Regularized Least Squares

Consider the problem

$$\min_{w} \|w\|_{\Pi}^{2} + \|y - Hw\|^{2} \qquad \Pi > 0$$

1. Articulate it as a LS problem

2. ~~Find~~ the corresponding normal equations

3. Show that the ~~normal~~ equations have a unique soln

$$\|w\|_{\Pi}^2 + \|y - Hw\|^2 = \|w\|^2_{\Pi^{*/2}\Pi^{1/2}} + \|y - Hw\|^2$$

$$= \|\Pi^{1/2} w\|^2 + \|y - Hw\|^2$$

$$= \|0 - \Pi^{1/2} w\|^2 + \|y - Hw\|^2$$

$$= \left\| \begin{bmatrix} y \\ 0 \end{bmatrix} - \begin{bmatrix} Hw \\ \Pi^{1/2} w \end{bmatrix} \right\|^2$$

$$= \left\| \begin{bmatrix} y \\ 0 \end{bmatrix} - \begin{bmatrix} H \\ \Pi^{1/2} \end{bmatrix} w \right\|^2$$

Corresponding normal equation:

$$\begin{bmatrix} H^* & \Pi^{*/2} \end{bmatrix} \begin{bmatrix} H \\ \Pi^{1/2} \end{bmatrix} w = \begin{bmatrix} H^* & \Pi^{*/2} \end{bmatrix} \begin{bmatrix} y \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(H^* H + \Pi^{*/2} \Pi^{1/2} \right) w = H^* y$$

We know that

$$H^* H \geq 0$$

Now $\quad H^* H + \Pi > 0$

To see this, note that

$$x^* (H^* H + \Pi) x = \underbrace{x^* (H^* H) x}_{\geq 0} + \underbrace{x^* \Pi x}_{> 0}$$

$\Rightarrow \quad x^* (H^* H + \Pi) x > 0$

$\Rightarrow \quad H^* H + \Pi$ is invertible & the solution of the regularized LS prob. is unique

$$\boxed{\hat{w} = (H^* H + \Pi)^{-1} H^* y}$$