Exact Tracking Analysis of the ϵ-NLMS Algorithm for Circular Complex Correlated Gaussian Input

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***Abstract*—This work presents exact tracking analysis of the ϵ-normalized least mean square (ϵ-NLMS) algorithm for circular complex correlated Gaussian input. The analysis is based on the derivation of a closed form expression for the cumulative distribution function (CDF) of random variables of the form $[||\mathbf{u}_i||^2_{\mathbf{D}_1}][\epsilon+||\mathbf{u}_i||^2_{\mathbf{D}_2}]^{-1}$. The CDF is then used to derive the first and second moments of these variables. These moments in turn completely characterize the tracking performance of the ϵ-NLMS algorithm in explicit closed form expressions. Consequently, new explicit closed-form expressions for the steady state tracking excess mean square error and optimum step size are derived. The simulation results of the tracking behavior of the filter match the expressions obtained theoretically for various degrees of input correlation and for various values of ϵ.**

***Index Terms*—Adaptive algorithms, tracking performance, indefinite quadratic forms.**

I. INTRODUCTION

It is well known that LMS suffers from slow convergence when the input is correlated. The ϵ-NLMS algorithm [1] is a variation of the LMS algorithm that exhibits faster convergence in the presence of correlated input. Though several works have attempted to study the performance of the ϵ-NLMS algorithm [1]–[12], their results are mostly approximate as they rely upon strong assumptions, e.g., separation principal [2], [3], white input [5], [6], [7], [8], specific structure of input regressor's distribution [3], [9], [10], small step size [3], long filters [7] and approximate solutions using Abelian integrals [2]. Closed form expressions for transient analysis and steady-state MSE of the ϵ-NLMS algorithm are derived in [12] but these are in terms of multidimensional moments for which no closed form solutions are available.

In [13], we present a novel approach for evaluating mean square and transient performance of the ϵ-NLMS algorithm for correlated complex Gaussian data. Here we use the same framework to perform its tracking analysis. Our approach relies on evaluating the CDF of random variables of the form[1] $[||\mathbf{u}_i||^2_{\mathbf{D}_1}][\epsilon + ||\mathbf{u}_i||^2_{\mathbf{D}_2}]^{-1}$ where $\mathbf{D}_1$ and $\mathbf{D}_2$ are diagonal matrices. This is done by rewriting these variable as indefinite quadratic forms in Gaussian random variables and using complex integration to evaluate the CDF *directly*[2]. Using this CDF, we can evaluate all the moments that appear in the tracking analysis of ϵ-NLMS in closed form.

The main contributions of the paper are as follows:

a) Analysis presented in this work is generalized[3] as it is not restricted to specific input correlation matrix, small step-size, independent regressor elements, and white input.
b) Closed form expressions for tracking excess mean square error (EMSE) and optimum step size are developed for the ϵ-NLMS algorithm.

The paper is organized as follows. Following this introduction, the system model is described in Section II. The tracking performance analysis of the ϵ-NLMS algorithm is presented in Section III. Derivation for the CDF of the random variable of the form $[||\mathbf{u}_i||^2_{\mathbf{D}_1}][\epsilon + ||\mathbf{u}_i||^2_{\mathbf{D}_2}]^{-1}$ is carried out in Section IV which is used in Section V to evaluate the moments. Simulation results are presented in Section VI investigating the performance of the derived analytical model. Finally, concluding remarks are given in Section VII.

II. SYSTEM MODEL

Let $\mathbf{w}^o(i)$ be the unknown system to be tracked and i be the time index. It is assumed that $\mathbf{w}^o(i)$ is changing according to

$$\mathbf{w}^o(i) = \mathbf{w}^o(i-1) + \mathbf{q}(i) \qquad (1)$$

where $\mathbf{q}(i)$ is assumed to be i.i.d with mean zero and covariance matrix

$$E[\mathbf{q}(i)\mathbf{q}(i)^{\mathrm{H}}] = \mathbf{Q} \qquad (2)$$

where $(.)^{\mathrm{H}}$ stands for conjugate transpose. The update rule of ϵ-NLMS algorithm is given as

$$\mathbf{w}(i) = \mathbf{w}(i-1) + \mu\frac{\mathbf{u}(i)^{\mathrm{H}}}{\epsilon + ||\mathbf{u}(i)||^2}e(i) \qquad i \geq 0 \qquad (3)$$

where $\mathbf{u}(i)$ is input regression vector, $\mathbf{w}(i)$ is an estimate of the unknown system $\mathbf{w}^o(i)$, and $e(i)$ is the estimation error

[1] For any matrix $\mathbf{A}$, $||\mathbf{u}_i||^2_{\mathbf{A}} \triangleq \mathbf{u}_i\mathbf{A}\mathbf{u}_i^*$.

[2] This comes in direct contrast to the usual approach of evaluating the characteristic function first and inverting it to obtain the pdf.

[3] Our analysis is exact up to the independence assumption.

defined by

$$\begin{aligned} e(i) &= d(i) - \mathbf{u}(i)\mathbf{w}(i-1) \\ &= \mathbf{u}(i)\mathbf{w}^o - \mathbf{u}(i)\mathbf{w}(i-1) + v(i) \\ &= e_a(i) + v(i) \end{aligned} \quad (4)$$

where $d(i)$ is the desired response, $v(i)$ is a zero mean i.i.d noise with variance σ_v^2 that is independent of the input and $e_a(i)$ is excess mean square error. Thus (3) can be written as

$$\mathbf{w}(i) = \mathbf{w}(i-1) + \mu\mathbf{u}(i)^{\mathrm{H}} g[e(i)] \qquad i \geq 0 \quad (5)$$

where

$$g[e(i)] = \frac{e_a(i) + v(i)}{\epsilon + ||\mathbf{u}(i)||^2} \quad (6)$$

Let $\tilde{\mathbf{w}}(i) = \mathbf{w}^o(i) - \mathbf{w}(i)$ denote the weight error vector, subtracting both sides of (5) from $\mathbf{w}^o(i)$, the update rule of ϵ-NLMS can be restated as

$$\tilde{\mathbf{w}}(i) = \tilde{\mathbf{w}}(i-1) - \mu\mathbf{u}(i)^{\mathrm{H}} g[e(i)] \qquad i \geq 0. \quad (7)$$

Here we will restrict our attention to circularly symmetric Gaussian inputs, i.e. $\mathbf{u}(i) \sim \mathcal{CN}\ (\mathbf{0}, \mathbf{\Lambda})$. For the sake of tracking analysis, the autocorrelation matrix $\mathbf{\Lambda}$ can be assumed to be diagonal, that is, $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_M)$, without any loss of generality [12].

III. Tracking analysis of the ϵ-NLMS algorithm

An energy conservation approach was used in [12] to study the performance of data normalized adaptive filters. There it was shown that for ϵ-NLMS, the tracking EMSE is given by

$$e_a(i) = \frac{\mu\alpha_u\sigma_v^2 + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{2\eta_u - \mu\alpha_u} \quad (8)$$

where

$$\alpha_u \triangleq E\left[\frac{||\mathbf{u}(i)||^2}{(\epsilon + ||\mathbf{u}(i)||^2)^2}\right] \quad (9)$$

and

$$\eta_u \triangleq E\left[\frac{1}{\epsilon + ||\mathbf{u}(i)||^2}\right] \quad (10)$$

Thus the tracking EMSE of the ϵ-NLMS is completely characterized by these multidimensional moments.

Differentiating (8) with respect to μ and equating it to zero leads to the following expression for optimum step size

$$\mu_{\mathrm{opt}} = \frac{1}{2\eta_u\sigma_v^2}\sqrt{\frac{\mathrm{Tr}(\mathbf{Q})\left[\alpha_u\mathrm{Tr}(\mathbf{Q}) - 4\eta_u^2\sigma_v^2\right]}{\alpha_u}} - \frac{\mathrm{Tr}(\mathbf{Q})}{2\eta_u\sigma_v^2} \quad (11)$$

By inspecting equations (8) and (11), we conclude that the tracking performance is completely characterized by the multidimensional-moment terms (9)-(10). These moments are in turn completely determined by the mean of the following random variables

$$\frac{1}{\epsilon + ||\mathbf{u}||^2},\ \frac{|u_k|^2}{(\epsilon + ||\mathbf{u}||^2)^2},\ (k \neq \bar{k}) \quad (12)$$

Consider the following generalized random variable

$$r_{k\bar{k}}(\alpha, \bar{\alpha}, \gamma) \triangleq \frac{\alpha|u_k|^2 + \bar{\alpha}|u_{\bar{k}}|^2 + \gamma}{\epsilon + ||\mathbf{u}||^2},\ k \neq \bar{k} \quad (13)$$

where α, $\bar{\alpha}$, and γ are constants appropriately chosen from the set $\{0, 1\}$ in order to retrieve required random variables from (12). The dependence of the input regressor vector $\mathbf{u}(i)$ on the time index i has been dropped to simplify the notation. A close examination of (13) shows that the moment in (10) can be obtained by setting $\alpha = 0$, $\bar{\alpha} = 0$, and $\gamma = 1$.

$$\eta_u = E\left[\frac{1}{\epsilon + ||\mathbf{u}(i)||^2}\right] = E[r_{k\bar{k}}(0,0,1)] \quad (14)$$

Further, from (9) and (12), α_u can be expressed as a summation of random variables given as

$$\alpha_u = E\left[\frac{||\mathbf{u}(i)||^2}{(\epsilon + ||\mathbf{u}(i)||^2)^2}\right] = \sum_{k=1}^{M} E\left[\frac{|u_k(i)|^2}{(\epsilon + ||\mathbf{u}(i)||^2)^2}\right] \quad (15)$$

Using binomial expansion, we can rewrite each term of the summation as

$$\begin{aligned} \frac{|u_k|^2}{(\epsilon + ||\mathbf{u}||^2)^2} &= \frac{1}{2}\left(\frac{|u_k|^2 + 1}{\epsilon + ||\mathbf{u}||^2}\right)^2 - \frac{1}{2}\left(\frac{|u_k|^2}{\epsilon + ||\mathbf{u}||^2}\right)^2 \\ &\quad -\frac{1}{2}\left(\frac{1}{\epsilon + ||\mathbf{u}||^2}\right)^2 \\ \frac{|u_k|^2}{(\epsilon + ||\mathbf{u}||^2)^2} &= \frac{1}{2}r_{k\bar{k}}^2(1,0,1) - \frac{1}{2}r_{k\bar{k}}^2(1,0,0) \\ &\quad -\frac{1}{2}r_{k\bar{k}}^2(0,0,1). \end{aligned} \quad (16)$$

The off-diagonal elements (when $k \neq \bar{k}$) are zero; as $[u_k^* u_{\bar{k}}][\epsilon + ||\mathbf{u}_i||^2]^{-1}$ has a symmetric pdf and is an odd function of $u_{\bar{k}}$ where $u_{\bar{k}}$ is independent of the remaining elements of $\mathbf{u}$. Thus all that is needed to evaluate the multidimensional moments of (9) and (10) is to evaluate the first two moments of the random variable $r_{k\bar{k}}(\alpha, \bar{\alpha}, \gamma)$. We will find these moments from the CDF of $r_{k\bar{k}}(\alpha, \bar{\alpha}, \gamma)$ which is derived subsequently.

IV. The CDF of Indefinite Quadratic Forms

In this section, we derive the CDF of random variable $r_{k\bar{k}}$ (which is an Indefinite Quadratic form) for correlated circular complex input data with Gaussian distribution. For notational convenience, we suppress the dependence of $r_{k\bar{k}}$ on its arguments. The CDF of $r_{k\bar{k}}$, denoted by $F_{r_{k\bar{k}}}(x)$, is defined as

$$F_{r_{k\bar{k}}}(x) = P(r_{k\bar{k}} \leq x). \quad (17)$$

From equation (13), we can easily see that the above CDF can be equivalently expressed as

$$F_{r_{k\bar{k}}}(x) = P(\epsilon x + x||\mathbf{u}||^2 - \alpha|u(k)|^2 - \bar{\alpha}|u(\bar{k})|^2 - \gamma \geq 0). \quad (18)$$

Consequently, the CDF of $r_{k\bar{k}}$ can be obtained using the integral

$$F_{r_{k\bar{k}}}(x) = \int p(\mathbf{u})\tilde{u}(\epsilon x + x||\mathbf{u}||^2 - \alpha|u(k)|^2 - \bar{\alpha}|u(\bar{k})|^2 - \gamma)d\mathbf{u} \quad (19)$$

where $p(\mathbf{u})$ is the pdf of $\mathbf{u}$ and $\tilde{u}(\cdot)$ is the unit step function. Since we are dealing with M-dimensional circular Gaussian input vector, the pdf of $\mathbf{u}$ with a diagonal covariance matrix $\mathbf{\Lambda}$, is given by

$$p(\mathbf{u}) = \frac{1}{\pi^M|\mathbf{\Lambda}|}e^{-\mathbf{u}\mathbf{\Lambda}^{-1}\mathbf{u}^*}. \quad (20)$$

As done in [14], we can represent the unit step function using the following integral

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{x(j\omega+\beta)}}{(j\omega+\beta)}d\omega. \quad (21)$$

Substituting equations (20) and (21) in (19) yields the following multidimensional integral

$$F_{r_{k\bar{k}}}(x) = \frac{1}{2\pi^{M+1}|\mathbf{\Lambda}|}\int_{-\infty}^{\infty}\times \int e^{-\mathbf{u}\left[\mathbf{\Lambda}^{-1}-(x\mathbf{I}-\mathbf{D}_{k\bar{k}})(j\omega+\beta)\right]\mathbf{u}^*}d\mathbf{u}\frac{e^{-(\gamma-\epsilon x)(j\omega+\beta)}}{(j\omega+\beta)}\, d\omega \quad (22)$$

where $\mathbf{D}_{k\bar{k}}$ is an $M\times M$ matrix with all elements equal to zero except the k^{th} and $\bar{k}^{th}$ elements in the main diagonal which are equal to α and $\bar{\alpha}$, respectively. In the above equation, the inner integral is nothing but the Gaussian integral. Thus, it can be shown that the CDF of $r_{k\bar{k}}$ is reduced to the following one - dimensional complex integral

$$F_{r_{k\bar{k}}}(x) = \frac{1}{\pi|\mathbf{\Lambda}|}\int_{-\infty}^{\infty}\frac{e^{-(\gamma-\epsilon x)(j\omega+\beta)}d\omega}{\left|\mathbf{\Lambda}^{-1}-(x\mathbf{I}-\mathbf{D}_{k\bar{k}})(j\omega+\beta)\right|(j\omega+\beta)}. \quad (23)$$

Since the matrices involved in the determinant are diagonal, we can expand the fraction inside the integral using partial fraction expansion. Specifically, we can show that the determinant in the above equation is equal to

$$\frac{A_k}{\left[\frac{1}{\lambda_k}-(x-\alpha)(j\omega+\beta)\right]}+\frac{A_{\bar{k}}}{\left[\frac{1}{\lambda_{\bar{k}}}-(x-\bar{\alpha})(j\omega+\beta)\right]} + \frac{A_o}{(j\omega+\beta)}+\sum_{m=1,m\neq k,\bar{k}}^{M}\frac{A_m}{\left[\frac{1}{\lambda_m}-x(j\omega+\beta)\right]} \quad (24)$$

where the constants A_0, A_k, $A_{\bar{k}}$, and $A_m(k\neq\bar{k}, m = 1,2,\cdots,M, m\neq k,\bar{k})$ are given by

$$A_0 = \frac{1}{|\mathbf{\Lambda}^{-1}|},\quad A_k = \frac{\lambda_k^M(x-\alpha)\delta(\alpha-1)}{\left[\zeta_{k\bar{k}}-\frac{(x-\bar{\alpha})}{(x-\alpha)}\right]\Pi_{i=1,i\neq k,\bar{k}}^{M}\left[\zeta_{ki}-\frac{x}{(x-\alpha)}\right]},$$

$$A_{\bar{k}} = \frac{\lambda_{\bar{k}}^M(x-\bar{\alpha})\delta(\bar{\alpha}-1)}{\left[\zeta_{\bar{k}k}-\frac{(x-\alpha)}{(x-\bar{\alpha})}\right]\Pi_{i=1,i\neq k,\bar{k}}^{M}\left[\zeta_{\bar{k}i}-\frac{x}{(x-\bar{\alpha})}\right]} \quad (25)$$

$$\text{and } A_m = \frac{\lambda_m^M x}{p_1(\alpha)p_2(\bar{\alpha})p_3} \quad (26)$$

where $\zeta_{ij} = \frac{\lambda_i}{\lambda_j},\ \ \forall\ \ i,j$ while

$$\begin{aligned} p_1(\alpha) &= \left\{(\zeta_{mk}-1)\,\delta(\alpha)+\left[\zeta_{mk}-\frac{(x-\alpha)}{x}\right]\delta(\alpha-1)\right\}\\ p_2(\bar{\alpha}) &= \left\{(\zeta_{m\bar{k}}-1)\,\delta(\bar{\alpha})+\left[\zeta_{m\bar{k}}-\frac{(x-\bar{\alpha})}{x}\right]\delta(\bar{\alpha}-1)\right\}\\ p_3 &= \prod_{i=1,i\neq k,\bar{k},m}^{M}(\zeta_{mi}-1) \end{aligned} \quad (27)$$

and $\delta(.)$ represents the impulse function. Thus reducing the integral in equation (23) into a sum of $M+1$ simple complex integrals. Ultimately, the CDF of $r_{k\bar{k}}$ can be expressed in closed form given by (28).

V. Evaluating the Moments

In this section, we use the CDF of $r_{k\bar{k}}$ to evaluate the first two moments of $r_{k\bar{k}}$. The detailed derivation is provided in [13]. Here we reproduce part of the derivation for the special case of $\alpha=0$, $\bar{\alpha}=0$, and $\gamma=1$

A. First Moment of $r_{k\bar{k}}$

The moment of the first random variable in (12) can be calculated directly from the first moment of $r_{k\bar{k}}$ by setting $\alpha=0$, $\bar{\alpha}=0$, and $\gamma=1$. Since $r_{k\bar{k}}$ is a positive random variable with support between zero and $[\delta(1)+\frac{1}{\epsilon}\delta(0)]$, its first moment can be expressed directly in terms of the CDF using integration by parts

$$\begin{aligned} E[r_{k\bar{k}}] &= \int_0^{[\delta(1)+\frac{1}{\epsilon}\delta(0)]}\left[1-F_{r_{k\bar{k}}}(x)\right]dx\\ &= \frac{1}{\epsilon}\delta(0)-I_1-\bar{I}_1\\ &\quad -\sum_{m=1,m\neq k,\bar{k}}^{M} I_{1m}. \end{aligned} \quad (29)$$

The above integrals are evaluated using partial fraction expansion. Due to lack of space, intermediate steps are omitted and only final solutions of these integrals are reported in (30)-(33).

1) Integral Solution for I_1: The integral solution of I_1 can be shown to be

$$\begin{aligned} I_1 &= \int_{\alpha}^{\frac{1}{\epsilon}}\frac{A_k e^{\frac{-(1-\epsilon x)}{\lambda_k(x)}}}{|\mathbf{\Lambda}|(x)}\delta(-1)dx\\ &= -\left[C_1\ e^{\frac{(\epsilon-1)}{\lambda_k}}\ E_1\left(\frac{\epsilon}{\lambda_k}\right)+\frac{(\epsilon-1)}{\epsilon}C_2\ e^{\frac{(\epsilon-1)}{\lambda_k}}\ E_2\left(\frac{\epsilon}{\lambda_k}\right)\right.\\ &\quad + C'_{\bar{k}}\ e^{\frac{(\epsilon-1)}{\lambda_{\bar{k}}}}\ E_1\left(\frac{1}{\lambda_k}-\frac{1-\epsilon}{\lambda_{\bar{k}}}\right)\\ &\quad \left.+\sum_{m=1,m\neq k,\bar{k}}^{M} C'_m\ e^{\frac{(\epsilon-1)}{\lambda_m}}\ E_1\left(\frac{1}{\lambda_k}-\frac{1-\epsilon}{\lambda_m}\right)\right]\delta(-1) \end{aligned} \quad (30)$$

where $E_n(x) \triangleq \int_1^{\infty}\frac{e^{-xt}}{t^n}dt$ is the generalized exponential integral function while the constants C_1, C_2, $C'_{\bar{k}}$, and C'_m $(m=1,2,\cdots,M,\ m\neq k,\bar{k})$ are defined as

$$C_1 = \frac{-C_2\delta(0)}{(\zeta_{k\bar{k}}-1)}+\frac{C_2 D'_-}{p_4}, C_2 = \frac{\lambda_k^M\ e^{\frac{1}{\lambda_k}}}{|\mathbf{\Lambda}|(\zeta_{k\bar{k}}-1)p_4},$$

$$C'_{\bar{k}} = \frac{\lambda_k^M\ e^{\frac{1}{\lambda_k}}\delta(0)}{|\mathbf{\Lambda}|(1-\zeta_{k\bar{k}})^2 p_5},$$

$$C'_m = \frac{\lambda_k^M\ e^{\frac{1}{\lambda_k}}}{|\mathbf{\Lambda}|(1-\zeta_{km})^2\left[\zeta_{k\bar{k}}-\delta(0)\zeta_{km}-\delta(-1)\right]p_6} \quad (31)$$

$$F_{r_{k\bar{k}}}(x) = \tilde{u}(\epsilon x-\gamma) + \frac{A_k e^{\frac{-(1-\epsilon x)}{\lambda_k(x-\alpha)}}}{|\mathbf{\Lambda}|(x-\alpha)}[\tilde{u}(x-\alpha)-\tilde{u}(\epsilon x-1)]\delta(\gamma-1) + \frac{A_k e^{\frac{-\epsilon x}{\lambda_k(\alpha-x)}}}{|\mathbf{\Lambda}|(\alpha-x)}[\tilde{u}(x)-\tilde{u}(x-\alpha)]\delta(\gamma)$$
$$+\frac{A_{\bar{k}} e^{\frac{-(1-\epsilon x)}{\lambda_{\bar{k}}(x-\bar{\alpha})}}}{|\mathbf{\Lambda}|(x-\bar{\alpha})}[\tilde{u}(x-\bar{\alpha})-\tilde{u}(\epsilon x-1)]\delta(\gamma-1) + \frac{A_{\bar{k}} e^{\frac{-\epsilon x}{\lambda_{\bar{k}}(\bar{\alpha}-x)}}}{|\mathbf{\Lambda}|(\bar{\alpha}-x)}[\tilde{u}(x)-\tilde{u}(x-\bar{\alpha})]\delta(\gamma)$$
$$+\sum_{m=1,m\neq k,\bar{k}}^{M} \frac{A_m}{|\mathbf{\Lambda}|x} e^{\frac{-(\gamma-\epsilon x)}{\lambda_m x}}\delta(\gamma-1)[\tilde{u}(x)-\tilde{u}(\gamma-\epsilon x)] \tag{28}$$

where

$$p_4 = \prod_{i=1,i\neq k,\bar{k}}^{M}(\zeta_{ki}-1), \quad p_5 = \prod_{i=1,i\neq k,\bar{k}}^{M}(\zeta_{ki}-\zeta_{k\bar{k}}),$$
$$p_6 = \prod_{i=1,i\neq k,\bar{k},m}^{M}(\zeta_{ki}-\zeta_{km}),$$
$$\text{and} \quad D'_- = \left[\frac{d}{dv}\prod_{i=1,i\neq k,\bar{k}}^{M}(\zeta_{ki}-v)\right]_{v=1}. \tag{32}$$

2) Integral Solution for $\bar{I}_1$: Result of the integral $\bar{I}_1$ will be the same as those given by (30), except that the variable $\{\lambda_k\}$ and $\{\lambda_{\bar{k}}\}$ will be exchanged.

3) Integral Solution for I_{1m}: The integral I_{1m} evaluates to

$$I_{1m} = \int_0^{\frac{1}{\epsilon}} \frac{A_m}{|\mathbf{\Lambda}|x} e^{\frac{-(1-\epsilon x)}{\lambda_m x}} dx$$
$$= C_{m1}E_1\left(\frac{\epsilon}{\lambda_m}\right) + \frac{C_{m2}\lambda_m}{\epsilon}E_2\left(\frac{\epsilon}{\lambda_m}\right)$$
$$+C_{mk}e^{\left(\frac{1}{\lambda_k}-\frac{1}{\lambda_m}\right)}E_1\left(\frac{\zeta_{mk}-1+\epsilon}{\lambda_m}\right)$$
$$+C_{m\bar{k}}e^{\left(\frac{1}{\lambda_{\bar{k}}}-\frac{1}{\lambda_m}\right)}E_1\left(\frac{\zeta_{m\bar{k}}-1+\epsilon}{\lambda_m}\right) \tag{33}$$

where

$$C_{m1} = \frac{-C_{m2}\lambda_m\left[p_{m\bar{k}}\delta(-1)+p_{mk}\delta(-1)\right]}{p_{mk}p_{m\bar{k}}}$$
$$C_{m2} = \frac{\lambda_m^{M-1}e^{\epsilon/\lambda_m}}{|\mathbf{\Lambda}|p_{mk}p_{m\bar{k}}p_3}$$

$$C_{mk} = \frac{\lambda_m^M e^{\epsilon/\lambda_m}\delta(-1)}{|\mathbf{\Lambda}|p_{mk}^2[p_{m\bar{k}}\delta(0)+(p_{m\bar{k}}-p_{mk})\delta(-1)]p_3}$$
$$C_{m\bar{k}} = \frac{\lambda_m^M e^{\epsilon/\lambda_m}\delta(-1)}{|\mathbf{\Lambda}|p_{m\bar{k}}^2[p_{mk}\delta(0)+(p_{mk}-p_{m\bar{k}})\delta(-1)]p_3}$$
$$p_{mk} = (\zeta_{mk}-1), \text{ and } p_{m\bar{k}} = (\zeta_{m\bar{k}}-1). \tag{34}$$

B. Second Moment of $r_{k\bar{k}}$

The second moment of $r_{k\bar{k}}$ can be calculated in a manner similar to the first moment and can be written as

$$E[r_{k\bar{k}}^2] = \frac{1}{\epsilon^2}\delta(\gamma-1) - I_2(\alpha,\gamma) - \bar{I}_2(\bar{\alpha},\gamma)$$
$$-\sum_{m=1,m\neq k,\bar{k}}^{M} I_{2m}(\gamma). \tag{35}$$

Again, the above integrals are evaluated using partial fraction expansion. Details are omitted for brevity, however, the interested reader is encouraged to consult [13] for a detailed treatment of these integrals and further insight into the problem.

VI. Simulation Results

In this section, the performance analysis of the ϵ-NLMS algorithm is investigated for an unknown complex valued system identification problem. The system noise is taken as zero mean i.i.d. with variance 0.01. The length of the adaptive filter is taken to be equal to that of the unknown system, i.e., 5. The correlation matrix of the correlated complex Gaussian input to the adaptive filter and unknown system is

$$\mathbf{R} = \begin{bmatrix} 1 & \alpha_c & \alpha_c^2 & \cdots & \alpha_c^{M-1} \\ \alpha_c & 1 & \alpha_c & \cdots & \alpha_c^{M-2} \\ \alpha_c^2 & \alpha_c & 1 & \cdots & \alpha_c^{M-3} \\ \vdots & & & & \\ \alpha_c^{M-1} & \alpha_c^{M-2} & \alpha_c^{M-3} & \cdots & 1 \end{bmatrix}$$

where $0 < \alpha_c < 1$ is the factor that controls the correlation between the regressor elements.

Figures 1 and 2 show the plots of the steady-state tracking EMSE for a wide range of step size values $[10^{-3}, 1]$. Two values of $\text{Tr}(\mathbf{Q})$ are considered (10^{-4} and 10^{-7}). The value of epsilon is taken as 0.5. For Figure 1 the correlation factor α_c is 0.2 while for Figure 2 its value is 0.7. The simulated results are obtained by averaging over 200 experiments while the analytical results are obtained by plotting the steady-state tracking EMSE expression given in equation (8). Close agreement between the theoretical and simulated results is obtained as can be seen from these figures. We also note that the analytical and simulation results agree for a wide range of correlation factor.

Figure 3 plots the steady-state tracking EMSE for various values of ϵ with a fixed step size $\mu = 0.6$, $\text{Tr}(\mathbf{Q}) = 10^{-5}$ and correlation factor $\alpha_c = 0.2$. We see that the steady state EMSE is a decreasing function of ϵ. The ϵ-NLMS algorithm can be viewed as the LMS algorithm with a variable step-size $\mu(i) = \frac{1}{\epsilon+||\mathbf{u}||^2}$. Consequently, by increasing ϵ, the effective step-size $\mu(i)$ decreases which results in a smaller steady-state EMSE.

Next the theoretical value of optimum step size μ_{opt} is compared with the one obtained by simulation for various

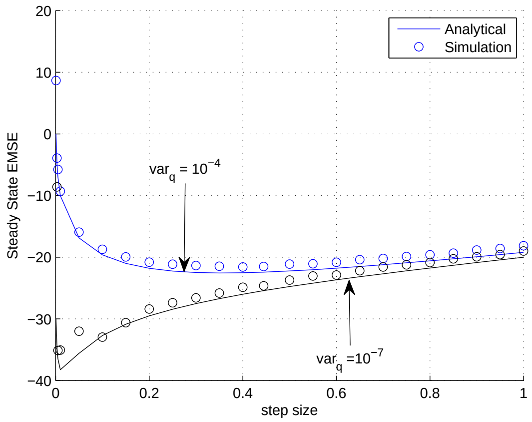

Fig. 1. Analytical and experimental steady state EMSE vs step sizes with $\mathrm{Tr}(\mathbf{Q}) = 10^{-4}, 10^{-7}$ with $\alpha_c = 0.2$ and $\epsilon = 0.05$.

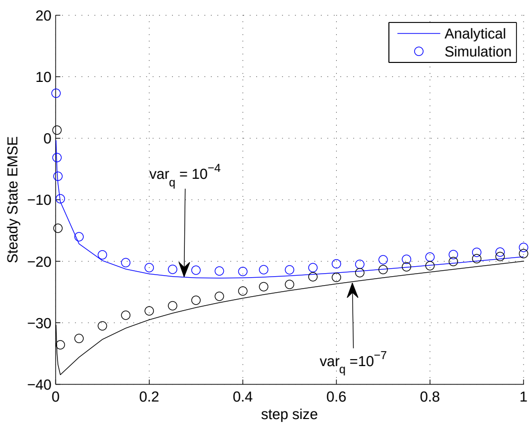

Fig. 2. Analytical and experimental steady state EMSE vs step sizes with $\mathrm{Tr}(\mathbf{Q}) = 10^{-4}, 10^{-7}$ with $\alpha_c = 0.7$ and $\epsilon = 0.05$.

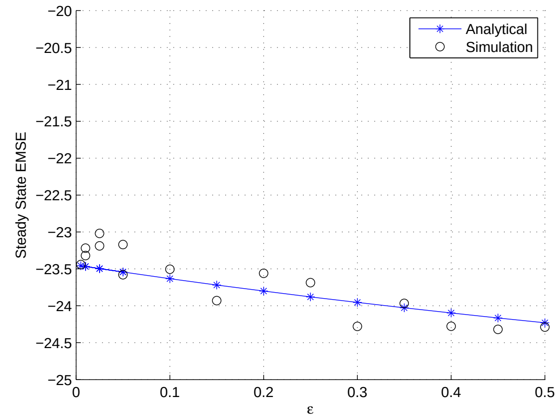

Fig. 3. Analytical and experimental steady state EMSE vs epsilon $\mathrm{Tr}(\mathbf{Q}) = 10^{-5}$, $\alpha_c = 0.2$ and $\mu = 0.6$.

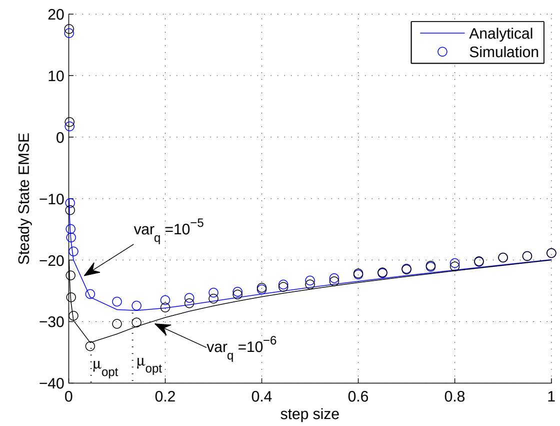

Fig. 4. Analytical and experimental steady state EMSE vs step sizes indicating μ_{opt} for $\mathrm{Tr}(\mathbf{Q}) = 10^{-5}, 10^{-6}$ with $\alpha_c = 0.2$ and $\epsilon = 0.001$.

values of $\mathrm{Tr}(\mathbf{Q})$. Figure 4 indicates the optimum step size μ_{opt} with a dashed line for two values of $\mathrm{Tr}(\mathbf{Q})$. The figure shows close agreement between the theoretical values obtained from equation (11) and the ones obtained by simulation. The consistency of the agreement of analytical values and simulation results supports our analytical treatment of the problem.

From Figures 1, 2 and 4, we note that the steady state EMSE is a decreasing function of step size for small values of step sizes, and after μ_{opt}, it becomes an increasing function of step size. This is contrary to the stationary input case, where the EMSE was a monotonically increasing function of step size. This feature is easily recognizable from Figure 3 where the EMSE is also plotted for very small values of μ. The figures also show that the derived analytical results are valid for a wide range of step-size values.

VII. Conclusion

The paper presents exact tracking analysis of the ϵ-NLMS algorithm for correlated complex Gaussian input. The energy conservation approach reduces the tracking analysis to evaluating certain multidimensional moments. Our approach shows that these moments can be evaluated by first deriving the CDF of a variable of the form $[||\mathbf{u}(i)||^2_{\mathbf{D}_1}][\epsilon + ||\mathbf{u}(i)||^2_{\mathbf{D}_2}]^{-1}$ and then using this CDF to evaluate the first and second moments of this random variable. The advantage of this approach is its transparency and its ability to evaluate performance in closed form. Though the tracking analysis for the NLMS algorithm can not be obtained directly from this approach by setting $\epsilon = 0$, it can easily be done in a similar manner by following the framework of [15].

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