Scaling Laws of Multiple Antenna (Group) Broadcast Channels

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Introduction to broadcast channels

- Multiple antennas add tremendous value to point to point systems
- Research shifted recently to the role of multiple antennas in multiuser systems
  - (Uplink) Multiple Access (MAC)
  - (Downlink) Broadcast (BC)
- Broadcast scenarios (point to multi-point) are especially important because downlink scheduling is the major bottleneck for broadband wireless networks

Outline

- **Effect of Transmit Correlation on Sum-Rate of MIMO Downlink Channels**
- **Scaling Laws of Multiple-Antenna Group Broadcast Channels**

Part I

Opportunistic Beamforming with Precoding for Spatially Correlated Broadcast Channels

Introduction to broadcast channels

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Three Main Questions in a Broadcast Scenario (1)

Q1) Quantify the maximum sum rate possible to all users

A1) Sum-rate is achieved using dirty paper coding (DPC) (Caire and Shamai '02, Viswanath and Tse '02, Vishwanath et al. '02, Yu and Cioffi '02)

- **(-)** DPC is computationally complex at both Tx and Rx
- **(-)** Requires a great deal of Feedback (CSI for all users at Tx)

Three Main Questions in a Broadcast Scenario (2)

The second question is motivated by the drawbacks of DPC

Q2) Devise computationally efficient algorithms for capturing capacity

A2) Utilize multi-user diversity to achieve performance close to capacity

- (+) Opportunist multiple random beamforming coincides asymptotically with DPC (Sharif and Hassibi '06)

$$R = M \log\log n + M \log \frac{P}{M} + o(1)$$

- (+) Requires simply SINR feedback to Tx

Three Main Questions in a Broadcast Scenario (3)

Q3) With this promising performance, how does opportunist beam-forming perform under various non-idealities

A3) **(i)** Time correlation (Kountouris and Gesbert '05)

(ii) Frequency correlation (Fakhereddin, Sharif, and Hassibi'06)

(iii) Channel estimation error (Vikali, Sharif, and Hassibi '06)

(iv) Spatial correlation (D. Park and S Y. Park '05)

Main problem to be addressed:

- For a Gaussian broadcast channel, we would like to quantify the hit that transmit correlation causes to scaling laws of the sum-rate capacity. We consider DPC and various beamforming schemes.

System Model

- Base station with M antennas broadcasting to n single-antenna users
- Received signal at each antenna

$$Y_i = \sqrt{P} H_i S + W_i, \qquad i = 1, \ldots, n$$

  with $E[S^* S] = 1$ and Gaussian noise $W_i \sim CN(0, I)$

- Channel H_i of i-th user is $1 \times M$ vector
  - Distributed as $CN(0, R)$; R is nonsingular with $\mathrm{tr}(R) = M$
  - Known perfectly at receiver
  - Follows a bock fading model (with coherence interval T)
  - H_i is independent from one user to another

Scaling of DPC under Correlation

- Sum-rate capacity of DPC

$$R_{DPC} = E\left\{ \max_{\{P_1,\ldots,P_n,\sum P_i = P\}} \log\det\left(I + \sum_{i=1}^{n} H_i^* P_i H_i\right)\right\}$$

- For large n we can show that RHS is both an upper and lower bound

$$\boxed{R_{DPC} = M\log\log n + M\log\left(\tfrac{P}{M}\right) + M\log\sqrt[M]{\det R}}$$

Since $\mathrm{tr}(R) = M$, the geometric mean satisfies

$$\sqrt[M]{\det(R)} \le \frac{\mathrm{tr}(R)}{M} \le 1$$

- Compare with rate for spatially uncorrelated channel

$$\boxed{R_{DPC} = M\log\log n + M\log\left(\tfrac{P}{M}\right)}$$

What is Random Beam Forming?

- Choose M random orthonormal vectors ϕ_m, $m = 1, \dots, M$ (according to an isotropic distribution)
- Construct the signal

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \qquad t = 1, \dots, T$$

  where T is less than the coherence interval of the channel.

- After T channel uses we independently choose another isotropic set of orthonormal vectors $\{\phi_m\}$, and so on. So we are transmitting M random beams.
- This is a generalization of the scheme "Opportunistic Beamforming" (Viswanath et al. '02) in which only one random beam is transmitted and proportional fairness is guaranteed.

Exploit Multi-User Diversity

- Each receiver $i = 1, \ldots, n$ computes the following M SINRs

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \qquad m = 1, \ldots, M$$

  and feeds back the best SINR

- Rather than randomly assigning the beams, the transmitter assigns signal s_m to the user with the best SINR for that signal. Therefore

$$C = E \sum_{m=1}^{M} \log \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \right)$$

- Due to the symmetry of all the random variables involved:

$$C = M E \log \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,1} \right)$$

Other Beamforming Schemes

- Random Beam forming (RBF) $S(t) = \sum_{m=1}^{M} \phi_m s_m(t)$
- RBF with Channel whitening

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)$$

- RBF with general precoding

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} A \phi_m s_m(t)$$

- Deterministic beamforming

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \quad \phi_m\text{'s are fixed}$$

How to Determine Scaling of BF Schemes

1. Sum rate

$$
\begin{aligned}
R_{\text{BF}} &= E\sum_{m=1}^{M}\log\left(1+\max_{i=1,\ldots,n}\text{SINR}_{i,m}\right)\\
&= ME\left(1+\max_{i=1,\ldots,n}\text{SINR}_{i,m}\right)
\end{aligned}
$$

2. To calculate expectation, condition on beams

$$
R_{\text{BF}|\Phi} = ME_{H_i|\Phi}\left(1+\max_{i=1,\ldots,n}\text{SINR}_{i,m}\right)
$$

- $\text{SINR}_{i,m}|\Phi$ is iid over i
- Find the distribution of $\text{SINR}_{i,m}|\Phi$
- Employ extreme value theory to find $\max_{i=1,\ldots,n}\text{SINR}_{i,m}$

3. Average $R_{\text{BF}|\Phi}$ over Φ

Statistics of $\mathrm{SINR}_{i,m}$ (White Channel)

- $\mathrm{SINR}_{i,m}$ is defined by

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \qquad m = 1, \ldots, M$$

- Easy to find distribution of $\mathrm{SINR}_{i,m}|\Phi$ when H_i is white

$$\begin{aligned} f(x) &= \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \left(\frac{1}{\rho}(1+x) + M - 1 \right) \\ F(x) &= 1 - \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \end{aligned}$$

- Finding these statistics in the correlated case is challenging

Statistics of $\mathrm{SINR}_{i,m}$ given Φ (Correlated Case)

- We can show that the CDF of SINR in the correlated case

$$F(x) = 1 - \frac{1}{2\pi^M \det(R)} \lambda_M \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(\lambda_i - \lambda_M)} e^{-\frac{1}{\rho}\frac{x}{\lambda_M}}$$

where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ are the eigenvalues of the matrix

$$A = (1+x)\Lambda^{1/2}\phi_m\phi_m^*\Lambda^{1/2} - x\Lambda \qquad \rho = \frac{P}{M}$$

Note that eigenvalues are a function of x.

- pdf is given by

$$f(x) = \frac{1}{2\pi^M \det(R)} e^{-\frac{1}{\rho}\frac{x}{\lambda_M}} \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(\lambda_i - \lambda_M)} \times$$
$$\left\{ \frac{1}{\rho}\frac{\|q_M\|_C^2}{\lambda_M} - \|q_M\|_B^2 - \sum_{i=1}^{M} \frac{1}{\lambda_i} \frac{\lambda_M^2 \|q_i\|_C^2 - \lambda_i^2 \|q_M\|_C^2}{x(\lambda_i - \lambda_M)} \right\}$$

where $B = \Lambda^{1/2}(\phi_m\phi_m^* - I)\Lambda^{1/2} \qquad C = \Lambda^{1/2}\phi_m\phi_m^*\Lambda^{1/2}$

Scaling of maximum SINR

- Can now show

$$\lim_{x\to\infty} \frac{1-F(x)}{f(x)} = \frac{P}{M}\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}$$

- Using extreme value theory, we can show that for large n

$$\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} = \frac{P}{M}\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\log n$$

- Conditional sum-rate capacity scales as

$$R_{\mathrm{BF}|\Phi} = M\log\log n + M\log\frac{P}{M} + M\log\left(\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\right)$$

- Sum-rate capacity of random beam-forming

$$\boxed{R_{\mathrm{RBF}} = M\log\log n + M\log\tfrac{P}{M} + ME_{\Phi}\log\left(\tfrac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\right)}$$

Averaging over the random beams

- Need to obtain CDF of $\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}$ which is challenging.
- The CDF of $y = \frac{1}{\|\phi\|^2_{\Lambda^{-1}}}$ is given by

$$G(x) = Pr(\frac{1}{\|\phi\|^2_{\Lambda^{-1}}} < x) = 1 - \sum_i \eta_i \left(\frac{1}{x} - \frac{1}{\lambda_i(\Lambda)}\right)^{M-1} u\left(1 - \frac{x}{\lambda_i(\Lambda)}\right)$$

where $\eta_i = \frac{1}{\prod_{j \neq i}(\frac{1}{\lambda_j(\Lambda)} - \frac{1}{\lambda_i(\Lambda)})}$

- Use CDF to show that

$$\begin{aligned} R_{RBF} &= M \log\log n + M \log \frac{P}{M} + \\ & \log \lambda_1(\Lambda) + \sum_{i=1}^{M} \eta_i \log\left(\frac{\lambda_i}{\lambda_1}\right) \sum_{k=1}^{M-1} \frac{1}{k+2} (\frac{-1}{\lambda_i})^{M-1-k} \left.\frac{1}{y^{k+2}}\right|_{\lambda_1}^{\lambda_i} \end{aligned}$$

Sum rate of Deterministic Beam Forming

- Sum-rate of deterministic beam forming

$$R_{BF-D} = M \log\log n + M \log \frac{P}{M} + \sum_{i=1}^{M} \log\left(\frac{1}{\phi_i^* U^* \Lambda^{-1} U \phi_i}\right)$$

$U^* \Lambda^{-1} U$ is the eigenvalue decomposition of R^{-1}

- Special case: $U\phi_i$'s are the columns of identity matrix

$$\boxed{R_{BF-D} = M \log\log n + M \log \tfrac{P}{M} + M \log \sqrt[M]{\det R}}$$

Since $\mathrm{tr}(R) = M$, the geometric mean satisfies

$$\sqrt[M]{\det(R)} \leq \frac{\mathrm{tr}(R)}{M} \leq 1$$

Sum rate of RBF with Channel Whitening

- For random beam forming with channel whitening,

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)$$

- Set $\alpha = \frac{\text{tr}(R^{-1})}{M}$ to guarantee $E[S^* S] \leq 1$

- Scaling becomes the same as for white channel case with reduced signal power

$$\boxed{R_{BF-W} = M \log \log n + M \log \frac{P}{M} + M \log \frac{M}{\text{tr}(R^{-1})}}$$

Simulations

- Consider a base station with $M = 2$ and $M = 3$ antennas
- The corresponding correlation matrix is

$$R = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}$$

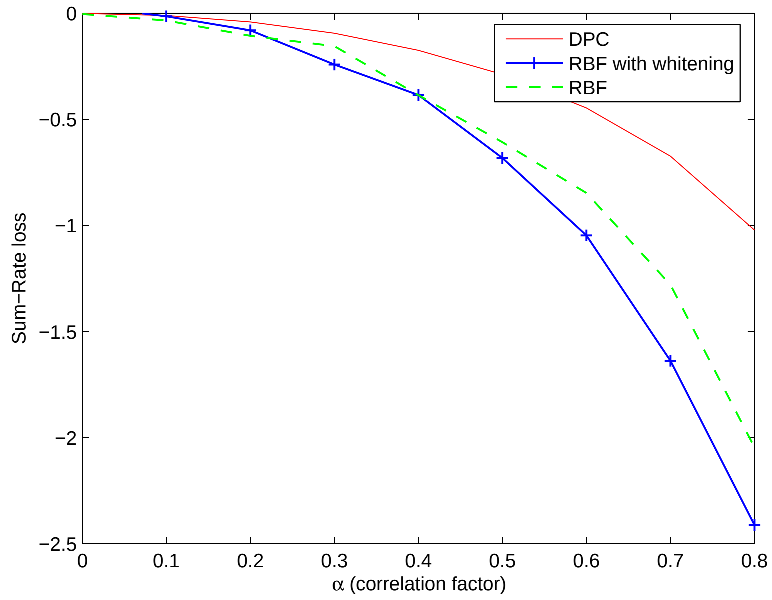


Figure 1: Sum-rate loss versus the correlation factor α for a system with $M = 2$ and $n = 100$.

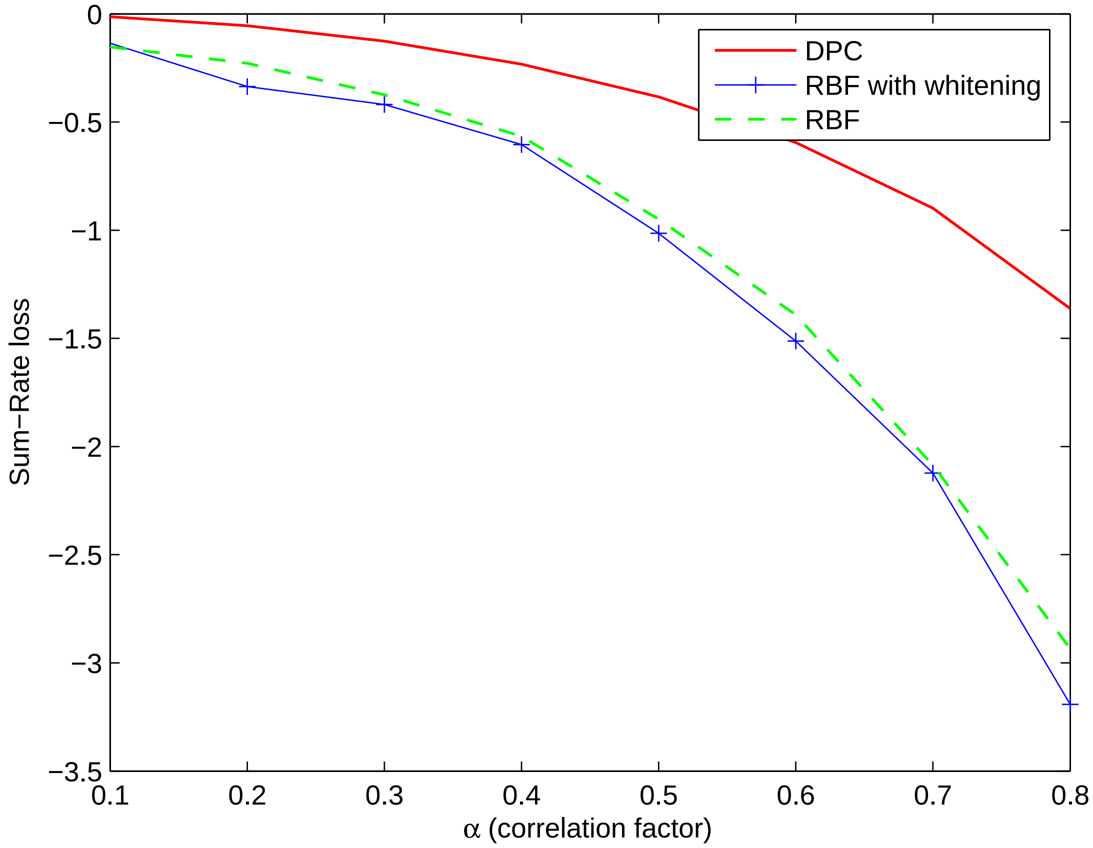


Figure 2: Sum-rate loss versus the correlation factor α for a system with $M = 3$ and $n = 100$.

Can we do better?

- Apply a general precoding matrix

$$\alpha A S(t) = \alpha A \sum_{m=1}^{M} \phi_m(t) s_m(t), \qquad t = 1, \ldots, T$$

- The factor α ensures that we have a fixed power constraint

$$\alpha \le \sqrt{\frac{M}{\mathrm{tr}(A^* A)}}$$

- This produces the effective channel

$$\tilde{H}_i = \alpha H_i A$$

with correlation $\alpha^2 \tilde{R} = \alpha^2 A^* R A$.

What is sum-rate with a general precoding?

- Sum-rate is given by

$$R_{\mathrm{PC}} = M \log\log n + M \log \frac{P}{M} - h_{\mathrm{PC}} \tag{1}$$

where h_{PC} is the hit incurred by using a general precoding matrix A

$$\boxed{h_{\mathrm{PC}} = M \log \frac{\mathrm{tr}(A^* A)}{M} + M E \log \|\phi_m\|^2_{\tilde{R}^{-1}}} \tag{2}$$

- Finding the optimum A that minimizes hit is difficult. But we can show that optimum precoding matrix A_{opt} can be written as

$$A_{opt} = Q_{Aopt} D_{Aopt}$$

where Q_{Aopt} is an orthornormal matrix and D_{opt} is diag with positive entries.

Special choices of A

- Difficult to optimize Q_{opt} and D_{opt} jointly.
- Set $Q_{opt} = Q_R$ as this will diagonalize R and optimize over D_{opt}.
- Zero forcing

  $$A_{\mathrm{ZF}} = Q_R \Lambda_R^{-\frac{1}{2}}$$

  and resulting hit

  $$h_{ZF} = M \log \frac{tr(R^{-1})}{M}$$

Special choices of A

- MMSE precoding

$$A = Q_R(\Lambda + \beta I)^{-\frac{1}{2}}$$

with β obtained as a solution to a fixed-pt problem

$$\frac{Tr(\Lambda + \beta^* I)^{-2}}{Tr(\Lambda + \beta^* I)^{-1}} = E\left(\frac{1}{\beta^* + \frac{1}{\|\Phi_m\|^2_{\Lambda^{-1}}}}\right)$$

- More generally, we can set $Q_{opt} = Q_R$ and find the optimum D_{opt}. Need to solve a set of M implicit equations

$$\frac{1}{d_i} E\left[\frac{\frac{1}{d_i \lambda_i}|\phi(i)|^2}{\|\phi\|^2_{D^{-1}\Lambda^{-1}}}\right] = \frac{1}{tr(D)}$$

Minimize an upper bound instead

- Difficulty in minimizing h_{PC} due to the *phi* term

$$h_{\mathrm{PC}} = M \log \frac{\mathrm{tr}(A^* A)}{M} + M E \log \|\phi_m\|^2_{\tilde{R}^{-1}}$$

- Minimize an upper bound

$$h \leq M \log \mathrm{tr}(A^* A) + M \log \mathrm{tr}((A^* R A)^{-1})$$

- Can show that optimum A in this case is

$$\boxed{A = Q_R \Lambda_R^{-1/4}}$$

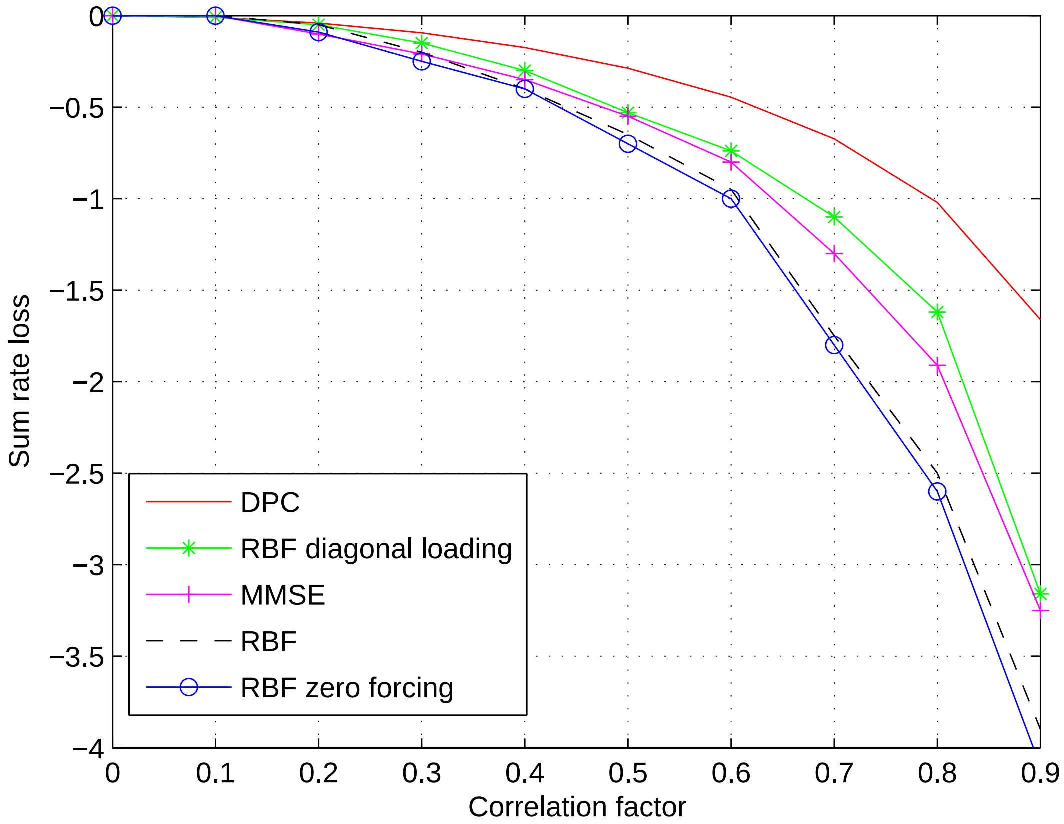


Figure 5: Sum-rate loss versus correlation factor α for a system with M = 3, P=10 and n = 200.

Conclusion

- Studied the effect of spatial correlation on various multiuser schemes for MIMO broadcast channels.
- Considered DPC and random, deterministic, and channel whitening schemes.
- All these techniques exhibit the same scaling for iid channels

$$\boxed{R_{\text{sum}-\text{rate}} = M \log\log n + M \log \tfrac{P}{M}}$$

- In the presence of correlation between transmit antennas, scaling is

$$\boxed{R_{\text{sum}-\text{rate}} = M \log\log n + M \log \tfrac{P}{M} + M \log c}$$

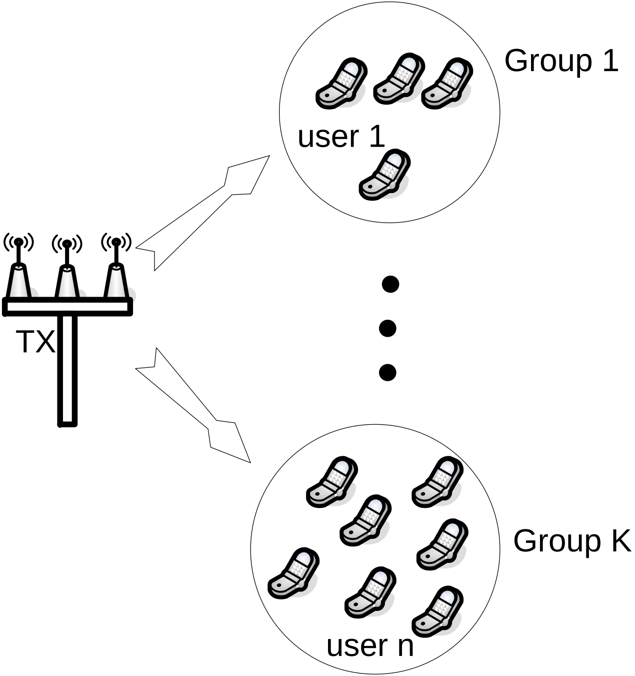
  The constant $0 < c \leq 1$ depends on the scheduling scheme and the eigenvalues of the correlation matrix R.

Part II:

Scaling Laws of Multiple-Antenna Group Broadcast Channels

Group broadcast scenario

- Broadcast problem: users interested in *independent* information
- Group Broadcast: Groups of users, each group of users interested in the same information
  - e.g. DAB/DVB with limited shows; users classified according to shows they are interested in
  - Single group: multicast problem (Khitsi et. al. 06, Jindal and Luo 06)
  - Multiple-groups each consisting of one user: broadcast problem


Group 1
user 1
TX
Group K
user n

Three main questions in a broadcast scenario

Q1) Quantify the maximum sum rate possible to all users

Q2) Quantify the asymptotic behavior in regimes of interest

Q3) How do scheduling schemes performs under various non-idealities

Would like to answer **Q2)**: Asymptotic behavior in various regimes (large number of users and antennas)

System model

- Base station equipped with M antennas
- n users each equipped with a single receive antenna.
- n single-antenna users with received signal

$$y_i = h_i^* s + \nu_i$$

  - Input satisfies $E[s^* s] \leq P$
  - Noise is white Gaussian $\nu \sim CN(0, I_M)$
  - User channels are independent and distributed as $CN(0, I_M)$
- Users are partitioned into K groups of $\frac{n}{K}$ users each; each group is interested in the same data.

Group broadcast capacity: Formal expression

- When there is one user only

$$C_{\text{one user}} = E \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \log\det\left(1 + \|h\|_B^2\right)$$

- Single group broadcast

$$\boxed{C_{\text{single group}} = E \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \log\det\left(1 + \|h_i\|_B^2\right)}$$

- Group broadcast eventually limited by the worst user

Group broadcast capacity: Formal expression (2)

- Multiple groups broadcast: K power matrices $B_1, \ldots, B_K$, one for each group.
- Matrices should maximize sum-rate under total power constraint

$$C_{\text{multiplegroups}} = E \max_{B_k \geq 0 \ \sum_{k=1}^{K} \operatorname{Tr}(B_k) \leq P} \log\det\left(1 + \sum_{k=1}^{K} \|h_k\|_{B_k}^2\right)$$

- With K *user* groups, we need to take care of the “worst” user of each group

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Study behavior of C_{GB} for large number of users n and antennas M
  - Large n and fixed M
  - Large M and fixed n
  - Large M and n with $M = \beta n$
  - Large M and n with $M = \log n$

Capacity bounding techniques (1)

Upper bounds

1. K times rate of one group

$$\begin{aligned} C_{GB} &\leq KC_{\text{single group}} \\ &= K\log(1 + \max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B^2) \end{aligned}$$

2. MAC-BC duality

- Maximum sum rate for K users, chosen one from each group

$$C_{K\text{ users}} = \max_{\substack{b_k \geq 0 \\ \sum_{k=1}^K b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right)$$

- Rate has to appeal to every user in every group

$$C_{K \text{ users}} \leq \min_{h_{i_1}} \cdots \min_{h_{i_K}} \max_{\substack{b_k \geq 0 \\ \sum_{k=1}^{K} b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right)$$

- Get rid of the determinant using AM-GM inequality $\det(A) \leq \left(\frac{\mathrm{tr}(A)}{M}\right)^M$ to write

$$C_{GB} \leq M \log\left(1 + \frac{P}{M} \max_k \min_{h_{i_1}} \cdots \min_{h_{i_K}} \{\|h_{i_1}\|^2, \cdots, \|h_{i_K}\|^2\}\right)$$

Capacity bounding techniques (2)

Lower Bounds

1. Time sharing

$$C_{GB} \geq \frac{1}{K} \sum_{k=1}^{K} \log \det \left(1 + \max_{B_k \geq 0 \ \mathrm{Tr}(B_k)=P} \ \min_{h_{i_k}} \|h_{i_k}\|_{B_k} \right)$$

2. Treating interference as noise

$$C_{GB} \geq K \log \left(\frac{\frac{1}{K} \frac{P}{M} \min_i \|h_i\|^2}{1 + \frac{K-1}{K} \frac{P}{M} \min_i \|h_i\|^2} \right)$$

Need to study scaling of the weighted max − min norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \ \min_i \|h_i\|_B^2$$

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Obtain upper and lower bounds on C_{GB}; bounds depend on the max-min weighted norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

- Find upper and lower bounds on the max-min in terms of the h_i's

Bounds on the max-min weighted Euclidean norm

Here we obtain upper and lower bounds on the weighted Euclidean norm for fixed M and n

Lower Bounds

1. max-min norm is greater than min norm

$$\max_{\mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2 \geq \frac{P}{M} \min_i \|h_i\|^2$$

2. h_i belongs to a finite set $\{h_1, \cdots, h_{\frac{n}{K}}\}$

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\frac{n}{K}} \min_i \|h_i\|^2$$

So

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\min\{M, \frac{n}{K}\}} \min_i \|h_i\|^2$$

3. Diagonal values and eigenvalues: Define $H = [h_1 \cdots h_{\frac{n}{K}}]$, then

$$\lambda_{min}(H^* H) \le \min_i \|h_i\|^2 \le \lambda_{max}(H^* H)$$

Upper Bounds

1. max-min is less than min-max

$$\max_{B \ge 0 \ \mathrm{Tr}(B) \le P} \min_i \|h_i\|_B^2 \le P \min_i \|h_i\|^2$$

2. Replace minimization with averaging (Jindal and Luo '06)

$$\begin{aligned} \max_{B \ge 0 \ \mathrm{Tr}(B) \le P} \min_i \|h_i\|_B^2 &\le \max_B \frac{1}{\frac{n}{K}} \sum_{i=1}^{\frac{n}{K}} \|h_i\|_B^2 \\ &\le P \lambda_{max}(H^* H) \end{aligned}$$

Study boils down to studying the scaling of

1) min norm $\min_i \|h_i\|^2$ 2) eigenvalues of $H^* H$

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Obtain upper and lower bounds on C_{GB}; bounds depend on the max-min weighted norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

- Find upper and lower bounds on the max-min in terms of the h_i's
- Find the asymptotics of $\min_i \|h_i\|^2$

Scaling of the Euclidean norm

In the rest of the presentation, we study the scaling of the minimum Euclidean norm $\min_i \|h_i\|^2$ for

- Large n and fixed M
- Large M and fixed n
- Large M and n with $M = \beta n$
- Large M and n with $M = \log n$

Behavior of the min Euclidean Norm

The behavior of $\min_i \|h_i\|^2$ looks like

Regime	**Asymptotic Value**	**Method**
large n	$\frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M} \frac{1}{n^{\frac{1}{M}}}$	min of iid r.v. Theorem
large M	M	Law of large numbers
$M = \beta \frac{n}{K}$	$\geq (1 - \sqrt{K\beta})^2$ $\leq (1 + \sqrt{K\beta})^2$	Random Matrix theory
$M = \log n$	$\mathcal{H} \in [1 - \epsilon_l, 1]$ $\epsilon \simeq .8414$	Chernof Bound

Behavior of the max min Euclidean Norm

The behavior of $\max_B \min_i \|h_i\|^2$ looks like

Regime	**Lower Bound**	**Upper Bound**
large n	$\frac{C_M}{M} \frac{1}{n^{\frac{1}{M}}}$	$C_M \frac{1}{n^{\frac{1}{M}}}$
large M	$P \frac{K}{n} M$	PM
$M = \beta \frac{n}{K}$	$P(1 - \sqrt{K\beta})^2$	$P(1 + \frac{1}{\sqrt{K\beta}})^2$
$M = \log n$	$P\mathcal{H}\ \ , \mathcal{H} \in [1 - \epsilon_l, 1]\ \ \epsilon \simeq .8414$	constant

$$C_M = \frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M}$$

GB scaling for large n, fixed M

- Group broadcast capacity scales as

$$\boxed{C_{GB} = \alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

  where

$$\frac{1}{M} \leq \alpha \leq 1$$

- For spatially correlated case, the capacity incurs a $\det(R)^{\frac{1}{M}}$ hit

$$\boxed{C_{GB} = \alpha \det(R)^{\frac{1}{M}} P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

- Unfortunate result: sum-rate decreases with the number of users.

- Counter this: increase the resources (i.e., number of antennas M).

GB scaling for large M, fixed n

- Upper bound: K times rate of single group

$$C_{GB} \quad \leq \quad K \max_{B\geq 0 \ \mathrm{Tr}(B)\leq P} \log(1+\min_i \|h_i\|_B^2)$$

$$\boxed{C_{GB} \leq K\log(1+PM) \quad \text{(law of large numbers)}}$$

- Lower bound: Use time sharing

$$C_{GB} \geq \log(1+\max_{B\geq 0 \ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B)$$

$$\boxed{C_{GB} \geq \log(1+P\tfrac{K}{n}M)}$$

GB scaling with M and n, $M = \beta n$

- Number of users and antennas grow to infinity while their ratio remains constant $\frac{M}{n} = \frac{\beta}{K}$.
- Lower bound: Use time sharing

$$\boxed{C \geq \log\left(1 + P(1 - \sqrt{K\beta})^2\right)}$$

- To obtain an upper bound, we start with the bound

$$C_{GB} \leq K \log(1 + \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2)$$

  to show

$$\boxed{C_{GB} \leq K \log(1 + P(1 + \tfrac{1}{\sqrt{\beta}})^2)}$$

- If we allow the number of antennas to grow linearly with the number of users, we can guarantee a constant sum rate.

Can we have constant rate with sublinear growth?

- But is it still possible to do so without straining the resources as much?
- We showed that for large n

$$
\begin{aligned}
C &= \alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}} \qquad \frac{C_M}{M} \simeq 1 \\
&= \alpha P \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}
\end{aligned}
$$

- To guarantee a constant rate, intuition requires to set $M = \log n$

GB scaling with M and n, $M = \log n$

- Use the Chernof bound, we show that

$$\lim_{M=\log n, n\to\infty} \min_i \frac{\|h_i\|^2}{M} = \mathcal{H} \in [1-\epsilon_l, 1] \quad \text{w.p.1}$$

  where $\epsilon_l \simeq .8414$.

- Capacity is lower-bounded by a constant

$$\boxed{C \geq \log(1 + P\mathcal{H})} \tag{3}$$

- Capacity is also upper bounded by a constant because it is for $M = \beta n$

Can we do any better?

- Can we guarantee a constant capacity per stream without straining resources as much?
- Can show that number of transmit antennas, M, should grow faster than $(\log n)^{\frac{1}{2}-\epsilon(n)}$,

$$\epsilon(n) = \frac{\log\log\log n}{\log\log n}$$

to guarantee constant rate per stream

Conclusion for Part II

- Studied the scaling law of the group broadcast problem
- Capacity decreases as $n^{-\frac{1}{M}}$ with number of users
- To guarantee a constant rate if we allow M to grow as $\log n$
- As a by-product (or a prerequisite), we studied the scaling of
  - Minimum Euclidean norm $\min_i \|h_i\|^2$
  - Max min Euclidean norm $\max_B \min_i \|h_i\|_B^2$

  in various regimes
- Most results apply for general distributions on h_i

Other Research Interests

- Compressive sensing for impulsive noise in OFDM (Giuseppe Caire)
- Adaptive filtering analysis and design (Babak Hassibi and Vitor Nascimento; previously with Ali Sayed)
- Receiver design for (MIMO) OFDM in (block) time-variant channels (Naofal Al-Dhahir + Students) (previously with A. Paulraj)
- Blind channel estimation (Students)