Home

Probability of Error for BPSK Modulation in Presence of Noise and Co-channel Interference

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***Abstract*—New expressions are introduced for the probability of error as a function of the carrier to co-channel interference (CCI) ratio for Binary Phase Shift Keying (BPSK) modulation scheme. The interferer's signal strength is assumed to follow Nakagami distribution. The interferer's phase distribution is considered for both uniform and non uniform cases. The addressed non uniform phase distribution encompasses the effect of noise in addition to interference. Results show that for the Rayleigh interference case a phase margin of at least 1 dB has to be considered to compensate for the phase uncertainty. Average probability of error is then numerically obtained assuming that the desired signal strength is also Nakagami distributed. Results show that a saturation effect from the average probability of error perspective is expected when the Nakagami parameter of the desired signal exceeds the value of 2.**

I. Introduction

In a cellular network system, scarcity of spectrum is conceived by a *re-use frequency* concept [1]. This in turn implies having two or more cells operating at the same frequency. Residuals of the signal power from a cell(s) may interfere with the intended cell causing what is called *co-channel interference*. In this paper, one dominant interferer is assumed while using 60 degree sectorized antennas. Normally, this model is used to describe high traffic density areas. Many researches have been concerned with studying the outage probability for different fading models [2] [3] [4]. The outage probability has also been considered for interference cancellation systems [5]. On the other hand, the average probability of error is a crucial criterion for the system performance yet it is more complicated to evaluate. The effect of the CCI on the average probability of error has been addressed in [6]. In this paper, we present another look to the analysis of the probability of error in presence of interference for binary phase shift keying (BPSK) modulation scheme starting from a geometric representation of the problem. We also consider both the noise and phase distribution effects. We are first interested in deriving the probability of error for BPSK as a function of both carrier to co-channel interference (CCI) and average interference signal-to-noise ratio (SNR). The average probability of error is then obtained by averaging this conditional probability over the CCI ratio distribution for Nakagami carrier and Nakagami interferer.

The rest of this paper is organized as follows. The probability of error is calculated in section II. The average probability of error is studied in section III. Finally, numerical examples and concluding remarks are given in section IV.

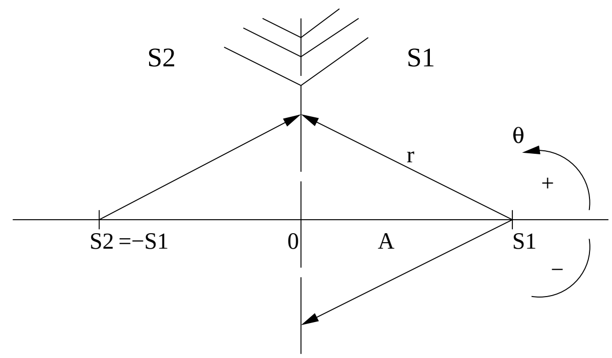

Fig. 1. Signal and interferer representation in BPSK modulation scheme.

II. Probability of Error for BPSK

Figure 1 shows the effect of an interferer with strength r and phase θ on the original signal with amplitude A in the case of BPSK modulation scheme. From the geometry of figure 1, it can be shown that the total probability of error can be written as

$$\begin{aligned} P_e &= P(S_1)P(e|S_1) + P(S_2)P(e|S_2) \\ &= \frac{1}{2}\int_A^\infty \int_{\pi-\phi}^{\pi+\phi} f_r(r) f_\theta(\theta) d\theta dr \\ &+ \frac{1}{2}\int_A^\infty \int_{-\phi}^{\phi} f_r(r) f_\theta(\theta) d\theta dr, \end{aligned} \tag{1}$$

where $f_x(.)$ represents the probability density function (PDF) of x and

$$\phi = \cos^{-1}\left(\frac{A}{r}\right). \tag{2}$$

In the next subsections this probability of error will be investigated with different amplitude and phase distributions of the interference signal.

A. Rayleigh Interferer with Uniform Phase Distribution

If the interferer strength follows a Rayleigh distribution, then the its PDF can be expressed as

$$f_r(r) = \frac{2r}{\Omega_i}\exp\left(-\frac{r^2}{\Omega_i}\right), \tag{3}$$

where $\Omega_i =< r^2 >$ is the interfere average power. The phase in turn is known to be uniformly distributed such that

$$f_\theta(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta \leq 2\pi. \tag{4}$$

Substituting from (3) and (4) into (1) yields

$$P_e(z) = \frac{1}{2} erfc(\sqrt{z}), \tag{5}$$

where $erfc(.)$ is the complementary error function and z is the CCI ratio, namely

$$z = \frac{A^2}{\Omega_i}. \tag{6}$$

The expression given in (5) is similar to that in the case of additive white Gaussian noise channel (AWGN) with CCI replaced for SNR. This is expected since the signal here is perturbed by interference rather than noise.

B. Rayleigh Interferer with Non-uniform Phase Distribution

The cumulative distribution function (CDF) of the phase θ between a noiseless reference and a vector perturbed by Gaussian noise is given by [7]

$$\begin{aligned} F(\theta, \gamma_i) &= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}+\theta} \exp[\gamma_i \sin^2(\theta) \sec^2(\eta)] d\eta, \\ & -\pi \leq \theta \leq 0, \end{aligned} \tag{7}$$

where γ_i is the instantaneous SNR of the interferer. Averaging (7) over the PDF of γ_i for the Rayleigh case of interference yields [8]

$$\begin{aligned} F(\theta) &= \frac{1}{2} \Bigg\{ 1 + \frac{\theta}{\pi} - \sqrt{\frac{a-1}{a+1}} \\ & \times \left[\frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left(\sqrt{\frac{a-1}{a+1}} \tan(\pi/2 + \theta) \right) \right] \Bigg\}, \end{aligned} \tag{8}$$

where $a = 1 + 2\overline{\gamma}_i$ and $\overline{\gamma}_i$ is the average SNR of the interferer.

In order to evaluate the probability of error in (1), it seems that we first need to evaluate the following probabilities: $p_\theta(\pi - \phi \leq \theta \leq \pi + \phi)$ and $p_\theta(-\phi \leq \theta \leq \phi)$. Actually both of these events are equal as a result of the symmetry of the problem. And any of them can be described as

$$\begin{aligned} p_\theta(-\phi \leq \theta \leq \phi) &= F(-(\pi - \phi)) + [F(0) - F(-\phi)] \\ &= 0.5 + F(-(\pi - \phi)) - F(-\phi), \end{aligned} \tag{9}$$

where $F(.)$ is the CDF of the phase θ. Thereupon, the probability of error in (1) can be written as

$$P_e = \int_A^\infty f_r(r) \left[0.5 + F(-(\pi - \phi)) - F(-\phi) \right] dr. \tag{10}$$

With the aid of (8), the probability of error in (10) can be evaluated as a function of the CCI ratio , z, yielding

$$\begin{aligned} P_e(z) &= \frac{1}{2} erfc(\sqrt{z}) + \frac{\sqrt{\overline{\gamma}_i}}{\pi} \int_z^\infty \frac{e^{-x} \sin(\phi)}{\sqrt{1 + \overline{\gamma}_i \sin^2(\phi)}} \\ & \times \tan^{-1} \left(\frac{\sqrt{\overline{\gamma}_i} \sin(\phi) \tan(\pi/2 - \phi)}{\sqrt{1 + \overline{\gamma}_i \sin^2(\phi)}} \right) dx, \end{aligned} \tag{11}$$

where

$$\phi = \cos^{-1} \left(\sqrt{\frac{z}{x}} \right). \tag{12}$$

C. Nakagami Interference with Uniform Phase Distribution

If the interferer amplitude follows Nakagami distribution, then the PDF of r will be

$$f_r(r) = \frac{2}{\Gamma(m_i)} \left(\frac{m_i}{\Omega_i} \right)^{m_i} r^{2m_i - 1} \exp \left(-\frac{m_i r^2}{\Omega_i} \right), \tag{13}$$

where m_i is the Nakagami parameter of the interferer and $\Gamma(.)$ is the Gamma function [9]. The problem of defining the phase distribution associated with Nakagami distributed amplitude is still open. For the assumption of uniform phase distribution, the probability of error as a function of the CCI ratio has been found by substituting for the amplitude and phase distributions into (1) and using the series expansion of $\cos^{-1}(.)$ [9] yielding

$$\begin{aligned} P_e(z) = & \frac{1}{\pi \Gamma(m_i)} \Bigg[\frac{\pi}{2} \Gamma(m_i, m_i z) \\ & - \sum_{k=0}^{\infty} \frac{(2k)! (m_i z)^{k+0.5} \Gamma(m_i - k - 0.5, m_i z)}{2^{2k} (k!)^2 (2k+1)} \Bigg], \end{aligned} \tag{14}$$

where $\Gamma(.,.)$ is the incomplete Gamma function [9].

D. Nakagami Interference with Non-uniform Phase Distribution

In this subsection we tackle the problem of non uniformly phase distribution taking into account the effect of noise. Following the same argument as in [8] and as presented for the Rayleigh case in section II-B, $F(\theta)$ for Nakagami case can be obtained as

$$F(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi + 2\theta} \left(\frac{1 + \cos x}{1 + b + \cos x} \right)^{m_i} dx, \tag{15}$$

where

$$b = 2 \frac{\overline{\gamma}_i}{m_i} \sin^2(\theta). \tag{16}$$

By suitable and careful substitution (to keep the principle cosine interval), the CDF of θ can be finally written as

$$F(\theta) = \frac{1}{2\pi} \int_{-\cos\theta}^{1} \left(\frac{1 + x^2}{1 + b/2 + x^2} \right)^{m_i} \frac{dx}{\sqrt{1 - x^2}}. \tag{17}$$

Top

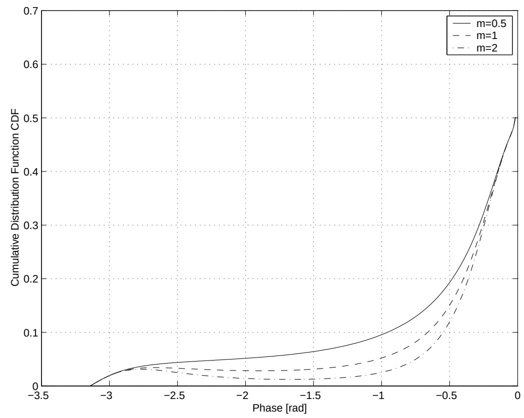

Fig. 2. The CDF of Nakagami phase for different m values

Figure 2 illustrates the CDF of the phase for different values of Nakagami parameter m.

The case for $m_i = 1$ reduces to the Rayleigh distribution which has been addressed in section II-B. The cases for m_i=0.5 and 2 will be addressed in the next subsections since they represent limiting bounds on the system performance in many scenarios of interest [10].

1) Case $m_i = 0.5$: In this case, $F(\theta)$ in (17) can be approximated using series expansion. Hence, the probability of error in (10) can be evaluated as

$$\begin{aligned} P_e(z) =& \frac{1}{2\sqrt{2}\Gamma(0.5)} \int_z^{\infty} \frac{e^{-0.5x}}{\sqrt{x}} \Bigg\{ 1 - \frac{2}{\pi}\sqrt{\frac{2}{a}} \\ & \times \Bigg[\cos\phi + \frac{a-1}{3a} cos^3\phi + \frac{1}{5}(\frac{3}{a^2} - \frac{1}{a} + \frac{1}{2}) \cos^5\phi \\ & + \frac{1}{7}(\frac{-5}{2a^3} + \frac{3}{2a^2} - \frac{1}{2a} + \frac{1}{2}) \cos^7\phi \\ & + \frac{1}{9}(\frac{35}{8a^4} - \frac{-5}{2a^3} + \frac{3}{4a^2} - \frac{1}{2a} + \frac{3}{8}) \cos^9\phi \Bigg] \Bigg\} dx, \end{aligned} \quad (18)$$

where ϕ is given in (12) and

$$a = 2 + 4\overline{\gamma}_i \sin^2\phi. \quad (19)$$

2) Case $m_i = 2$: In this case, $F(\theta)$ in (17) can be evaluated in closed form. Then the probability of error can be finally given as

$$\begin{aligned} P_e(z) =& \int_z^{\infty} 2xe^{-2x} \Bigg[1 + \frac{2}{\pi} \frac{a^2 + a - 2}{(a+1)\sqrt{a^2-1}} \\ & \times \ \tan^{-1}\left(\frac{\tan(\pi/2 - \phi)}{a+1}\right) \\ & - \frac{(\pi - 2\phi)(a^2 + a - (a+1)\cos 2\phi) - (a-1)\sin 2\phi}{\pi(a+1)(a - \cos 2\phi)} \Bigg] dx, \end{aligned} \quad (20)$$

where a in this case is given by

$$a = 2 + \overline{\gamma}_i \sin^2\phi. \quad (21)$$

III. Average Probability of Error

Assuming that both the signal and interferer strengths have Nakagami characteristics, the PDF of the CCI ratio follows an *F*-distribution given by [11]

$$f_z(z) = K_s^{m_s} K_i^{m_i} \frac{\Gamma(m_s + m_i)}{\Gamma(m_s)\Gamma(m_i)} \frac{z^{m_s - 1}}{(K_s z + K_i)^{m_s + m_i}}, \quad (22)$$

where suffices s and i refer to the signal and interferer, respectively and for each of them, $K = \frac{m}{\Omega}$ where Ω denotes the average power. With the aid of numerical methods, $f_z(z)$ in (22) is used to average $P_e(z)$ expressions given in section II.

IV. Numerical Examples and Conclusions

Figure 3 illustrates the effect of the instantaneous CCI on the calculated probability of error (P_e) for Rayleigh interferer scenario. It can be inferred that there is a degradation of about a half order of magnitude from the P_e perspective for the case of non uniform phase distribution and noise effect compared to that for uniform phase distribution and absence of noise. In other words, a phase margin of at least 1 dB is required to compensate for the phase uncertainty compared to the uniform distribution assumption. For the same CCI value, it is also shown that a change in the average SNR of the interferer has a very subtle effect on the system performance; namely the noise effect is very subtle.

Figure 4 shows the effect of the interferer Nakagami parameter on the P_e performance with respect to the CCI ratio for $\overline{\gamma}_i$=10 dB. It is shown that at very low CCI values, the higher the value of m_i the worse the performance is and vice versa for high CCI values. This can be interpreted as follows: low CCI values implies high interferer power while high m_i values implies small interference power variance. Hence, low CCI values and high m_i values imply high interfering power values most of the time. There is also a clear gain of about two orders of magnitude from the P_e perspective as m_i increases from 1 to 2 for CCI values ranging from 8 to 10 dB.

The performance of the average probability of error $\overline{P}_e$ is studied in figures 5 and 6. In figure 5 the effect of the average signal power is studied for the worst fading conditions ($m_i = m_s = 0.5$) and $\overline{\gamma}_i$=10 dB. In such a scenario, it is shown that an average P_e of about 10^{-2} is expected for average CCI of about 30 dB. Figure 6 illustrates the effect of the signal Nakagami parameter (m_s) on the $\overline{P}_e$ for $m_i = 0.5$ and $\Omega_i = 10$ dB. It is shown that an improvement up to one order of magnitude from the $\overline{P}_e$ perspective is expected as Ω_s increases from 20 to 30 dB. Moreover, it can be inferred that a saturation effect from the $\overline{P}_e$ perspective is expected for m_s values grater than 2.

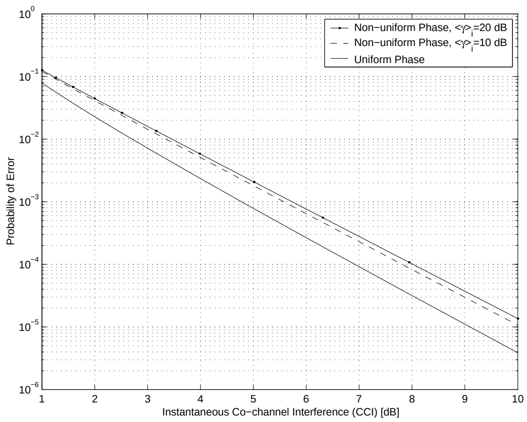

Fig. 3. Probability of error for Rayleigh interferer

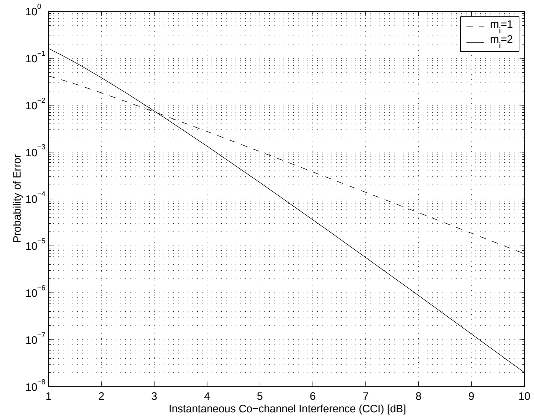

Fig. 4. Probability of error for Nakagami interferer

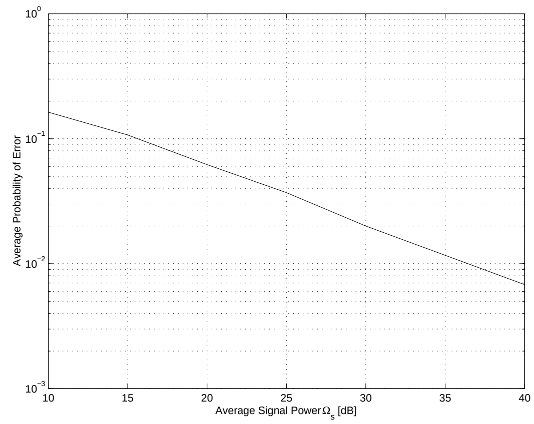

Fig. 5. Average Probability of error versus the desired signal power ($m_s = m_i = 0.5$ and $\overline{\gamma}_i = 10$ dB)

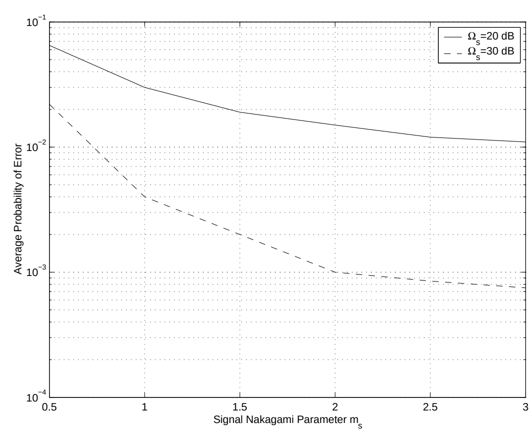

Fig. 6. Average Probability of error versus the Nakagami parameter of the desired signal ($m_i = 0.5$ and $\Omega_i = 10$ dB).

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Top