Scaling Laws of Multiple Antenna Group-Broadcast Channels

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***Abstract*— Broadcast (or point to multipoint) communication has attracted a lot of research recently. In this paper, we consider the group broadcast channel where the users' pool is divided into groups, each of which is interested in common information. Such a situation occurs for example in digital audio and video broadcast where the users are divided into various groups according to the shows they are interested in. The paper obtains upper and lower bounds for the sum rate capacity in the large number of users regime and quantifies the effect of spatial correlation on the system capacity. The paper also studies the scaling of the system capacity when the number of users and antennas grow simultaneously.**

I. Introduction

Future breakthroughs in wireless communications will be mostly driven by applications that require high data rates [1]. While increasing the link budget and/or bandwidth can accommodate this increase in data rate, such a solution would not be economical. A more cost effective solution is to exploit the space dimension by employing multiple antennas at the transmitter and receiver. Multiple input multiple output (MIMO) communication has thus been the focus of a lot of research [1], [2], [3] which basically demonstrated that the capacity of a point to point MIMO link increases linearly with the number of transmit and receive antennas (an excellent overview of the research on this problem can be found in [4]).

Research focus has shifted recently to the role of multiple antennas in multiuser systems, especially broadcast scenarios (i.e., *point to multipoint* communication) as downlink scheduling is the major bottleneck for future broadband wireless networks. The broadcast channel resembles downlink communication in a cellular system, where the base station is to transmit to a group of users. In these and other broadcast scenarios, one is usually interested in 1) quantifying the system capacity or the maximum possible sum rate to all users, 2) quantifying the scaling behavior of the sum rate for *large* number of users, and 3) devising computationally efficient algorithms for capturing most of the sum rate in the *large* number of users regime. In this paper, we distinguish between two types of broadcast scenarios depending on whether the users are interested in common information or not

A. The broadcast problem: The independent users case

In this problem, users are interested in independent information. Much work has been devoted to answer the three questions raised above for this problem. The capacity region question was recently settled by a technique similar to writing on dirty paper and hence bearing the name dirty paper coding (DPC). Specifically, [5] and [6] have shown that DPC is capable of achieving the maximum possible sum-rate capacity. Subsequently, [7] showed that DPC is able to achieve any point in the capacity region.

While DPC solves the broadcast problem optimally, it is computationally expensive and requires a great deal of feedback as the transmitter needs perfect channel state information of all users [8]. Thus, there has been increased interest to match the DPC capacity for large number of users using simple techniques. In [9] and [10], Sharif and Hassibi showed that the sum rate capacity behaves like $\rho \log n \log n$ for large n (where n is the total number of users and ρ is the signal to noise ratio). They also showed that opportunistic beam forming matches this limit. Other promising techniques for capturing most of the DPC capacity were proposed in [7], [10], [11], [12], [13], [14]. They all rely on multiuser diversity to match the DPC for large number of users. Here, each user experiences a different channel and therefore the transmitter can exploit this diversity and choose the set of users that have the best channel conditions.

B. The broadcast problem: The group of users case

The broadcast scenario considered above assumes that the various users are interested in *independent* streams of data. More common is the situation that one group of users would be interested in one stream of data, another group with another stream, and so on. An example where this might occur is digital audio and video broadcast where there is a limited number of shows and users are classified according to the shows they are interested in [15], [16], [17]. Here, similar questions to the (independent user) broadcast problem would be relevant.

To make the discussion more rigorous, assume that there are n users each equipped with a single antenna. The users are partitioned into K groups where each group is interested in the same stream of data. The transmitter, which is equipped with M antennas, is to schedule transmission to these groups so as to maximize the sum-rate capacity of the system. If the transmitter had one antenna only, this would be a trivial problem. For in this case, all channels involved would be single input single output. Thus, to transmit to any group of users, one simply needs to take care of the user with the weakest link (i.e., the one with the least channel gain). Such ordering of users, however, is not possible in the multiple antenna case and the problem becomes more challenging.

The group broadcast problem includes the (independent users) broadcast problem as a special case. Specifically, the independent users broadcast is a group broadcast problem in which each group consists of one user only. The other extreme is when all users belong to one group and are all interested in the same information. This is known as a multicast problem and has started to attract some attention recently. In [18], Khitsi et. al. characterized the system capacity for the two user binary multicast problem. When multicasting to more than 2 users or for Gaussian multicast, [18] was only able to obtain upper and lower bounds. In [19], Steinberg and Shamai considered a two user situation with a hybrid of broadcast and multicast in which the two users can have common as well as independent messages. Exact capacity expressions were only possible in the degraded message sets case (which is similar to having one antenna only at the base station).

By examining the techniques used in [18] and [19] and the results arrived at, one can be convinced that finding the exact capacity for multicast (let alone the more general group broadcast problem) can be quite challenging. As such, several authors have resorted instead to evaluate the capacity *asymptotes*. In his Masters Thesis [20], Khitsi considered the multicast problem where there is exactly one group of n users interested in a common message transmitted from a base station with M antennas. He showed that for *large* n, the capacity decreases in proportion to $n^{-\frac{1}{M}}$. In a recent paper [21], Jindal and Luo built on the work of Khisti and obtained the scaling order of various techniques when used in a multicast scenario. These techniques include transmit beamforming and group broadcast using spatially white or orthogonal signaling.

In this paper, we consider the multiple group in the large number of users and antennas regime. In contrast to [20] and [21] which consider the multicast problem, we consider the *multiple* group broadcast problem. Moreover, in a further contrast to [20] and [21] which obtain order relationships for the multicast problem, we obtain upper and lower bounds that more tightly characterize the system sum-rate capacity.

The paper is organized as follows. In the first part of the paper, we consider the large number of users (n) case and obtain upper bounds (in Section IV) and lower bounds (in Section V). In the rest of the paper, we consider the scaling for the large number of antennas (M) regime. We do so for $n = \beta M$ (Section VI) and for $n = e^M$ (Section VII). We set the stage, however, by introducing the system model.

II. System Model

Consider a broadcast channel with a base station equiped with M antennas and n users each equipped with a single receive antenna. The received signal at the ith user is given by

$$y_i = h_i^* s + n_i$$

where $h_i \sim CN(0, I_M)$ and is assumed to be iid over the users, s is the transmitted signal, and $n_i \sim CN(0, I_M)$ is the additive Gaussian noise. For simplicity of exposition, we will assume that the number of users in each group is $\frac{n}{K}$. The different number of users case can be treated similarly.

III. Scaling law for the minimum of a number of random variables

Group broadcast is intuitively limited by the worst of otherwise identical users. As such, we digress in this section to develop a theory for finding the minimum of a large number of random variables. To this end, let $x_1, x_2, \cdots, x_n$ be iid nonnegative random variables with CDF $F(x)$ and characteristic function $\phi(x)$. We would like to find the scaling law of the minimum of these random variables, $x_{\min} = \{x_1, x_2, \cdots, x_n\}$. The CDF of the mimimum is given by

$$F_{\min}(x) = 1 - (1 - F(x))^n$$

Now let's expand $F(x)$ in a Taylor series

$$F(x) = \sum_{i=0}^{\infty} F^{(i)}(0) \frac{x^i}{i!} \tag{1}$$

Note that $F(0) = 0$ and let $F_{\min}^{(i_0)}(0)$ be the first nonzero derivative of $F(x)$, then

$$F_{\min}(x) = 1 - \left(1 - \frac{F^{(i_0)}(0)}{i_0!} x^{i_0} - \sum_{i=i_0+1}^{\infty} F^{(i)}(0) \frac{x^i}{i!}\right)^n$$

Now, for large enough n, we have

$$F_{\min}\left(\frac{x}{n^{\frac{1}{i_0}}}\right) = 1 - \left(1 - \frac{F^{(i_0)}(0)}{i_0!} \frac{x^{i_0}}{n}\right)^n$$

Taking the limit as n grows yields

$$\lim_{n \to \infty} F_{\min}\left(\frac{x}{n^{\frac{1}{i_0}}}\right) = 1 - \exp\left(-\frac{F^{(i_0)}(0)}{i_0!} x^{i_0}\right) \tag{2}$$

The above expression shows that $F_{\min}(\frac{x}{n^{\frac{1}{i_0}}})$ is not concentrated. Rather, it converges to a distribution which is independent of n. We thus say that

$$\boxed{x_{\min} \text{ converges to } \frac{E}{n^{\frac{1}{i_0}}}}$$

where E is the expectation that arises from the distribution in (2).

$$E = \int_0^{\infty} \exp\left(-\frac{F^{(i_0)}(0)}{i_0!} x^{i_0}\right) \tag{3}$$

$$= \frac{C_{i_0}}{F^{(i_0)}(0)^{\frac{1}{i_0}}} \tag{4}$$

and where

$$C_{i_0} = \frac{\Gamma(\frac{1}{i_0})(i_0!)^{\frac{1}{i_0}}}{i_0}$$

It remains to find the least i_0 such that $F^{i_0}(0) \neq 0$. Fortunately, we can do so without having to explicitly find the CDF and its derivatives by relying instead on the initial value theorem

$$\boxed{\lim_{x \to 0} F^{(i_0)}(x) = \lim_{s \to \infty} s^{i_0} \phi(s)}$$

a) Example: Let's find the scaling laws for $\min_{h_i} \|h_i\|^2$ when h_i are iid $CN(0, R)$. The pdf and CDF of $\|h_i\|^2$ will both have different forms depending on the whether some of the eigenvalues λ_l of R are the same or different. On the other hand, the characteristic function takes one form and is given by

$$\phi(s) = \prod_{l=1}^{M} \frac{1}{1 + \lambda_l s}$$

From this, it is easy to see that

$$\lim_{s \to \infty} s^i \phi(s) = F^{(i)}(0) = 0 \quad \text{for } i < M$$

and that

$$\lim s^M \phi(s) = F^{(M)}(0) = \frac{1}{\prod_{l=1}^{M}} \frac{1}{\lambda_l} = \frac{1}{\det(R)}$$

We thus conclude that

$$\boxed{\min_i \|h_i\|^2 \text{ scales as } C_M \det(R)^{\frac{1}{M}} \frac{1}{n^{\frac{1}{M}}}}$$

IV. Upper Bounds

A. An Upper Bound Using the MAC-BC Duality

We use the MAC-BC duality [22] to obtain an upper bound on the group broadcast problem. Specifically, the maximum sum rate for K users, chosen one from each group, is given by

$$C_K \text{ users} = \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right) \tag{5}$$

Since, this rate has to appeal to all user groups, we can write

$$C \le \min_{h_{i_1}} \cdots \min_{h_{i_K}} \max_{\substack{b_k \ge 0 \\ \sum_{k=1}^{K} b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right) \tag{6}$$

This is an upper bound because the b_k's are optimized for each set of K users when in the group broadcast problem the b_k's should be fixed over all user groups. Now, to get rid of the determinant in (6), we use the arithmetic-mean geometric-mean (AM-GM) inequality $\det(A) \leq \left(\frac{\mathrm{tr}(A)}{M}\right)^M$ to write

$$\begin{aligned} C &\leq M \log(1 + \min_{h_{i_1}} \cdots \min_{h_{i_K}} \max_{b_k} \frac{1}{M} \sum_{k=1}^{K} b_k \|h_i\|^2) && (7) \\ &= M \log(1 + \frac{P}{M} \min_{h_{i_1}} \cdots \min_{h_{i_K}} \max_{k} \{\|h_{i_1}\|^2, \cdots, \|h_{i_K}\|^2\}) && (8) \\ &= M \log(1 + \frac{P}{M} \max_{k} \min_{h_{i_1}} \cdots \min_{h_{i_K}} \{\|h_{i_1}\|^2, \cdots, \|h_{i_K}\|^2\}) && (9) \\ &= M \log\left(1 + \frac{P}{M} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}\right) && (10) \end{aligned}$$

where the 3rd line follows from the Neuman-Peterson theorem. Alternatively, and with the aid of the relationship $\det(I + AA^*) = \det(I + A^*A)$, we can show that

$$C \leq K \log\left(1 + \frac{P}{K} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}\right) \tag{11}$$

From (10) and (11), we conclude that

$$\boxed{C \leq \min\{M, K\} \log\left(1 + \frac{P}{\min\{M,K\}} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}\right)} \tag{12}$$

Using the approximation that for small x, $\log(1 + x) \simeq x$, we can write

$$\boxed{C \leq P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}} \tag{13}$$

The above results apply for the iid case. In the correlated case, the maximization in (8) is done over h_i's with autocorrelation R and that results in a hit $\det(R)^{\frac{1}{M}}$ on the upper bound

$$\boxed{C \leq P C_M \det(R)^{\frac{1}{M}} \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}} \tag{14}$$

B. How loose is the upper bound?

The use of the AM-GM inequality might raise some concern about how tight the upper bound is. So, instead of using the AM-GM in (6), we use the approximation $\det(I + A) \sim 1 + tr(A)$. By doing so and going through the same arguments in (7)-(10), we obtain

$$C = \log\left(1 + P \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}\right) \tag{15}$$

or using the $\log(1 + x) \simeq x$ approximation, we get

$$\boxed{C = P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}} \tag{16}$$

which is the same as the bound (13). So we don't loose much by applying the AM-GM inequality.

C. An Alternative Upper Bound

An alternative upper bound is obtained by allocating all available power to one group of users only. The attainable rate in this case is given by

$$\begin{aligned} C_{\text{one group}} &= \max_{\mathrm{Tr}(B) \leq P} \min_{h_i} \log\left(1 + h_i^* B h_i\right) \\ &= \log\left(1 + \max_{\mathrm{Tr}(B) \leq P} \min_{h_i} h_i^* B h_i\right) \\ &= \log(1 + \frac{P}{M} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}) \\ &\simeq \frac{P}{M} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}} && (17) \end{aligned}$$

where in (17), we used the fact that the maximum in $\max_{\mathrm{Tr}(B) \leq P} \min_{h_i} h_i^* B h_i$ is attained at $B = \frac{P}{M} I$ (see [23]). Thus, the achievable rate for K such groups is upper bounded by K times the rate of (17)

$$C \leq K \frac{P}{M} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}$$

Combining this with (13) yields

$$\boxed{C \leq \min\{1, \tfrac{K}{M}\} P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

In a similar manner, we can easily obtain the effect of spatial correlation as

$$\boxed{C_{\text{corr}} \leq \min\{1, \tfrac{K}{M}\} \det(R)^{\frac{1}{M}} P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

V. Lower Bound

Having obtained an upper bound, we now quantify how various methods for scheduling (or resource allocation) behave for large number of users. This would give us an idea about the achievable rates and also provides lower bounds on the group broadcast problem. In what follows we consider the following scheduling schemes

1) Opportunistic beamforming
2) Scheduling by treating interference as noise
3) Time sharing

A. Opportunist beamforming

In random beamforming the transmitter attempts to choose the best M out of K users to transmit to. To do this, the transmitter uses its M antennas to send M random beams. Each user calculates the M SINR's (signal to interference and noise ratio), one SINR for each beam, and feeds back the maximum SINR along with its index. The transmitter would in turn rank the K users according to their SINR's and transmits to the M best ones. Not only does this method require much less feedback than the DPC approach, but it also asymptotically (i.e., in the presence of large number of users) achieves the same performance [9].

To be more specific, the transmitter chooses M random orthonormal beam vectors ϕ_m (of size $M \times 1$) generated according to an isotropic distribution. Now these beams are used to transmit the symbols $s_1(t), s_2(t), \ldots, s_M(t)$ by constructing the transmitted vector

$$s(t) = \sqrt{P} \sum_{m=1}^{M} \phi_m(t) s_m(t), \qquad t = 1, \ldots, T \tag{18}$$

After T channel uses, the transmitter independently chooses another set of orthogonal vectors $\{\phi_m\}$ and constructs the signal vector (according to (18)) and so on. From now on and for simplicity, we will drop the time index. The signal y_k at some k'th receiver is given by

$$\begin{aligned} y_k &= h_k^* s + n_k && (19)\\ &= \sqrt{P}\sum_{m=1}^{M} h_k^*\phi_m s_m + n_k, \qquad k=1,\ldots,K && (20)\end{aligned}$$

Here, $E(ss^*) = \frac{P}{M}I$ since the s_m's are assumed to be identically distributed and independently assigned to different users. The k'th receiver estimates the effective channel gain $h_k^*\phi_m$, something that can be arranged by training, to calculate M SINR's, one for each transmitted beam

$$\mathrm{SINR}_{k,m} = \frac{|H_k\phi_m|^2}{\frac{M}{P} + \sum_{j\neq m}|H_k\phi_j|^2}, \qquad m = 1,\ldots,M \quad (21)$$

Each receiver then feeds back its maximum SINR, i.e. $\max\limits_{1\le m\le M}\mathrm{SINR}_{k,m}$, along with the maximizing index m. Thereafter, the transmitter assigns s_m to the user with the highest corresponding SINR, i.e. $\max\limits_{1\le k\le K}\mathrm{SINR}_{k,m}$. If we perform this kind of scheduling, the throughput for large n can be written as [10]

$$R_{RBF} = ME\log\left(1 + \max_{1\le i\le n}\mathrm{SINR}_{i,m}\right) + o(1) \quad (22)$$

where the term $o(1)$ accounts for the small probability that user k may be the strongest user for more than one beam ϕ_m [10].
In the group broadcast scenario, we replace each beam's SINR by the minimum SINR over all users in the group

$$R_{RBF} = ME\log\left(1 + \max_k\left\{\min_{i_1}\mathrm{SINR}_{i_1},\ldots,\min_{i_K}\mathrm{SINR}_{i_k}\right\}\right)$$

The SINR for the ith user of the kth group is given by

$$\mathrm{SINR}_{i_k} = \frac{|h_{i_k}^*\phi_1|^2}{\frac{M}{P} + \sum_{m=2}^{M}|h_{i_k}^*\phi_m|^2}$$

It is easy to show that SNIR pdf is given by

$$f(x) \quad = \quad \frac{e^{-\frac{x}{\rho}}}{(1+x)^M}\left(\frac{1}{\rho}(1+x) + M - 1\right)$$

from which we conclude that

$$F'_{\mathrm{SINR}}(0) = f_{\mathrm{SINR}}(0) = \frac{M}{P} + M - 1$$

It thus follows that the minimum SINR scales as

$$\begin{aligned}\min_{i_k}\mathrm{SINR}_{i_k} &= \frac{C_1}{F^{(1)}(0)}\\ &= \frac{1}{\frac{M}{P}+M-1}\frac{K}{n}\end{aligned}$$

and the sum-rate capacity would be

$$\boxed{R_{RBF} = M\log\left(1 + \frac{1}{\frac{M}{P}+M-1}\frac{K}{n}\right)} \quad (23)$$

B. Time Sharing

A tighter lower bound is obtained by time sharing. Thus, assume all groups take the same time share, then

$$\begin{aligned} R_{\mathrm{TS}} &\geq \frac{1}{K}\max_{\mathrm{Tr}(B)=P}\sum_{k=1}^{K}\min_{h_{i_k}}\log\det\left(1 + h_{i_k}^* B h_{i_k}\right) && (24)\\ &= \frac{1}{K}\max_{\mathrm{Tr}(B)=P}\sum_{k=1}^{K}\log\left(1 + \min_{h_{i_k}} h_{i_k}^* B h_{i_k}\right) && (25)\end{aligned}$$

We now relax the problem further by setting $B = \frac{P}{M}I$, from which we conclude that

$$R_{\mathrm{TS}} \geq \log\left(1 + \frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}\right)$$

or using the approximation $\log(1+x) = x$,

$$\boxed{R_{\mathrm{TS}} \geq \frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

Just like the upper bound, correlation results in a hit $\det(R)^{\frac{1}{M}}$ on the lower bound

$$\boxed{R_{\mathrm{TS}} \geq \frac{P}{M}C_M\det(R)^{\frac{1}{M}}\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

C. Treating Interference as Noise

The other extreme would be to allow all groups to transmit simultaneously. Each group would then ignore signals that are meant for the other groups, treating them as additive noise. The rate that the 1st group achieves with this strategy would be

$$R_1 = \min_{h_{i_1}}\log\left(\frac{h_{i_1}^* B_{1_{i_1}} h_{i_1}}{1 + h_{i_1}^*\sum_{k=2}^{K} B_{k_{i_1}} h_{i_1}}\right)$$

Now, relax the problem further by assuming equal isotropic covariances for all user groups, i.e. set

$$B_k = \frac{1}{K}\frac{P}{M}I$$

then

$$\begin{aligned} R_1 &= \log\min_{i_1}\frac{\frac{1}{K}\frac{P}{M}\|h_{i_1}\|^2}{1 + \frac{K-1}{K}\frac{P}{M}\|h_{i_1}\|^2}\\ &= \log\min_{i_1}\frac{1}{K-1}\left(1 - \frac{1}{1 + \frac{K-1}{K}\frac{P}{M}\|h_{i_1}\|^2}\right)\\ &= \log\frac{1}{K-1}\left(1 - \frac{1}{1 + \frac{K-1}{K}\frac{P}{M}\min_{i_1}\|h_{i_1}\|^2}\right)\\ &= \log\left(\frac{\frac{1}{K}\frac{P}{M}\min_{i_1}\|h_{i_1}\|^2}{1 + \frac{K-1}{K}\frac{P}{M}\min_{i_1}\|h_{i_1}\|^2}\right)\\ &= \log\left(\frac{\frac{1}{K}\frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}{1 + \frac{K-1}{K}\frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}\right)\\ &\simeq \frac{1}{K}\frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}\end{aligned}$$

Thus, the sum rate for K such user groups is upper bounded according to

$$\boxed{R_{\mathrm{Interf}} \geq \frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

Correlation will again introduce a hit $\det(R)^{\frac{1}{M}}$ on the lower bound.

$$\boxed{R_{\mathrm{Interf,\ corr}} \geq \det(R)^{\frac{1}{M}}\frac{P}{M}C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

From the bounds obtained in this section and the previous section, we conclude that the group broadcast capacity scales as

$$\boxed{C = \alpha P C_M\frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

where

$$\frac{1}{M} \leq \alpha \leq \min\{1, \frac{K}{M}\}$$

For the spatially correlated case, the capacity incurs a $\det(R)^{\frac{1}{M}}$ hit on the SINR

$$\boxed{C_{\text{corr}} = \alpha \det(R)^{\frac{1}{M}} P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

This is an unfortunate result as it shows that the sum-rate decreases with the number of users. To counter this, we increase the resources (i.e., number of antennas M). In the rest of this paper, we study the scaling of group broadcast capacity with the number of antennas for 1) $\frac{M}{n} = \beta$ and 2) $M = \log n$.

VI. Scaling with M and n, $\frac{M}{n} = \beta$

Here we consider the scaling of the upper and lower bounds when both the number of users and antennas grow to infinity while their ratio remains constant $\frac{M}{n} = \beta$. To this end, note first that both the upper and the lower bounds depend on the value of $\min_{h_i} \frac{\|h_i\|^2}{M}$ and so we need to evaluate the scaling of this quantity as $n, M \to \infty$. To do this, define the matrix

$$\Psi = H_i^* H_i$$

where

$$H_i = \begin{bmatrix} h_1 & h_2 & \cdots & h_{\frac{n}{K}} \end{bmatrix}$$

Note that $\text{diag}(\Psi) = \begin{bmatrix} \|h_1\|^2 & \|h_2\|^2 & \cdots & \|h_{\frac{n}{K}}\|^2 \end{bmatrix}^T$. Note also that

$$\lambda_{min}(\frac{\Psi}{M}) \leq \min_i \frac{\|h_i\|^2}{M} \leq \lambda_{max}(\frac{\Psi}{M}) \leq \max_i \frac{\|h_i\|^2}{M} \tag{26}$$

Moreover as $n, M \to \infty$ with $\frac{M}{\frac{n}{K}} = K\beta$, the eigenvalues of $\frac{\Psi}{M}$ become uniformly distributed in the range $[(1-\sqrt{K\beta})^2, (1+\sqrt{K\beta})^2]$. We can thus write

$$(1 - \sqrt{K\beta})^2 \leq \lim_{n,M\to\infty} \min_i \frac{\|h_i\|^2}{M} \leq (1 + \sqrt{K\beta})^2$$

This allows us to get a lower bound on capacity which is obtained using time-sharing

$$C \geq K \max_{B\geq 0 \ \text{Tr}(B)\leq P} \frac{1}{K} \log(1 + \min_i h_i^* B h_i) \tag{27}$$

$$C \geq \log\left(1 + P \min_i \frac{\|h_i\|^2}{M}\right) \tag{28}$$

i.e.,

$$\boxed{C \geq \log\left(1 + P(1 - \sqrt{K\beta})^2\right)} \tag{29}$$

We obtain the upper bound through another matrix construction. Our starting point is the bound (see Subsection IV-C)

$$\begin{aligned} C &\leq K \max_{B\geq 0 \ \text{Tr}(B)\leq P} \log(1 + \min_i h_i^* B h_i) \\ &= K \log(1 + \max_{B\geq 0 \ \text{Tr}(B)\leq P} \min_i h_i^* B h_i) \end{aligned}$$

We need an upper bound for $\max_{B\geq 0 \ \text{Tr}(B)\leq P} \min_i h_i^* B h_i$. To do so, we replace the minimization over the h_i's with the sum average (as done in [21])

$$\begin{aligned} \max_{B\geq 0 \ \text{Tr}(B)\leq P} \min_i h_i^* B h_i &\leq \frac{1}{n} \max_{B\geq 0 \ \text{Tr}(B)\leq P} \sum_{i=1}^{n} h_i^* B h_i \\ &= \frac{1}{n} \max_{B\geq 0 \ \text{Tr}(B)\leq P} \sum_{i=1}^{n} \text{Tr}(B h_i h_i^*) \\ &= \frac{1}{n} \max_{B\geq 0 \ \text{Tr}(B)\leq P} \text{Tr}\left(B \sum_{i=1}^{n} h_i h_i^*\right) \\ &= \frac{1}{n} \max_{B\geq 0 \ \text{Tr}(B)\leq P} \text{Tr}\left(B H_i H_i^*\right) \\ &= \frac{1}{n} P \lambda_{max}(H_i H_i^*) \end{aligned}$$

Now, as $n, M \to \infty$ with $\frac{M}{\frac{n}{K}} = K\beta$, the eigenvalues of $\frac{H_i H_i^*}{n}$ will be confined to the range $[(1 - \frac{1}{\sqrt{K\beta}})^2, (1 + \frac{1}{\sqrt{K\beta}})^2]$. We can thus obtain the following upper bound on capacity

$$\boxed{C \leq K \log(1 + P(1 + \tfrac{1}{\sqrt{K\beta}})^2)}$$

Thus, if we allow the number of antennas to grow linearly with the number of users, we can guarantee a constant sum rate. But is it still possible to do so without straining the resources as much?

VII. Scaling with M and n, $M = \log n$

For $M \gg K$, the group capacity sales as $\alpha \frac{P}{M} C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}$. Now it is easy to see that $C_M \simeq M$ for large M, so the sum-rate scales approximately as $\alpha \frac{P}{n^{\frac{1}{M}}}$. Thus, to guarantee constant rate, we need to set $M = \log n$. To prove this rigourously, let's study the behavior of $\min_i \frac{\|h_i\|^2}{M}$ for $M = \log n$ which we do using the Chernof bound. To this end, let $Y = \frac{\|h_i\|^2}{M}$, and define $g(Y)$ by

$$g(Y) = \begin{cases} 1 & \text{if } Y \leq 1 - \epsilon \\ 0 & \text{if } Y > 1 - \epsilon \end{cases}$$

Then for $\nu \geq 0$

$$g(Y) \leq e^{-\nu(Y-(1-\epsilon))}$$

and hence

$$E[g(Y)] = P(Y \leq 1 - \epsilon) \leq e^{\nu(1-\epsilon)} E[e^{-\nu Y}]$$

or

$$P(\frac{\|h_i\|^2}{M} \leq 1 - \epsilon) = e^{\nu(1-\epsilon)} \frac{1}{\left(1 + \frac{\nu}{M}\right)^M} \tag{30}$$

Now we can tighten the upper bound by choosing the optimum ν, which, upon setting the first derivative to zero, turns out to be

$$\nu = M \frac{\epsilon}{1 - \epsilon} > 0$$

and the bound reads

$$P(\frac{\|h_i\|^2}{M} \leq 1 - \epsilon) \leq e^{M\epsilon}(1 - \epsilon)^M \tag{31}$$

$$= e^{M(\epsilon + \log(1-\epsilon))} \tag{32}$$

We can use this to bound the probability $P(\min_i \frac{\|h_i\|^2}{M} \leq 1 - \epsilon)$

$$P(\min_i \frac{\|h_i\|^2}{M} \leq 1 - \epsilon) = 1 - (1 - P(\frac{\|h_i\|^2}{M} \leq 1 - \epsilon))^n \tag{33}$$

$$\leq 1 - (1 - e^{M(\epsilon + \log(1-\epsilon))})^n \tag{34}$$

$$= 1 - (1 - n^{\epsilon + \log(1-\epsilon)})^n \tag{35}$$

where the last line follows from the fact that $M = \log n$. For the above probability to vanish as n grows, we require that

$$\epsilon + \log(1-\epsilon) < -1$$

Let ϵ_l be the infimum of the set $\{\epsilon : \epsilon + \log(1-\epsilon) < -1\}$, (i.e. ϵ_l satisfies $\epsilon_l + \log(1-\epsilon_l) = -1$ or $\epsilon_l \simeq .8414$). Then,

$$\boxed{\lim_{n\to\infty} P(\min_i \frac{\|h_i\|^2}{M} \geq 1-\epsilon_l) = 1} \tag{36}$$

Now let's obtain an upper bound for $\min_i \frac{\|h_i\|^2}{M}$. Employing Chernof bound again, it is easy to show that for $\nu \geq 0$

$$P(\frac{\|h_i\|^2}{M} \geq 1+\epsilon) \leq e^{-\nu(1+\epsilon)} E[e^{\nu \frac{\|h_i\|^2}{M}}] \tag{37}$$

$$= e^{-\nu(1+\epsilon)} \frac{1}{(1-\frac{\nu}{M})^M} \tag{38}$$

Moreover, the upper bound is tightest for

$$\nu = M\frac{\epsilon}{1+\epsilon}$$

We thus have

$$\begin{aligned} P(\frac{\|h_i\|^2}{M} \geq 1+\epsilon) &\leq e^{-M\epsilon}(1+\epsilon)^M \\ &= e^{M(-\epsilon+\log(1+\epsilon))} \end{aligned}$$

or

$$P(\min_{h_i} \frac{\|h_i\|^2}{M} \geq 1+\epsilon) \leq (n^{(-\epsilon+\log(1+\epsilon))})^n$$

where we used the fact that $n = \log M$. This probability vanishes provided that $-\epsilon + \log(1+\epsilon) < 0$ and the infimum for which this is true is $\epsilon_u = 0$. We can thus write

$$\boxed{\lim_{n\to\infty} P\left(\min_i \frac{\|h_i\|^2}{M} \leq 1\right) = 1} \tag{39}$$

From (36) and (39), we see that

$$\boxed{\lim_{n\to\infty} P\left(\min_i \frac{\|h_i\|^2}{M}\right) = \mathcal{H} \in [1-\epsilon_l, 1] \quad \text{w.p.1}} \tag{40}$$

A. Lower bound for sum-rate capacity ($M = \log n$)

We are now ready to derive the lower bound for the sum rate capacity which we obtain through time sharing. Specifically, we have

$$\max_{B\geq 0 \ \mathrm{Tr}(B)\leq P} \log(1+\min_i h_i^* B h_i) = \log(1+P\min_i \frac{\|h_i\|^2}{M})$$

where the second inequality follows by setting $B = \frac{P}{M}I$. Or with $M = \log n$ and as $n \to \infty$

$$\boxed{C \geq \log(1+P\mathcal{H})} \tag{41}$$

This lower bound shows that a growth of $M = \log n$ will guarantee a constant capacity because for $M = \beta n$ the sum rate is upper bounded by a constant.

VIII. Conclusion

In this paper, we studied the scaling of multigroup broadcast for large number of users. Specifically, we obtained upper and lower bounds for the sum-rate capacity in the large number of users regime. We showed that the sum rate capacity scales as $\alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}$. We also quantified the effect of the spatial correlation as a hit $\det(R)^{\frac{1}{M}}$ on the SNR. This is an unfortunate result as it shows that the capacity decreases with the number of users. To go around this, we studied the scaling of the group broadcast capacity with the number of users and antennas. Specifically, we showed that if we set $M = \log n$, we can guarantee a constant rate for each user in spite of the increase in the number of users.

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