An L_2 Gap Metric Identification Algorithm

Samir Hasan Al-Amer
Systems Engineering Department
King Fahd University of Petroleum & Minerals
Dhahran 31261 Saudi Arabia
samir@ccse.kfupm.edu.sa

Abstract

In this paper, an L_2-gap metric identification algorithm is proposed. The algorithm uses a set of frequency response samples to come up with rational models so that the L_2-gap between the observations and the identified model is minimized. Each iteration of the algorithm involves solving a weighted least squares approximation problem. Illustrative examples are provided.

1. Notation

L_{∞} denotes the space of functions that are essentially bounded on the imaginary axis. RL_{∞} is a subspace of L_{∞} whose elements are rational functions. *G*(s)* denotes complex conjugate transpose of *G(s)*, $h(G)$ denotes the number of open right half plane poles of *G(s)*. The winding number *won(G)* denotes the number of counterclockwise encirclement around the origin by *G(s)* evaluated on the Nyquist contour.

2. Introduction

The behavior of two systems connected in feedback configuration with identical compensator can be very close even for cases where the norm of the difference is arbitrarily large. The gap metric was introduced by Zames and El-Sakkary[1,2] to study approximation and robustness of stability of systems with feedback interconnections. It captures the closeness of closed loop systems and is generally considered very useful for the analysis of uncertain feedback systems.

Let $P_1(s) = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ be normalize coprime factorization. The gap between the two systems $P_1(s)$ and $P_2(s)$ can be computed using [3]

$$d(P_1, P_2) = \max\{\vec{d}(P_1, P_2), \vec{d}(P_2, P_1)\}$$

where

$$\vec{d}(P_1, P_2) = \inf_{Q \in H_\infty} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty$$

El-Sakkary[2] proposed the following metric for single-input-single-output systems

$$d_L(P_1, P_2) = \sup_{w \in \Omega} \frac{|P_1(jw) - P_2(jw)|}{\sqrt{1 + |P_1(jw)|^2}\sqrt{1 + |P_2(jw)|^2}} \qquad (1)$$

This is known as the L_2-gap metric and in general it is different from the gap metric. This metric can also be interpreted as the maximum chordal distance between the projections of the frequency responses of the two systems on the Riemann sphere [4]. A closely related metric is the *v*-gap. The *v*-gap metric and L_2-gap metric have identical values if the following conditions are satisfied.

$$1 + P_2^*(jw)P_1(jw) \neq 0 \quad \forall w \qquad (2)$$

$$won(1 + P_2^* P_1) + h(P_1) - h(P_2) = 0 \qquad (3)$$

The *v*-gap metric can be defined as

$$d_n(P_1, P_2) = \begin{cases} d_L(P_1, P_2) & if\,(2)\,and\,(3)\,are\,satisfied \\ 1 & otherwise \end{cases}$$

Properties of *v*-gap are extensively studied by Vinnicombe[5,6].

Date and Vinnicombe[7] proposed an algorithm for identification in *í*-gap metric. The algorithm involves solving a series of LMI optimization problems followed by Hankel approximation. Georgiou et al [8] uses a set of input-output data to generate an estimate of the power spectrum and uses it to obtain an identified model with an upper bound on the L_2-gap of the error. The fact that under appropriate conditions both the v-gap metric and L_2-gap metric have identical values is used to obtain the bound in the v-gap. In this paper, we propose a new algorithm for the L_2-gap metric identification. The objective is to identify a linear time-invariant SISO system such that the L_2-gap between the system described by the available data and the identified model is reduced. In the proposed scheme, the solution is obtained by solving a sequence of weighted least squares problems where the weight is updated in each iteration based on the error in the proceeding iteration.

The remaining part of the paper is organized as follows. The problem statement is given in Section 3. The proposed algorithm is presented in Section 4 and illustrative examples are given in Section 5.

3. Problem Statement

In this section, we present the L_2-gap metric identification problem. The process to be identified, G(s), is assumed to be a single-input-single-output continuous system. It is also assumed that G(s) belongs to L_∞ and is stabilizable by unity feedback. The identification problem assumes the availability of a set of frequency response samples at the frequencies $\Omega = \{w_1, w_2, ..., w_N\}$ which are not necessarily equally spaced. The set Ω is selected so that it covers the frequency intervals of interest. The frequency response samples are assumed to be corrupted by random noise

.
$$P_k = G(jw_k) + e_k$$

where

$$k \in [1, N],\; w_k \in \Omega, \left| e_k \right| \le \bar{e}$$

The value of $\bar{e}$ is assumed to be finite. The objective is to find an r th order rational transfer function G_r such that

$$d_{L,S}(P, G_r) = \max_{w_k \in \Omega} \frac{\left| P(jw_k) - G_r(jw_k) \right|}{\sqrt{1 + \left| P(jw_k) \right|^2} \sqrt{1 + \left| G_r(jw_k) \right|^2}} \quad (3)$$

is minimized. Note that $d_{L,S}$ in equation (3) is a lower bound to d_L defined in equation (1) and they become closer as N increases. For the discrete-time case, the gap metric in the definition (1) is replaced by

$$d_L(P_1, P_2) = \sup_{w \in \Omega} \frac{\left| P_1(e^{jw}) - P_2(e^{jw}) \right|}{\sqrt{1 + \left| P_1(e^{jw}) \right|^2} \sqrt{1 + \left| P_2(e^{jw}) \right|^2}} \quad (4)$$

and the frequency points are selected to be in the interval $[0, 2p]$. The frequency response is sampled on the unit circle instead of the imaginary axis.

4. Proposed Algorithm

In this section, we present a new L_2-gap metric identification algorithm. The identification problem is posed as an approximation problem. Looking at the definition (1) of the L_2-gap one can view that the L_2-gap as a weighted $L_\square$-norm with the frequency weight given by

$$\frac{1}{\sqrt{1 + \left| P_1(jw) \right|^2} \sqrt{1 + \left| P_2(jw) \right|^2}}$$

The approximation in the L_2-gap can therefore be viewed as an approximation in the $L_\square$-norm. Suboptimal techniques to solve the weighted $L_\square$-approximation such as [9] cannot be used since the weight is not known in advance. To solve this problem we will use an approach similar to the one in [10,11]. The proposed algorithm tries to obtain a weight such that the solution of the weighted least squares problem is also the optimal solution of the L_2-gap approximation problem. If the optimal L_2-gap approximation is known, it is an easy exercise to obtain the frequency dependent weight so that the solutions of the both approximation problems are identical. There is no direct way of obtaining the weight. An iterative approach is used to obtain the weights. An initial guess of weight is taken as

$$U^0(w_k) = 1 \quad \forall w_k \in \Omega$$

The identified model $G_r(s) = \dfrac{N_r(s)}{D_r(s)}$ is obtained by solving the following least squares problem.

$$\min_{D_r, N_r} \left\| U^0(w_k) \left[P(w_k) D_r(w_k) - N_r(w_k) \right] \right\|_2$$

The error is computed using

$$U^1(w_k) = \frac{U^0(w_k)}{a} \frac{\left| P(jw_k) - \dfrac{N_r(jw_k)}{D_r(jw_k)} \right|}{\sqrt{1 + \left| P(jw_k) \right|^2} \sqrt{1 + \left| \dfrac{N_r(jw_k)}{D_r(jw_k)} \right|^2}}$$

As in the earlier work [10,11], the new weight is obtained as the product of the previous weight and the error in the current iteration. The scalar á is used to keep the magnitude of the weight within an acceptable range (usually between zero and one). And the procedure is repeated. The algorithm is stopped after a fixed number of iterations and the model that gives the least L_2-gap is taken as the identified model. A summary of the proposed identification algorithm is given below.

Summary of the L_2-gap metric identification algorithm:

Given the set $\Omega = \{w_1, w_2, ..., w_N\}$, $\{P(w_k)\}$ and r

Step 1: Let $\ell = 0;\ U^0(w_k) = 1 \quad \forall w_k \in \Omega$

Step 2: Solve the weighted least squares approximation

$$\min_{D_r, N_r} \left\| U^1(w_k)\left[P(w_k)D_r(w_k) - N_r(w_k)\right]\right\|_2$$

Step 3: Update the weight

$$U^{l+1}(w_k) = \frac{U^l(w_k)}{a} \frac{\left|P(jw_k) - \frac{N_r(jw_k)}{D_r(jw_k)}\right|}{\sqrt{1+|P(jw_k)|^2}\sqrt{1+\left|\frac{N_r(jw_k)}{D_r(jw_k)}\right|^2}}$$

where a is a scaling factor so that $\|U^{\ell}\| = 1$

Step 4: $\ell = \ell + 1$; go to Step 2.

The proposed algorithm is a generalization of the Lawson's algorithm for solving ℓ_∞ approximation problem. This algorithm reduces to the Lawson's algorithm if the approximating function is restricted to finite impulse response transfer functions. It is known that in the limit the Lawson's algorithm converges to the optimal solution of the ℓ_∞ approximation problem [14]. Unfortunately, for the general rational approximation, no proof of convergence is available.

4.1 Identification in the *v*-Gap Metric

The *v*-gap metric have more useful properties in designing robust control systems. The above algorithm can be used for identification in the *v*-gap metric sense. Recall that if the conditions (2) and (3) are satisfied then both the L_2-gap and *v*-gap have identical values.
To obtain an identified model in the v-gap metric one needs extra assumptions on the number of right half plane poles and the winding numbers so that the conditions (2) ans (3) can be checked.
Once, an identified model is obtained, the conditions (2) and (3) are checked. If they hold, then the identified model is also the best with respect to the *v*-gap metric and the useful properties of the *v*-gap metric can be used.

5. Illustrative Examples

In this section we consider two examples. In the first example we will try to identify first and second order models where the data is assumed to come from the third order continuous-time system used by Date and Vinnicombe[7]. In the second example the algorithm is used to identify a second order discrete-time system.

5.1 Example 1

The data is assumed to come from the true model given by

$$\frac{0.125s^2 + 1.25s + 3.25}{s^3 + 2.4s^2 + 7.082s + 12.564}$$

The frequency response is sampled at frequency points that are equally spaced on the logarithmic scale between 0.001 and 1000. In the first part a set of noise-free samples are obtained with N= 1024. The proposed identification algorithm was used to identify models of orders 1, 2 and 3. The first and second order models are given by

$$G_1(s) = \frac{-0.1125s + 0.8088}{s + 0.7449}$$

$$G_2(s) = \frac{0.0210s^2 + 0.0040s + 1.4851}{s^2 + 0.4544s + 6.2830}$$

and the L_2-gap are 0.5486 and 0.0210 respectively. The third order model is exactly the same as the true model. Thirty iterations was used in obtaining the above results.

In the second part of this example, Noise-corrupted frequency response samples are used. The measurement noise e_k is a randomly generated complex number such that $|e_k| \le 0.1$ for all k. The proposed algorithm is used to identify models of order 1,2 and 3. The identified models are

$$G_1(s) = \frac{-0.1646s + 0.7448}{s + 0.7497}$$

$$G_2(s) = \frac{-0.0060s^2 + 0.0351s + 1.2074}{s^2 + 0.4751s + 6.1332}$$

$$G_3(s) = \frac{0.0004s^3 + 0.0803s^2 + 1.2442s + 0.0004}{s^3 + 0.4138s^2 + 6.0116s + 0.0004}$$

and the corresponding L_2 –gap of the difference between the true and the identified models are 0.5675, 0.0843 and 0.0703 respectively. Computation of the identified models can be done very quickly. Thirty iterations used in identifying a third order model took less than 2 seconds on Pentium computer with 250 MHz clock speed.

Note also that the conditions (2) and (3) are satisfied for this example and consequently the L_2 – gap and the ν-gap are identical.

5.2 Example 2

The data is assumed to come form the discrete-time system.

$$G(z) = \frac{z(z+1.2)}{z^2 + 1.1z + 2}$$

This system is assumed to be connected in feedback configuration with the controller

$$C(z) = \frac{-0.6608z - 0.7571}{z^2 + 0.6602z + 0.1246}$$

The proposed algorithm is used for L_2-gap metric identification. A set of 128 frequencies are selected to be equally spaced on the unit circle. The frequency response is assumed to be corrupted by randomly generated complex additive perturbation with magnitude less than 0.05. First and second order models are identified and the identified models are given by

$$G_1(z) = \frac{0.6994z + 0.3425}{z + 0.5698}$$

$$G2(z) = \frac{1.1487z^2 + 0.9903z + 0.0294}{z^2 + 1.1023z + 1.9090}$$

The corresponding L_2- gap are given by 0.5544 and 0.0120 respectively.

6. Conclusion

In this paper a frequency domain algorithm to identify discrete-time and continuous time systems is proposed. The proposed algorithm formulates the identification problem as an L_2-gap metric approximation problem. The identified models are rational transfer function that minimizes the L_2-gap. The main step in each iteration of the algorithm is a least squares approximation that can be efficiently solved. Two examples were used to illustrate the algorithm. The identified models were very close to the true models.

References

1. G. Zames and A. El-Sakkary, Unstable systems and feedback: the gap metric, in; Proc. Allerton Conference, 1980, pp. 380- 385.
2. El-Sakkary, The gap metric: robustness of stabilization of feedback systems, IEEE Trans. Automatic Control, Vol. 30, 1985, pp 240-247.
3. T. Georgiou, On the computation of the gap metric, Systems and Control Letters, Vol. 11, 1988, pp 253-257.
4. El-Sakkary, Estimating robustness on the Riemann sphere, International Journal Control, Vol. 49, 1989, pp 561-567.
5. Vinnicombe G., Frequency domain uncertainty and the graph topology, IEEE Trans. Automat. Contr., Vol. 38, 1993, pp 1371-1383.
6. Vinnicombe G., Uncertainty and Feedback: H∝ loop-shaping and the v-gap metric, Imperial College Press, 2001.
7. Date P. and G. Vinnicombe, An algorithm for Identification in the v-gap metric , in; Proc. CDC 1999, Phoenix, Arizona, pp. 3230-3235.
8. T. Georgiou C.R. Shankwitz and M. Smith, Identification of linear systems, A graph point of view, in Proc. American Control Conf., 1992, pp 307-312.
9. K. Glover, All Optimal Hankel Norm Approximation of Linear Multivariable Systems and Their L_{∞} Error Bounds, International Journal of Control, Vol 39, pp. 1115-1193, 1984.
10. D. Kavranglue and S.H. Al-Amer, A new frequency domain technique for H_{∞} norm approximation, in; Proc. CDC, 1996, Kobe, Japan, pp. 4574-4579.
11. D. Kavranglue and S.H. Al-Amer, New efficient frequency domain algorithm for H_{∞}-approximation with applications to controller reduction, IEE Proc. Vol. 148, 2001, Pt. D. pp. 383-390.
12. J. R. Rice and K. Usow, The Lawson Algorithm and Extension, Mathematics of Computations, Vol 22, No. 101,pp. 118-127,1968.