequencies $f_1, f_2, \ldots, f_N$. The signals then pass through the multipath fading channel to the receiver, which operates in the reverse order of the transmitter to get the estimate of x. The length of the codeword is $N_c = \frac{K}{R_c}$, where R_c denotes the code rate.

The delay power profile of the multipath channel has the form

$$P_h(t) = \sum_{l=1}^{L} P_l \; \delta(t - \tau_l), \tag{1}$$

where L denotes the number of paths, P_l is the power, and τ_l is the delay of the l^{th} path, $0 < \tau_1 < \cdots < \tau_L$. The channel is modelled as a linear time invariant filter with impulse response

$$h(t) = \sum_{l=1}^{L} \alpha_l \; \delta(t - \tau_l), \tag{2}$$

where fading levels $\{\alpha_l\}$ are independent complex Gaussian random variables with $\sigma^2_{\alpha_l} = P_l$. The frequency response of the channel is

$$H(f) = \sum_{l=1}^{L} \alpha_l \; e^{-j2\pi f \tau_l}. \tag{3}$$

The bandwidth of each carrier are assumed to be smaller than the coherence bandwidth ($\approx 1/\tau_L$) so that the fading is flat within the bandwidth of each carrier. The fading level experienced by the i^{th} carrier is $H_i \triangleq H(f_i)$ which is complex Gaussian with mean 0 and variance $\sum_{l=1}^{L} P_l$. If an encoded bit b_i is transmitted by the i^{th} carrier, the received decision statistic of the bit would be

$$y_i = (-1)^{b_i} \cdot |H_i| \cdot \sqrt{E_b R_c} + n_i, \tag{4}$$

were E_b is the bit energy and $n_i \sim \mathcal{N}(0, \frac{N_0}{2})$ the Gaussian noise. Note that while $\{n_i\}$ are independent, the fading levels $\{H_i\}$ are not, which introduces the main obstacle in analyzing the system performance.

The main focus of this work is on the effect of the multipath fading on the performance of a coded MC system. Hence, we assume that the receiver has perfect channel information, and interchannel interference is negligible due to the time invariance of the channel. However, the work can be generalized to include interference analysis as well as the effect of channel estimation errors.

III. Performance Analysis

A. Packet Error Rate Approximation

The channel coding considered in this paper can be either a convolutional code or a turbo code. The weight spectrum of the code is defined as

$$A(D) = \sum_{d>0} a_d D^d, \tag{5}$$

where $a_d D^d$ indicates that there are a_d different codewords with Hamming weight d (the term "weight" will refer to Hamming weight hereafter). Note that for convolutional codes, if an input sequence generates a code weight of d, all of the shifts of the input sequence also generate the same code weight. In other words, this group of almost K input sequences generate the same code weight. Thus for most well-behaved convolutional codes, the coefficients $\{a_d\}$ of the *dominant code weights*, i.e., small code weights, are usually only of order K[1]. On the other hand for turbo codes, due to the bit interleaving in turbo encoding, an input sequence that generates a small code weight does not imply all of its shifts would generate the same turbo code weight. Thus $\{a_d\}$ of the dominant code weights are smaller than the case of convolutional codes. We can conclude that for most well-behaved convolutional codes and turbo codes, $\{a_d\}$ of the dominant code weights are at most of order K.

[1]This statement is true unless there exists a loop with zero-input and zero-output in the state diagram of the convolutional code. This sort of a code should be avoided in practice since it brings out very small weight codewords.

The PER of the system is upper bounded by union bound [14]. We can approximate the PER by the sum of the dominant terms in the union bound. Since the dominant terms are associated with small code weights, we are only interested in small weight codewords. Assume that only W smallest code weights $d_1 < d_2 < \cdots < d_W$ are of interest. Let C_s be the set of such codewords, the cardinality of C_s is $|C_s| = \sum_{i=1}^{W} a_{d_i}$. For convolutional codes, codewords in C_s can be obtained easily through the Viterbi algorithm; while for turbo codes, we should apply efficient algorithms in [15]–[17].

For each $\mathbf{c} \in C_s$, if we can compute the pairwise error probability $P_r\{\mathbf{0} \to \mathbf{c}\}$ between the all-zero codeword $\mathbf{0}$ and $\mathbf{c}$, the PER approximation can be obtained as

$$\text{PER} \approx \sum_{c \in C_s} P_r\{\mathbf{0} \to \mathbf{c}\}. \tag{6}$$

Since for most well-behaved convolutional codes and turbo codes, a_d is only of order K for small d. This implies $|C_s|$ is at almost of order K. As we will see in Section III-B, $P_r\{\mathbf{0} \to \mathbf{c}\}$ can be quickly approximated through simple expressions. Therefore the computation complexity of the PER approximation is also at most of order K, which is very efficient.

B. Pairwise Error Probability

Before we derive the approximation for the pairwise error probability $P_r\{\mathbf{0} \to \mathbf{c}\}$ with a specific interleaver π prior to the transmission, we first define $\pi_i(\mathbf{c})$ as the fraction of the interleaved codeword $\pi(\mathbf{c})$ transmitted during the i_{th} use of the channel. We can obtain $\{\pi_i(\mathbf{c})\}$ by directly feeding $\pi(\mathbf{c})$ to the serial-to-parallel operator without BPSK modulation, i.e.

$$[\ \pi(\mathbf{c})^T, 0, 0, \cdots, 0\ ] = [\ \pi_1(\mathbf{c})^T, \pi_2(\mathbf{c})^T, \cdots, \pi_M(\mathbf{c})^T\ ]^T. \tag{7}$$

Note that we might need to pad q zeros to $\pi(\mathbf{c})$ to make the overall length $N_c + q = MN$, where $0 \le q < N$ and recall N is the number of carriers. This implies that we need to use the channel M times to transmit the whole packet.

Now define $\mathbf{A} = [\ |H_1|^2, |H_2|^2, \cdots, |H_N|^2\ ]^T$, we have the following

$$\begin{aligned} & P_r\{\mathbf{0} \to \mathbf{c}\} \\ = & \int_{\mathrm{R}^N} P_r\{\mathbf{0} \to \mathbf{c} \mid \mathbf{H}\}\ f(\mathbf{H})\ d\mathbf{H} \\ = & \int_{\mathrm{R}^{+N}} P_r\{\mathbf{0} \to \mathbf{c} \mid \mathbf{A}\}\ f(\mathbf{A})\ d\mathbf{A} \\ = & \int_{\mathrm{R}^{+N}} Q\left(\sqrt{2\gamma_b R_c \sum_{i=1}^{M} \mathbf{A}^T \pi_i(\mathbf{c})}\right) f(\mathbf{A})\ d\mathbf{A}, \end{aligned} \tag{8}$$

where $f(\mathbf{H})$ and $f(\mathbf{A})$ are the probability density functions and $\gamma_b \triangleq \frac{E_b}{N_0}$. Note that $d_{\mathbf{c}}^2 \triangleq \sum_{i=1}^{M} \mathbf{A}^T \pi_i(\mathbf{c})$ is the square of the Euclidean distance between the noiseless receptions of $\pi(\mathbf{0})$ and $\pi(\mathbf{c})$. The main difficulty in evaluating (8) is that $\{|H_i|^2\}$ are correlated exponential random variables with the same mean $\sum_{l=1}^{L} P_l$. Currently there is no simple means to find the joint probability density function $f(\mathbf{A})$. Even if we know the form of $f(\mathbf{A})$, the integration in (8) is still untractable since N is typically not small in an ordinary MC system. To solve this problem, let us consider the Karhunen-Loève expansion [18] of $\mathbf{A}$:

$$\mathbf{A} = \sum_{l=1}^{P} \beta_l \Psi_l. \tag{9}$$

$\{\Psi_l\}$ is the set of orthonormal eigenvectors of matrix $\mathcal{K}_{\mathbf{A}} = [K_{i,j}]$ where

$$\begin{aligned} K_{i,j} &= \operatorname{Cov}\left(|H_i|^2, |H_j|^2\right) \\ &= \sum_{s=1}^{L} \sum_{t=1}^{L} P_s P_t \cos\left(2\pi(f_i - f_j)(\tau_s - \tau_t)\right) \end{aligned} \tag{10}$$

and P is the rank of $\mathcal{K}_{\mathbf{A}}$. On the other hand, $\{\beta_l\}$ are uncorrelated random variables with unknown distributions. Note that

$$\beta_l = \sum_{k=1}^{P} \beta_k \Psi_k^T \Psi_l = \mathbf{A}^T \Psi_l = \sum_{i=1}^{N} |H_i|^2 \Psi_{l,i}, \tag{11}$$

where $\Psi_{l,i}$ denotes the i^{th} element of Ψ_l. As we can see, β_l is a linear combination of N random variables $\{|H_i|^2\}$. Although $\{|H_i|^2\}$ are correlated, the correlation between $|H_i|^2$ and $|H_j|^2$ is not strong if $|f_i - f_j|$ is greater than the coherence bandwidth ($\approx 1/\tau_L$). In other words, the correlation dies out as $|i - j|$ increases. Furthermore, N is usually not small. According to Central Limit Theory for dependent variables [19], these observations suggest that β_l should behave very close to a Gaussian random variable. Thus we will assume $\beta_l \sim \mathcal{N}(\mu_l, \lambda_l)$, where $\mu_l = (\sum_{k=1}^{L} P_k)(\sum_{i=1}^{N} \Psi_{l,i})$ and λ_l is the eigenvalue associated with Ψ_l. Since $\{\beta_l\}$ are uncorrelated and Gaussian, they should be independent. Notice that the Euclidean distance square can be expressed as

$$d_{\mathbf{c}}^2 = \sum_{i=1}^{M} \mathbf{A}^T \pi_i(\mathbf{c}) = \sum_{l=1}^{P} \beta_l \sum_{i=1}^{M} \Psi_l^T \pi_i(\mathbf{c}). \tag{12}$$

We can compute the mean $\mu_{\mathbf{c}}$ and variance $\sigma_{\mathbf{c}}^2$ of $d_{\mathbf{c}}^2$ easily by

$$
\begin{aligned}
\mu_{\mathbf{c}} &= \sum_{l=1}^{P} \mu_l \sum_{i=1}^{M} \Psi_l^T \pi_i(\mathbf{c}), && (13) \\
\sigma_{\mathbf{c}}^2 &= \sum_{l=1}^{P} \lambda_l \left(\sum_{i=1}^{M} \Psi_l^T \pi_i(\mathbf{c}) \right)^2 . && (14)
\end{aligned}
$$

Since $\{\beta_l\}$ are independent Gaussian random variables, we should expect $d_{\mathbf{c}}^2$ to be Gaussian. However, since $d_{\mathbf{c}}^2$ is always nonnegative, the probability density of it should be zero for all negative realization. This is obviously not true for any Gaussian random variable. In numerical experiments, this would cause an unacceptable error in PER approximation even when the Gaussian random variable has a probability density very close to 0 in negative region. Since $d_{\mathbf{c}}^2$ is chi-square distributed if H_i's are all independent, chi-square distribution would be a better candidate to approximate the distribution of $d_{\mathbf{c}}^2$. The probability density function of $d_{\mathbf{c}}^2$ is approximated by

$$
f_{d_{\mathbf{c}}^2}(x) = \frac{1}{\sigma^n 2^{n/2} \Gamma(\frac{n}{2})} \, x^{n/2-1} \, \exp\{-\frac{y}{2\sigma^2}\}, \qquad (15)
$$

where n and σ can be computed by

$$
n = \lceil \frac{\sigma_{\mathbf{c}}^2}{2\mu_{\mathbf{c}}} \rceil, \; \sigma = \sqrt{\frac{\mu_{\mathbf{c}}}{n}}. \qquad (16)
$$

Hence the final form of the pairwise error probability can be obtained as

$$
\begin{aligned}
& P_r\{\mathbf{0} \to \mathbf{c}\} \\
&= \int_0^{\infty} Q\left(\sqrt{2\gamma_b R_c x}\right) \frac{x^{n/2-1}}{\sigma^n 2^{n/2} \Gamma(\frac{n}{2})} \, \exp\{-\frac{y}{2\sigma^2}\} \, dx. \\
& \hfill (17)
\end{aligned}
$$

Clearly, the chi-square approximation is the main source of error in our analysis. However since the parameters can be computed very efficiently and the resulting union bound is not very far from the actual performance, we regard it as a good choice. We also want to point out that although we can compute $\mu_{\mathbf{c}}$ and $\sigma_{\mathbf{c}}^2$ without the Karhunen-Loéve expansion, the expansion does help us reduce the amount of computation to get $\mu_{\mathbf{c}}$ and $\sigma_{\mathbf{c}}^2$. The Karhunen-Loève expansion only needs to be done once for the channel to obtain $\{\Psi_l\}$ and $\{\lambda_l\}$, which are independent of $\mathbf{c}$. After that, we can compute n and σ easily for any codeword $\mathbf{c}$ and any specific interleaver π. Since $Q(\cdot)$ decays very fast, (17) can be computed by numerically integrating over an finite interval. Hence we can obtain $P_r\{\mathbf{0} \to \mathbf{c}\}$ in a fast manner for any $\mathbf{c}$.

Path Index	1	2	3	4	5	6	7	8	9	10	11	12
Delay (μ s)	0.000	0.070	0.100	0.160	0.201	0.260	0.361	0.611	0.810	0.850	0.950	1.010
Power	0.140	0.166	0.217	0.236	0.024	0.009	0.005	0.009	0.119	0.019	0.048	0.009

Fig. 2. Delay power profile of the channel.

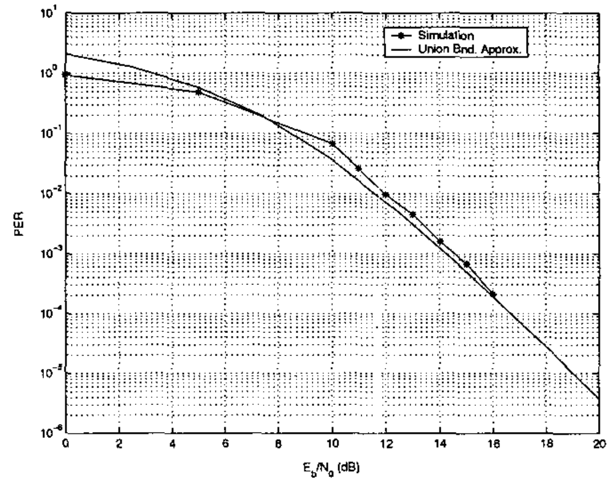

Fig. 3. Repetition coded system with 10 MHz bandwidth. $R_c = 1/16$, data length= 64, 128 carriers.

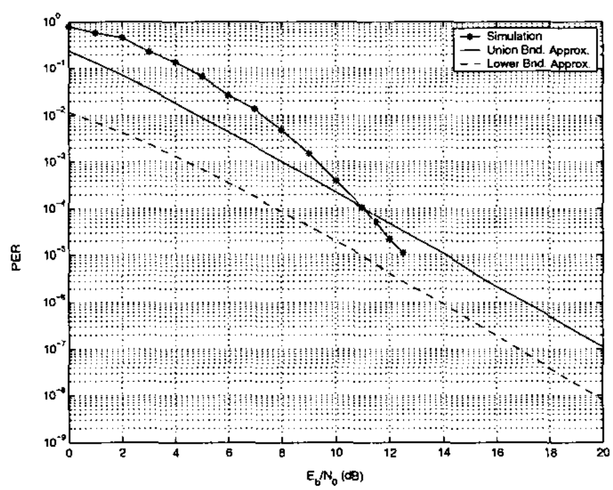

Fig. 4. $(1, 5/7, 5/7)$ turbo coded system with 10 MHz bandwidth. $R_c = 1/3$, data length= 1024, 128 carriers.

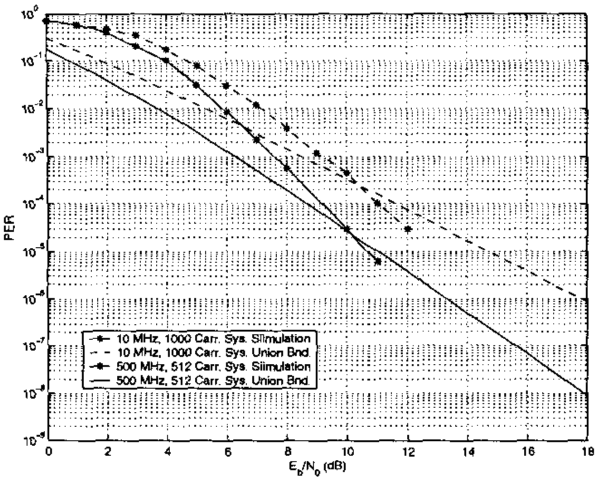

Fig. 5. 10 MHz and 500 MHz $(1, 33/37, 33/37)$ turbo coded systems. $R_c = 1/3$, data length= 1000.

IV. Numerical Examples

Throughout this section, we use the same channel model, a 12-path Rayleigh fading channel with delay spread equal to 1.01 μs. The delay power profile of the channel is shown in Table 1. The union bound approximation for turbo coded systems are all computed by summing up the first 10 dominant terms corresponds to the 10 smallest code weights. In Fig. 2 we consider a repetition coded system in which each data packet consists of 64 bits and each bit is repeated 16 times. In Fig. 3, we consider a turbo coded system. Both systems have the same bandwidth 10 MHz. As we can see, our union bound approximation works better for the repetition coded system. This is because the distance structure of the repetition is much simpler than the turbo code. Hence, after we transmit a codeword through the multipath fading channel and decode it, the Euclidean distance squared between the transmitted and decoded codewords (if distinct) has more variations in the turbo coded system than the repetition coded system. Hence the distribution of the Euclidean distance squared is more complicated, and the chi-square approximation would be less accurate. However, it can be seen that our union bound and lower bound (pairwise error probability between all-zero codeword and the minimum weight codeword) approximations differ by 3 dB with the simulation curve falling in between. This can give us a good idea about the performance of the turbo coded system in high SNR region.

In Fig. 4, we compare the performance of a turbo coded system in two extremely different setups. One has 10 MHz bandwidth with 1000 carriers, while the other one is a 500 MHz wideband system with 512 carriers. As we can see, the union bound approximation works better for the wideband system. This is because for a wideband system, the carriers are further away from each other. Hence the fading levels experienced by them are less correlated. As a result, chi-square approximation works better and thus our union bound approximation performs well for wideband systems. The graph shows that the wideband system clearly outperforms the 10 MHz system in all SNR region. This may not be the case if imperfect channel estimation is considered.

V. Conclusions

In this paper, we proposed an efficient method to analyze the performance of a coded multi-carrier system over multipath fading channels. The union bound approximation works especially well for wideband systems. This enables us to efficiently examine and optimize the system configuration for coded wideband systems over multipath fading channels in the future.

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