Scaling Laws of Multiple Antenna (Group) Broadcast Channels

Dr. Tareq Al-Naffouri

Electrical Engineering Department
King Fahd University of Petroleum and Minerals
Dhahran, Saudi Arabia

Fulbright Research Visitor
Electrical Engineering Department
USC

Joint work with Masoud Sharif, Amir Dana, and Babak Hassibi

Introduction to broadcast channels

- Multiple antennas add tremendous value to point to point systems
- Research shifted recently to the role of multiple antennas in multiuser systems
  - (Uplink) Multiple Access (MAC)
  - (Downlink) Broadcast (BC)
- Broadcast scenarios (point to multi-point) are especially important because downlink scheduling is the major bottleneck for broadband wireless networks

Outline

- **Effect of Transmit Correlation on Sum-Rate of MIMO Downlink Channels**
- **Scaling Laws of Multiple-Antenna Group Broadcast Channels**

Part I

How Much Does Transmit Correlation Affect the Sum-Rate of MIMO Downlink Channels?

Three Main Questions in a Broadcast Scenario (1)

Q1) Quantify the maximum sum rate possible to all users

A1) Sum-rate is achieved using dirty paper coding (DPC) (Caire and Shamai '02, Viswanath and Tse '02, Vishwanath et al. '02, Yu and Cioffi '02)

- **(-)** DPC is computationally complex at both Tx and Rx
- **(-)** Requires a great deal of Feedback (CSI for all users at Tx)

Three Main Questions in a Broadcast Scenario (2)

The second question is motivated by the drawbacks of DPC

Q2) Devise computationally efficient algorithms for capturing capacity

A2) Utilize multi-user diversity to achieve performance close to capacity

(+) Opportunist multiple random beamforming coincides asymptotically with DPC (Sharif and Hassibi '06)

$$R = M \log \log n + M \log \frac{P}{M} + o(1)$$

(+) Requires simply SINR feedback to Tx

Three Main Questions in a Broadcast Scenario (3)

Q3) With this promising performance, how does opportunist beam-forming perform under various non-idealities

A3) **(i)** Time correlation (Kountouris and Gesbert '05)

(ii) Frequency correlation (Fakhereddin, Sharif, and Hassibi'06)

(iii) Channel estimation error (Vikali, Sharif, and Hassibi '06)

(iv) Spatial correlation (D. Park and S Y. Park '05)

Main problem to be addressed:

- For a Gaussian broadcast channel, we would like to quantify the hit that transmit correlation causes to scaling laws of the sum-rate capacity. We consider DPC and various beamforming schemes.

System Model

- Base station with M antennas broadcasting to n single-antenna users
- Received signal at each antenna

$$Y_i = \sqrt{P} H_i S + W_i, \qquad i = 1, \ldots, n$$

with $E[S^* S] = 1$ and Gaussian noise $W_i \sim CN(0, I)$

- Channel H_i of i-th user is $1 \times M$ vector
  - Distributed as $CN(0, R)$; R is nonsingular with $\mathrm{tr}(R) = M$
  - Known perfectly at receiver
  - Follows a bock fading model (with coherence interval T)
  - H_i is independent from one user to another

Scaling of DPC under Correlation

- Sum-rate capacity of DPC

$$R_{DPC} = E\left\{ \max_{\{P_1,\ldots,P_n,\sum P_i = P\}} \log\det\left(I + \sum_{i=1}^{n} H_i^* P_i H_i \right) \right\}$$

- For large n we can show that RHS is both an upper and lower bound

$$\boxed{R_{DPC} = M \log\log n + M \log\left(\tfrac{P}{M}\right) + M \log \sqrt[M]{\det R}}$$

Since $\mathrm{tr}(R) = M$, the geometric mean satisfies

$$\sqrt[M]{\det(R)} \leq \frac{\mathrm{tr}(R)}{M} \leq 1$$

- Compare with rate for spatially uncorrelated channel

$$\boxed{R_{DPC} = M \log\log n + M \log\left(\tfrac{P}{M}\right)}$$

What is Random Beam Forming?

- Choose M random orthonormal vectors ϕ_m, $m = 1, \ldots, M$ (according to an isotropic distribution)

- Construct the signal

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \qquad t = 1, \ldots, T$$

  where T is less than the coherence interval of the channel.

- After T channel uses we independently choose another isotropic set of orthonormal vectors $\{\phi_m\}$, and so on. So we are transmitting M random beams.

- This is a generalization of the scheme "Opportunistic Beamforming" (Viswanath et al. '02) in which only one random beam is transmitted and proportional fairness is guaranteed.

Exploit Multi-User Diversity

- Each receiver $i = 1, \ldots, n$ computes the following M SINRs

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \qquad m = 1, \ldots, M$$

  and feeds back the best SINR

- Rather than randomly assigning the beams, the transmitter assigns signal s_m to the user with the best SINR for that signal. Therefore

$$C = E \sum_{m=1}^{M} \log \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \right)$$

- Due to the symmetry of all the random variables involved:

$$C = ME \log \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,1} \right)$$

Other Beamforming Schemes

- Random Beam forming (RBF) $S(t) = \sum_{m=1}^{M} \phi_m s_m(t)$
- RBF with Channel whitening

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)$$

- RBF with general precoding

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} A \phi_m s_m(t)$$

- Deterministic beamforming

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \quad \phi_m\text{'s are fixed}$$

How to Determine Scaling of BF Schemes

1. Sum rate

$$\begin{aligned} R_{\mathrm{BF}} &= E \sum_{m=1}^{M} \log \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \right) \\ &= M E \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \right) \end{aligned}$$

2. To calculate expectation, condition on beams

$$R_{\mathrm{BF}|\Phi} = M E_{H_i|\Phi} \left(1 + \max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} \right)$$

- $\mathrm{SINR}_{i,m}|\Phi$ is iid over i
- Find the distribution of $\mathrm{SINR}_{i,m}|\Phi$
- Employ extreme value theory to find $\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}$

3. Average $R_{\mathrm{BF}|\Phi}$ over Φ

Statistics of $\mathrm{SINR}_{i,m}$ (White Channel)

- $\mathrm{SINR}_{i,m}$ is defined by

$$\mathrm{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho + \sum_{n \neq m} |H_i \phi_n|^2}, \qquad m = 1, \ldots, M$$

- Easy to find distribution of $\mathrm{SINR}_{i,m}|\Phi$ when H_i is white

$$\begin{aligned} f(x) &= \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \left(\frac{1}{\rho}(1+x) + M - 1 \right) \\ F(x) &= 1 - \frac{e^{-\frac{x}{\rho}}}{(1+x)^M} \end{aligned}$$

- Finding these statistics in the correlated case is challenging

Statistics of $\mathrm{SINR}_{i,m}$ given Φ (Correlated Case)

- We can show that the CDF of SINR in the correlated case

$$\boxed{F(x) = 1 - \frac{1}{2\pi^M \det(R)} \lambda_M \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(\lambda_i - \lambda_M)} e^{-\frac{1}{\rho}\frac{x}{\lambda_M}}}$$

  where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$ are the eigenvalues of the matrix

$$A = (1+x)\Lambda^{1/2}\phi_m\phi_m^*\Lambda^{1/2} - x\Lambda \qquad \rho = \frac{P}{M}$$

  Note that eigenvalues are a function of x.

- pdf is given by

$$f(x) = \frac{1}{2\pi^M \det(R)} e^{-\frac{1}{\rho}\frac{x}{\lambda_M}} \prod_{i=1}^{M-1} \frac{\lambda_i \lambda_M}{x(\lambda_i - \lambda_M)} \times$$
$$\left\{ \frac{1}{\rho}\frac{\|q_M\|_C^2}{\lambda_M} - \|q_M\|_B^2 - \sum_{i=1}^{M} \frac{1}{\lambda_i} \frac{\lambda_M^2 \|q_i\|_C^2 - \lambda_i^2 \|q_M\|_C^2}{x(\lambda_i - \lambda_M)} \right\}$$

  where $B = \Lambda^{1/2}(\phi_m\phi_m^* - I)\Lambda^{1/2} \qquad C = \Lambda^{1/2}\phi_m\phi_m^*\Lambda^{1/2}$

Scaling of maximum SINR

- Can now show

$$\lim_{x\to\infty} \frac{1-F(x)}{f(x)} = \frac{P}{M} \frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}$$

- Using extreme value theory, we can show that for large n

$$\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m} = \frac{P}{M} \frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}} \log n$$

- Conditional sum-rate capacity scales as

$$R_{\mathrm{BF}|\Phi} = M \log\log n + M \log \frac{P}{M} + M \log\left(\frac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\right)$$

- Sum-rate capacity of random beam-forming

$$\boxed{R_{\mathrm{RBF}} = M \log\log n + M \log \tfrac{P}{M} + M E_\Phi \log\left(\tfrac{1}{\|\phi_m\|^2_{\Lambda^{-1}}}\right)}$$

Averaging over the random beams

- Need to obtain CDF of $\frac{1}{\|\phi_m\|^2_{\Lambda-1}}$ which is challenging.
- The CDF of $y = \frac{1}{\|\phi\|^2_{\Lambda-1}}$ is given by

$$\boxed{G(x) = Pr(\frac{1}{\|\phi\|^2_{\Lambda-1}} < x) = 1 - \sum_i \eta_i \left(\frac{1}{x} - \frac{1}{\lambda_i(\Lambda)}\right)^{M-1} u\left(1 - \frac{x}{\lambda_i(\Lambda)}\right)}$$

where $\eta_i = \frac{1}{\prod_{j\neq i}(\frac{1}{\lambda_j(\Lambda)} - \frac{1}{\lambda_i(\Lambda)})}$

- Use CDF to show that

$$R_{RBF} = M \log\log n + M \log \frac{P}{M} + \log \lambda_1(\Lambda) + \sum_{i=1}^{M} \eta_i \log\left(\frac{\lambda_i}{\lambda_1}\right) \sum_{k=1}^{M-1} \frac{1}{k+2} (\frac{-1}{\lambda_i})^{M-1-k} \left.\frac{1}{y^{k+2}}\right|_{\lambda_1}^{\lambda_i}$$

Sum rate of Deterministic Beam Forming

- Sum-rate of deterministic beam forming

$$R_{BF-D} = M \log \log n + M \log \frac{P}{M} + \sum_{i=1}^{M} \log \left(\frac{1}{\phi_i^* U^* \Lambda^{-1} U \phi_i} \right)$$

$U^* \Lambda^{-1} U$ is the eigenvalue decomposition of R^{-1}

- Special case: $U\phi_i$'s are the columns of identity matrix

$$\boxed{R_{BF-D} = M \log \log n + M \log \tfrac{P}{M} + M \log \sqrt[M]{\det R}}$$

Since $\mathrm{tr}(R) = M$, the geometric mean satisfies

$$\sqrt[M]{\det(R)} \leq \frac{\mathrm{tr}(R)}{M} \leq 1$$

Sum rate of RBF with Channel Whitening

- For random beam forming with channel whitening,

$$S(t) = \sum_{m=1}^{M} \sqrt{\alpha} R^{-1/2} \phi_m s_m(t)$$

- Set $\alpha = \frac{\mathrm{tr}(R^{-1})}{M}$ to guarantee $E[S^* S] \leq 1$

- Scaling becomes the same as for white channel case with reduced signal power

$$\boxed{R_{BF-W} = M \log \log n + M \log \frac{P}{M} + M \log \frac{M}{\mathrm{tr}(R^{-1})}}$$

Simulations

- Consider a base station with $M = 2$ and $M = 3$ antennas
- The corresponding correlation matrix is

$$R = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}$$

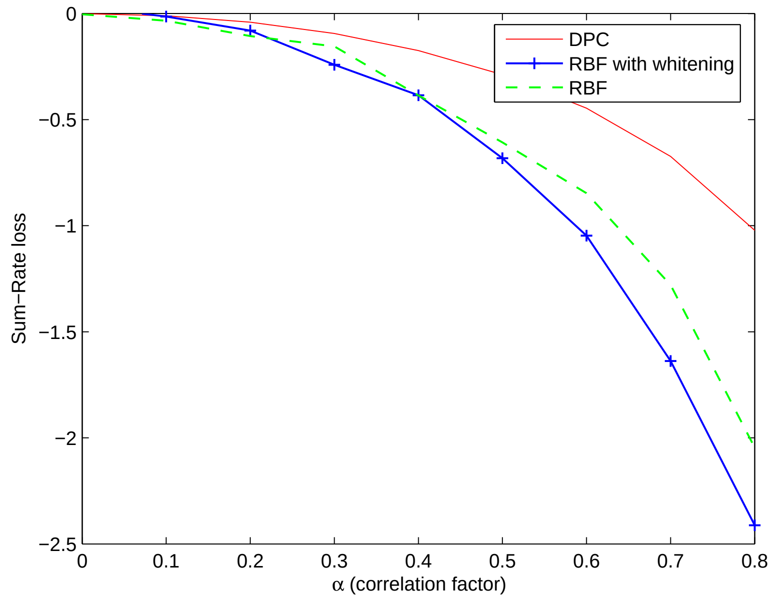


Figure 1: Sum-rate loss versus the correlation factor α for a system with $M = 2$ and $n = 100$.

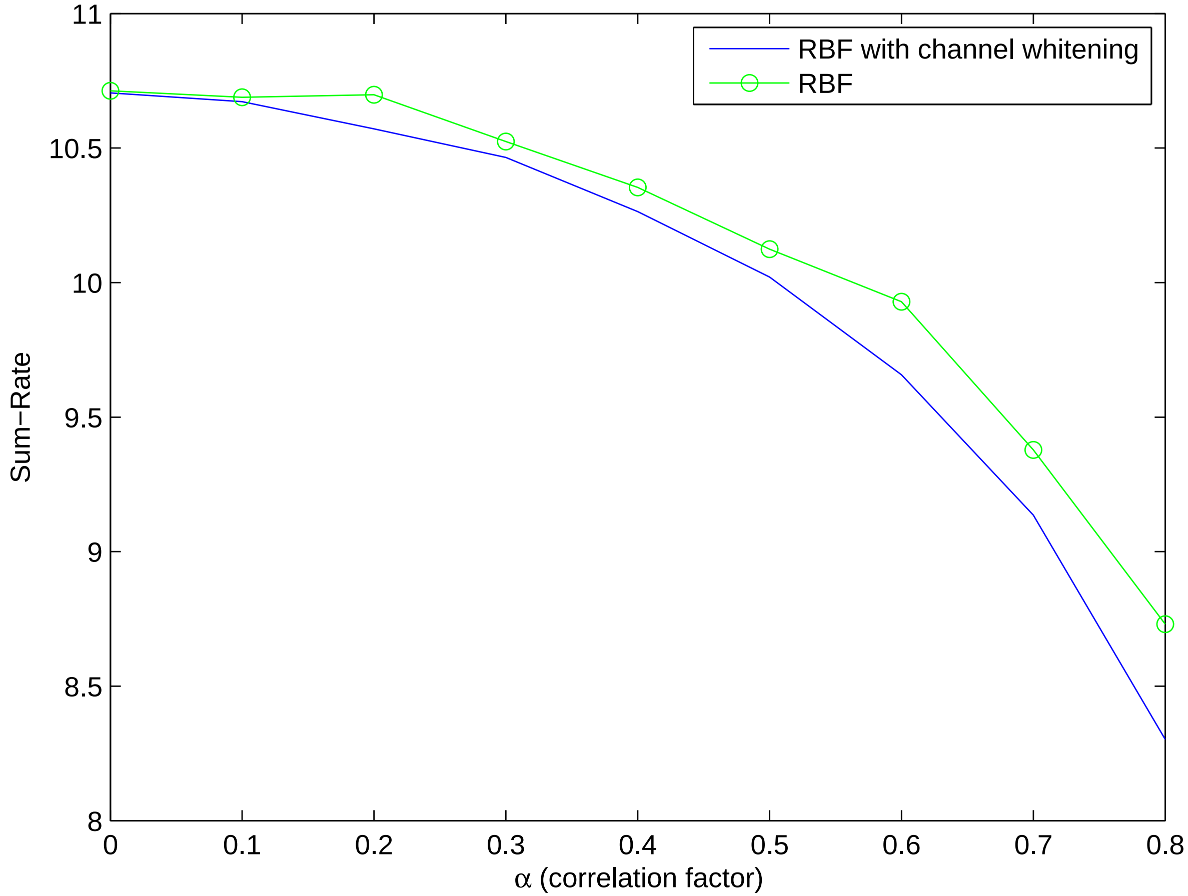


Figure 2: Sum-rate versus the correlation factor α for a system with $M = 2$, $P = 10$, and $n = 100$.

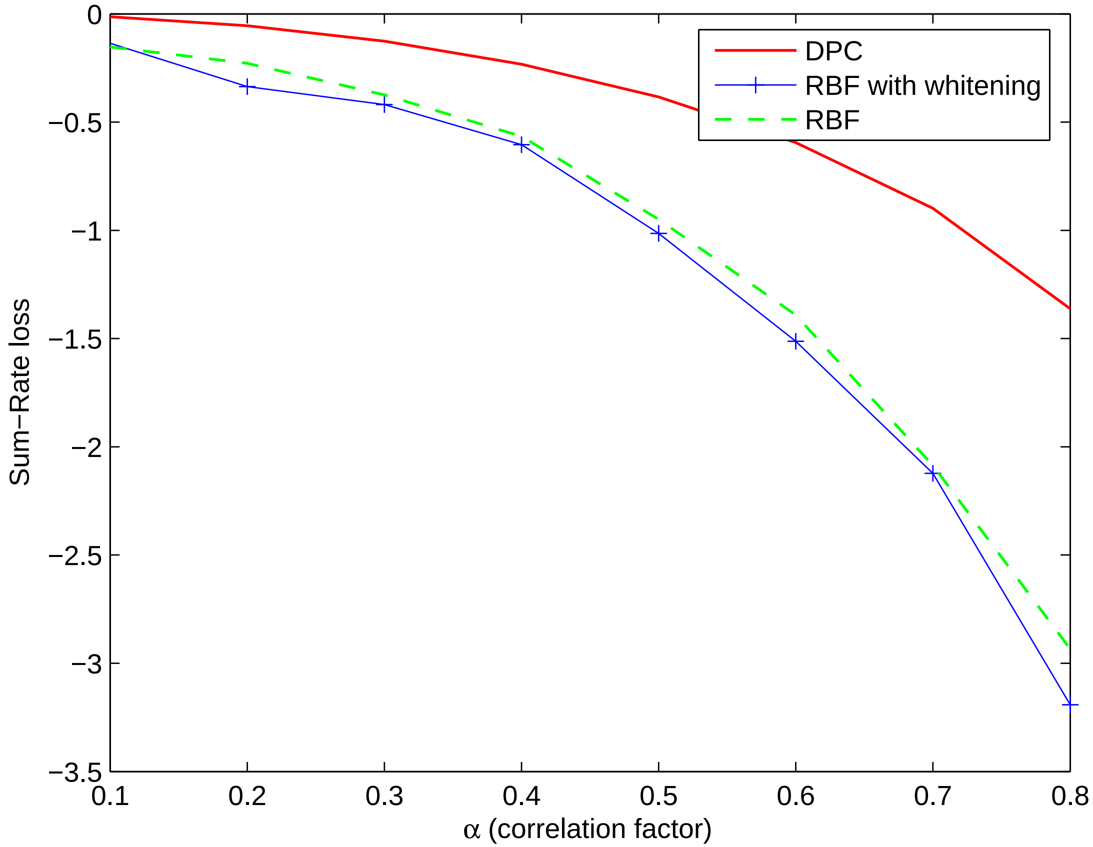


Figure 3: Sum-rate loss versus the correlation factor α for a system with $M = 3$ and $n = 100$.

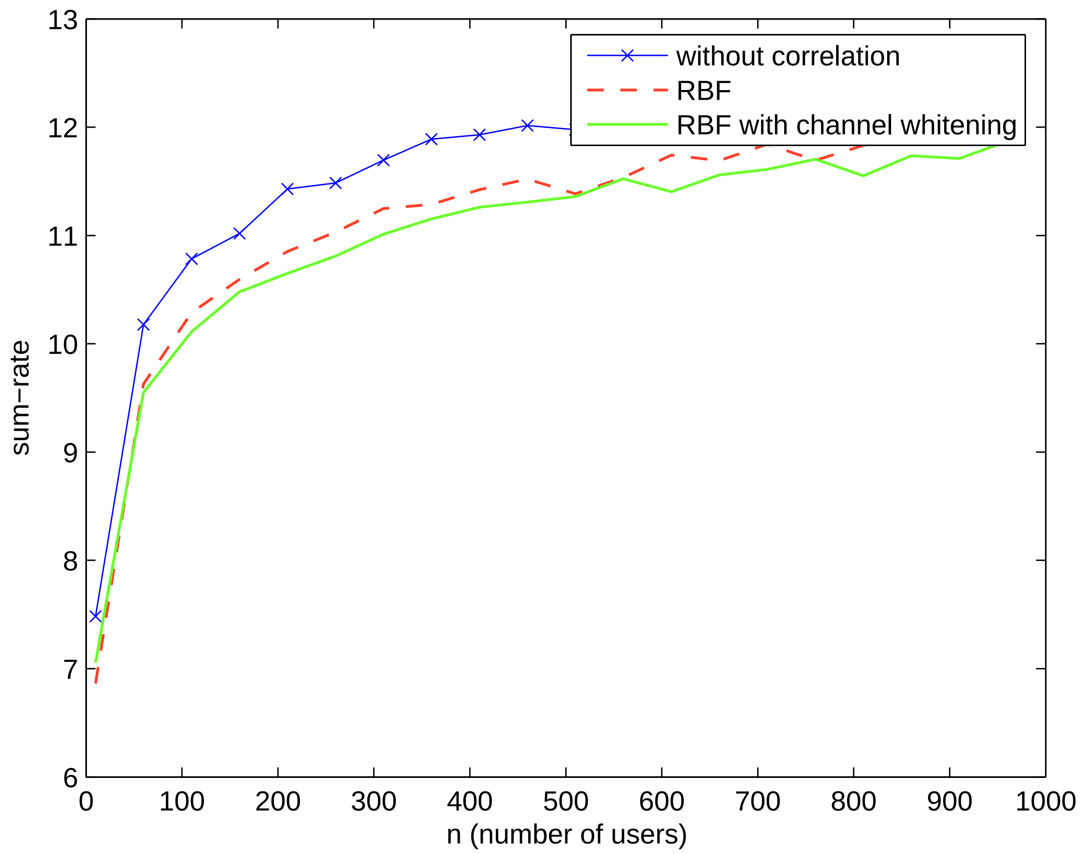


Figure 4: Sum-rate versus the number of users in a system with $M = 2$ and $\alpha = 0.5$

Conclusion for Part I

- Studied the effect of spatial correlation on various multiuser schemes for MIMO broadcast channels.
- Considered DPC and random, deterministic, and channel whitening schemes.
- All these techniques exhibit the same scaling for iid channels

$$\boxed{R_{\text{sum-rate}} = M \log\log n + M \log \tfrac{P}{M}}$$

- In the presence of correlation between transmit antennas, scaling is

$$\boxed{R_{\text{sum-rate}} = M \log\log n + M \log \tfrac{P}{M} + M \log c}$$

  The constant $0 < c \leq 1$ depends on the scheduling scheme and the eigenvalues of the correlation matrix R.

Extra Slide: Finding the Distribution of $SINR$

- Consider the $SINR$ for the first beam

$$\mathrm{SINR}_{i,1} = \frac{|H_i \phi_1|^2}{1/\rho + \sum_{n=2}^{M} |H_i \phi_n|^2},$$

- Define S by

$$S = -\frac{x}{\rho} + H_i^*((1+x)\phi_1\phi_1^* - xI)H_i$$

Then

$$\begin{aligned} P(\mathrm{SINR}_{i,1} > x) = P(S > 0) &= \int_{-\infty}^{\infty} P(H_i)u(S)dH_i \\ &= \frac{1}{\pi^M \det(R)} \int_{-\infty}^{\infty} e^{-H_i^* R^{-1} H_i} u(S) dH_i \end{aligned}$$

- To evaluate integral, use the integral representation of unit step

$$u(S) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(j\omega+\beta)S}}{j\omega + \beta} d\omega$$

- Desired probability becomes

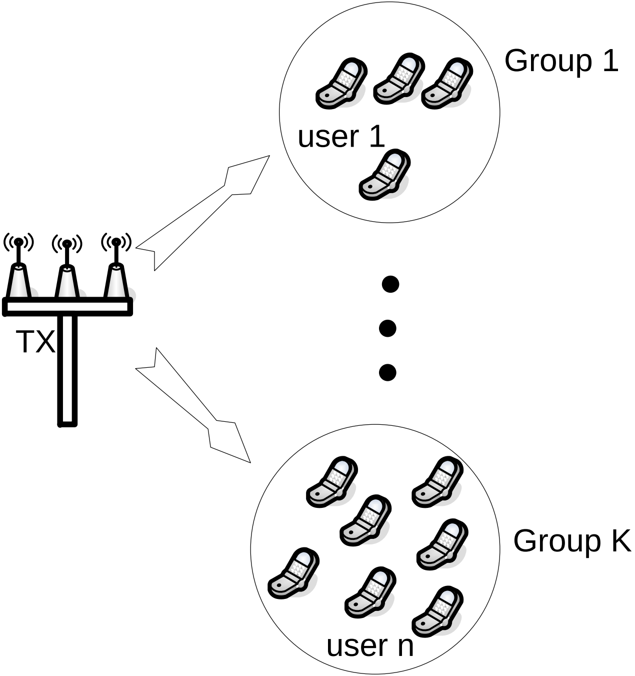
$$
\begin{aligned}
P(\ \mathrm{SINR}_{i,1} > x) & \\
= \quad & \frac{1}{2\pi^{M+1}\det(R)} \int_{-\infty}^{\infty} d\omega \frac{1}{j\omega+\beta} \int_{-\infty}^{\infty} dH_i e^{(j\omega+\beta)S - H_i^* R^{-1} H_i} \\
= \quad & \frac{1}{2\pi^{M+1}\det(R)} \int_{-\infty}^{\infty} d\omega \frac{e^{-(j\omega+\beta)\frac{x}{\rho}}}{j\omega+\beta} \int_{-\infty}^{\infty} dH_i e^{-H_i^* \tilde{R}^{-1} H_i} \\
= \quad & \frac{1}{2\pi^{M+1}\det(R)} \int_{-\infty}^{\infty} d\omega \frac{e^{-(j\omega+\beta)\frac{x}{\rho}}}{j\omega+\beta} \frac{1}{\det(\tilde{R})}
\end{aligned}
$$

Part II:

Scaling Laws of Multiple-Antenna Group Broadcast Channels

Group broadcast scenario

- Broadcast problem: users interested in *independent* information
- Group Broadcast: Groups of users, each group of users interested in the same information
  - e.g. DAB/DVB with limited shows; users classified according to shows they are interested in
  - Single group: multicast problem (Khitsi et. al. 06, Jindal and Luo 06)
  - Multiple-groups each consisting of one user: broadcast problem


Group 1
user 1
TX
Group K
user n

Three main questions in a broadcast scenario

Q1) Quantify the maximum sum rate possible to all users

Q2) Quantify the asymptotic behavior in regimes of interest

Q3) How do scheduling schemes performs under various non-idealities

Would like to answer **Q2)**: Asymptotic behavior in various regimes (large number of users and antennas)

System model

- Base station equipped with M antennas
- n users each equipped with a single receive antenna.
- n single-antenna users with received signal

$$y_i = h_i^* s + \nu_i$$

  - Input satisfies $E[s^* s] \leq P$
  - Noise is white Gaussian $\nu \sim CN(0, I_M)$
  - User channels are independent and distributed as $CN(0, I_M)$
- Users are partitioned into K groups of $\frac{n}{K}$ users each; each group is interested in the same data.

Group broadcast capacity: Formal expression

- When there is one user only

$$C_{\text{one user}} = E \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \log\det\left(1 + \|h\|_B^2\right)$$

- Single group broadcast

$$\boxed{C_{\text{single group}} = E \max\nolimits_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min\nolimits_i \log\det\left(1 + \|h_i\|_B^2\right)}$$

- Group broadcast eventually limited by the worst user

Group broadcast capacity: Formal expression (2)

- Multiple groups broadcast: K power matrices $B_1, \ldots, B_K$, one for each group.

- Matrices should maximize sum-rate under total power constraint

$$C_{\text{multiplegroups}} = E \max_{B_k \geq 0 \ \sum_{k=1}^{K} \mathrm{Tr}(B_k) \leq P} \log\det\left(1 + \sum_{k=1}^{K} \|h_k\|_{B_k}^2\right)$$

- With K *user* groups, we need to take care of the "worst" user of each group

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Study behavior of C_{GB} for large number of users n and antennas M
  - Large n and fixed M
  - Large M and fixed n
  - Large M and n with $M = \beta n$
  - Large M and n with $M = \log n$

Capacity bounding techniques (1)

Upper bounds

1. K times rate of one group

$$
\begin{aligned}
C_{GB} &\leq KC_{\text{single group}} \\
&= K\log(1 + \max_{B\geq 0\ \mathrm{Tr}(B)\leq P} \min_i \|h_i\|_B^2)
\end{aligned}
$$

2. MAC-BC duality

- Maximum sum rate for K users, chosen one from each group

$$
C_{K\text{ users}} = \max_{\substack{b_k \geq 0 \\ \sum_{k=1}^K b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right)
$$

- Rate has to appeal to every user in every group

$$C_{K \text{ users}} \leq \min_{h_{i_1}} \cdots \min_{h_{i_K}} \max_{\substack{b_k \geq 0 \\ \sum_{k=1}^{K} b_k = P}} \log\det\left(I + \sum_{k=1}^{K} h_{i_k} b_k h_{i_k}^*\right)$$

- Get rid of the determinant using AM-GM inequality $\det(A) \leq \left(\frac{\mathrm{tr}(A)}{M}\right)^M$ to write

$$C_{GB} \leq M \log\left(1 + \frac{P}{M} \max_k \min_{h_{i_1}} \cdots \min_{h_{i_K}} \{\|h_{i_1}\|^2, \cdots, \|h_{i_K}\|^2\}\right)$$

Capacity bounding techniques (2)

Lower Bounds

1. Time sharing

$$C_{GB} \geq \frac{1}{K} \sum_{k=1}^{K} \log\det \left(1 + \max_{B_k \geq 0 \ \mathrm{Tr}(B_k)=P} \min_{h_{i_k}} \|h_{i_k}\|_{B_k} \right)$$

2. Treating interference as noise

$$C_{GB} \geq K \log \left(\frac{\frac{1}{K} \frac{P}{M} \min_i \|h_i\|^2}{1 + \frac{K-1}{K} \frac{P}{M} \min_i \|h_i\|^2} \right)$$

$$\boxed{\text{Need to study scaling of the weighted max} - \text{min norm}}$$

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Obtain upper and lower bounds on C_{GB}; bounds depend on the max-min weighted norm

$$\max_{B \geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

- Find upper and lower bounds on the max-min in terms of the h_i's

Bounds on the max-min weighted Euclidean norm

Here we obtain upper and lower bounds on the weighted Euclidean norm for fixed M and n

<u>Lower Bounds</u>

1. max-min norm is greater than min norm

$$\max_{\mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2 \geq \frac{P}{M} \min_i \|h_i\|^2$$

2. h_i belongs to a finite set $\{h_1, \cdots, h_{\frac{n}{K}}\}$

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\frac{n}{K}} \min_i \|h_i\|^2$$

So

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\min\{M, \frac{n}{K}\}} \min_i \|h_i\|^2$$

3. Diagonal values and eigenvalues: Define $H = [h_1 \cdots h_{\frac{n}{K}}]$, then

$$\lambda_{min}(H^* H) \leq \min_i \|h_i\|^2 \leq \lambda_{max}(H^* H)$$

Upper Bounds

1. max-min is less than min-max

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \leq P \min_i \|h_i\|^2$$

2. Replace minimization with averaging (Jindal and Luo '06)

$$\begin{aligned} \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 &\leq \max_B \frac{1}{\frac{n}{K}} \sum_{i=1}^{\frac{n}{K}} \|h_i\|_B^2 \\ &\leq P \lambda_{max}(H^* H) \end{aligned}$$

Study boils down to studying the scaling of

1) min norm $\min_i \|h_i\|^2$ 2) eigenvalues of $H^* H$

Our Approach

- C_{GB} is difficult to calculate, so find the asymptotics
- Obtain upper and lower bounds on C_{GB}; bounds depend on the max-min weighted norm

$$\max_{B\geq 0 \ \mathrm{Tr}(B)=P} \min_i \|h_i\|_B^2$$

- Find upper and lower bounds on the max-min in terms of the h_i's
- Find the asymptotics of $\min_i \|h_i\|^2$

Scaling of the Euclidean norm

In the rest of the presentation, we study the scaling of the minimum Euclidean norm $\min_i \|h_i\|^2$ for

- Large n and fixed M
- Large M and fixed n
- Large M and n with $M = \beta n$
- Large M and n with $M = \log n$

Scaling of the minimum of iid variables

- Let $x_1, x_2, \cdots, x_n$ be nonnegative iid r. v.'s with CDF $F(x)$, and CF $\phi(x)$.
- Need to find scaling law of $x_{\min}(n) = \{x_1, x_2, \cdots, x_n\}$
- CDF of the mimimum is given by

$$F_{\min}(x) = 1 - (1 - F(x))^n$$

- Can show $n^{\frac{1}{i_0}} x_{\min}(n)$ converges in distribution to y with CDF

$$F_y(y) = 1 - \exp\left(-\frac{F^{(i_0)}(0)}{i_0!} y^{i_0}\right)$$

- We thus say that

$$\boxed{x_{\min} \text{ converges to } \frac{E}{n^{\frac{1}{i_0}}}}$$

where E is the expectation that arises from the distribution (1)

$$
\begin{aligned}
E &= \int_0^\infty \exp\left(-\frac{F^{(i_0)}(0)}{i_0!}x^{i_0}\right) \\
&= \frac{C_{i_0}}{F^{(i_0)}(0)^{\frac{1}{i_0}}} \quad C_{i_0} = \frac{\Gamma(\frac{1}{i_0})(i_0!)^{\frac{1}{i_0}}}{i_0}
\end{aligned}
$$

- The constant i_0 is the least i_0 for which $F^{(i_0)}(0) \neq 0$
- Can find i_0 and $F^{(i_0)}(0)$ using initial value theorem

$$\boxed{\lim_{x\to 0} F^{(i_0)}(x) = \lim_{s\to\infty} s^{i_0}\phi(s)}$$

- Note that there is no restriction on distribution $F(x)$

Scaling for large n, fixed M

- Scaling law for $\min_{h_i} \|h_i\|^2$, $h_i \sim CN(0, R)$.
- CDF of $\|h_i\|^2$ will have different forms depending on eigenvalues of R
- Characteristic function given by

$$\phi(s) = \prod_{l=1}^{M} \frac{1}{1 + \lambda_l s}$$

- It is easy to see that

$$F^{(i_0)}(0) = \lim_{s \to \infty} s^i \phi(s) = \begin{cases} 0 & \text{for } i < M \\ \frac{1}{\det(R)} & \text{for } i = M \end{cases}$$

- We thus conclude that

$$\boxed{\min_i \|h_i\|^2 \text{ scales as } C_M \det(R)^{\frac{1}{M}} \frac{1}{n^{\frac{1}{M}}}} \quad C_M = \frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M}$$

Scaling for large M, fixed n

- By the law of large numbers

$$\min_i \|h_i\|^2 = M$$

which implies

$$\boxed{\max \min \|h_i\|_B^2 \leq PM}$$

- Applying the law of large numbers to

$$\max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2 \geq \frac{P}{\min\{M, \frac{n}{K}\}} \min_i \|h_i\|$$

implies

$$\boxed{\max \min \|h_i\|_B^2 \geq P \tfrac{K}{n} M}$$

Scaling for large M and n, $M = \beta n$

We consider the regime: $M, n \to \infty$ with $M = \beta n$

- Use $\lambda_{min}(H_i^* H_i) \leq \min_i \|h_i\|^2$ to show

$$\min_i \frac{\|h_i\|^2}{M} \geq (1 - \sqrt{K\beta})^2$$

which implies

$$\boxed{\max\min \frac{\|h_i\|_B^2}{M} \geq P(1 - \sqrt{K\beta})^2}$$

- Use $\max \ \min_i \|h_i\|_B^2 \leq P\frac{K}{n}\lambda_{max}(H^* H)$ to show

$$\boxed{\max\min \frac{\|h_i\|_B^2}{M} \leq P(1 + \frac{1}{\sqrt{K\beta}})^2}$$

Behavior of the min Euclidean Norm

The behavior of $\min_i \|h_i\|^2$ looks like

Regime	**Asymptotic Value**	**Method**
large n	$\frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M} \frac{1}{n^{\frac{1}{M}}}$	min of iid r.v. Theorem
large M	M	Law of large numbers
$M = \beta \frac{n}{K}$	$\geq (1 - \sqrt{K\beta})^2$ $\leq (1 + \sqrt{K\beta})^2$	Random Matrix theory
$M = \log n$	$\mathcal{H} \in [1 - \epsilon_l, 1]$ $\epsilon \simeq .8414$	Chernof Bound

Behavior of the max min Euclidean Norm

The behavior of $\max_B \min_i \|h_i\|^2$ looks like

Regime	**Lower Bound**	**Upper Bound**
large n	$\frac{C_M}{M}\frac{1}{n^{\frac{1}{M}}}$	$C_M\frac{1}{n^{\frac{1}{M}}}$
large M	$P\frac{K}{n}M$	PM
$M=\beta\frac{n}{K}$	$P(1-\sqrt{K\beta})^2$	$P(1+\frac{1}{\sqrt{K\beta}})^2$
$M=\log n$	$P\mathcal{H}\ \ ,\mathcal{H}\in[1-\epsilon_l,1]\ \ \epsilon\simeq .8414$	constant

$C_M=\frac{\Gamma(\frac{1}{M})(M!)^{\frac{1}{M}}}{M}$

GB scaling for large n, fixed M

- Group broadcast capacity scales as

$$\boxed{C_{GB} = \alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

where

$$\frac{1}{M} \leq \alpha \leq 1$$

- For spatially correlated case, the capacity incurs a $\det(R)^{\frac{1}{M}}$ hit

$$\boxed{C_{GB} = \alpha \det(R)^{\frac{1}{M}} P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}}}$$

- Unfortunate result: sum-rate decreases with the number of users.
- Counter this: increase the resources (i.e., number of antennas M).

GB scaling for large M, fixed n

- Upper bound: K times rate of single group

$$C_{GB} \quad \leq \quad K \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \log(1 + \min_i \|h_i\|_B^2)$$

$$\boxed{C_{GB} \leq K \log(1 + PM) \quad \text{(law of large numbers)}}$$

- Lower bound: Use time sharing

$$C_{GB} \geq \log(1 + \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B)$$

$$\boxed{C_{GB} \geq \log(1 + P\tfrac{K}{n} M)}$$

GB scaling with M and n, $M = \beta n$

- Number of users and antennas grow to infinity while their ratio remains constant $\frac{M}{n} = \frac{\beta}{K}$.
- Lower bound: Use time sharing

$$\boxed{C \geq \log\left(1 + P(1 - \sqrt{K\beta})^2\right)}$$

- To obtain an upper bound, we start with the bound

$$C_{GB} \leq K \log(1 + \max_{B \geq 0 \ \mathrm{Tr}(B) \leq P} \min_i \|h_i\|_B^2)$$

to show

$$\boxed{C_{GB} \leq K \log(1 + P(1 + \tfrac{1}{\sqrt{\beta}})^2)}$$

- If we allow the number of antennas to grow linearly with the number of users, we can guarantee a constant sum rate.

Can we have constant rate with sublinear growth?

- But is it still possible to do so without straining the resources as much?
- We showed that for large n

$$\begin{aligned} C &= \alpha P C_M \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}} \qquad \frac{C_M}{M} \simeq 1 \\ &= \alpha P \frac{K^{\frac{1}{M}}}{n^{\frac{1}{M}}} \end{aligned}$$

- To guarantee a constant rate, intuition requires to set $M = \log n$

GB scaling with M and n, $M = \log n$

- Use the Chernof bound, we show that

$$\lim_{M=\log n, n\to\infty} \min_i \frac{\|h_i\|^2}{M} = \mathcal{H} \in [1-\epsilon_l, 1] \quad \text{w.p.1}$$

  where $\epsilon_l \simeq .8414$.

- Capacity is lower-bounded by a constant

$$\boxed{C \geq \log(1 + P\mathcal{H})} \tag{1}$$

- Capacity is also upper bounded by a constant because it is for $M = \beta n$

Can we do any better?

- Can we guarantee a constant capacity per stream without straining resources as much?
- Can show that number of transmit antennas, M, should grow faster than $(\log n)^{\frac{1}{2}-\epsilon(n)}$,

$$\epsilon(n) = \frac{\log\log\log n}{\log\log n}$$

to guarantee constant rate per stream

Conclusion for Part II

- Studied the scaling law of the group broadcast problem
- Capacity decreases as $n^{-\frac{1}{M}}$ with number of users
- To guarantee a constant rate if we allow M to grow as $\log n$
- As a by-product (or a prerequisite), we studied the scaling of
  - Minimum Euclidean norm $\min_i \|h_i\|^2$
  - Max min Euclidean norm $\max_B \min_i \|h_i\|_B^2$

  in various regimes
- Most results apply for general distributions on h_i