De-noising Compressed Estimates of Random Sparse Vectors
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1 Signal Model and problem formulation

Suppose we transmit a sparse vector x of length N, and assume this vector is sparse (i.e. the number of nonzero coefficients, say s is much smaller than the number of dimensions N). Furthermore, assume we cannot observe the entire vector x at the receiver, but rather we observe its projection onto a subspace contaminated by noise, $y = \Psi x + z$, where Ψ is an $m \times N$ random measurement matrix and z is AWGN. In general, $s < m << N$ and Ψ is typically either a Gaussian matrix or a random selection of m rows from the $N \times N$ Fourier Matrix.

This scenario appears frequently in communications and signal processing applications and yet we're unable to use a standard inversion method such as Least Squares (LS) to estimate x since we have less equations than unknowns -hence infinitely many solutions in terms of linear algebra. This is where compressive sensing comes in ([1],[2],[3]). One method of coping with this is to pose this problem as an optimization problem that selects among the infinite possible solutions the one that concentrates most of the energy of x on the fewest possible number of coefficients, i.e. the sparsest solution. A typical example is

$$\hat{x}^{cs} = \arg\min \|y - \Psi x\|_2^2 + \lambda \|x\|_1 \tag{1.1}$$

where λ is a user parameter.

After familiarizing yourself with compressive sensing, you will notice that most theories of recoverability of sparse vectors from incomplete measurements are expressed in terms of the mean square error (MSE) between the sparse vector and its estimate, $E[(x - \hat{x}^{cs})^2]$. The problem is that such a metric can be misleading if one is concerned primarily with exact support recovery for model selection, i.e. one wishes to minimize the ℓ_0 norm between the sparse vector and its estimate. The problem is amplified in practice when we have noisy measurements, since the estimation output of the compressive sensing algorithm typically returns a fuzzy vector that is not strictly sparse, and one is left with the problem of discerning true nonzero locations from false ones.

Another major problem we face in practice is when we have random sparse vectors and hence random sparsity levels in each sample or realization. Unfortunately papers on recoverability assume the sparsity level is known, and most papers proposing greedy recovery algorithms such as Orthogonal Matching Pursuit assume this as well, while in many cases in communications and signal processing this parameter is unknown and could be very difficult to optimally estimate.

2 Objective

Much research is going on currently to find techniques to eliminate false support positives and negatives from the compressive sensing output. Similar research is also being conducted for devising algorithm termination techniques for greedy methods, hence also providing a means to reduce false positives. You are to explore some of these techniques, conduct a literature review, and perhaps modify previous techniques or propose new ones.

Start by the tutorials given in [3], these will lead you to more sophisticated introductions to compressive sensing such as [1]. After you've understood the proposed problem introduced here, you can move on to references such as [4] and [6], and perhaps to [7] for a Bayesian approach, assuming the nonzero entries in x are Gaussian. Good luck!

References

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