KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS
ELECTRICAL ENGINEERING DEPARTMENT
SPRING 2010 (102)
EE 315 Probabilistic Methods in Electrical Engineering
QUIZ #3

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Q1. The joint pdf of the random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c(x+y), & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

a) Find the value of c such that $f_{X,Y}(x,y)$ is a valid pdf. (6 points)

We know that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy$ must be equal to 1 for $f_{X,Y}(x,y)$ to be a valid pdf function.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= c\int_0^1\int_0^1 (x+y)dxdy \\
&= c\int_0^1 \left[\frac{x^2}{2}+xy\right]_{x=0}^1 dy = c\int_0^1 \left[\frac{x^2}{2}+xy\right]_{x=0}^1 dy \\
&= c\int_0^1 \left(\frac{1}{2}+y\right)dy = c\left[\frac{1}{2}y+\frac{y^2}{2}\right]_{y=0}^1 \\
&= c\left[\frac{1}{2}+\frac{1}{2}\right] = c
\end{aligned}$$

So, $\boxed{c=1}$

b) Find the joint CDF of X and Y for $-\infty < x < \infty, \quad -\infty < y < \infty$. (12 points)

$$F_{X,Y}(x,y)=\begin{cases} 0, & -\infty<x<0,\quad -\infty<y<0 \\ 0, & -\infty<x<0,\quad 0\le y<1 \\ 0, & -\infty<x<0,\quad 1\le y<\infty \\ 0, & 0\le x<1,\quad -\infty<y<0 \\ 0, & 1\le x<\infty,\quad -\infty<y<0 \\ \int_0^x\int_0^y(\alpha+\beta)\,d\beta\, d\alpha & 0\le x<1,\quad 0\le y<1 \\ \int_0^x\int_0^1(\alpha+\beta)\,d\beta\, d\alpha & 0\le x<1,\quad 1\le y<\infty \\ \int_0^1\int_0^y(\alpha+\beta)\,d\beta\, d\alpha & 1\le x<\infty,\quad 0\le y<1 \\ \int_0^1\int_0^1(\alpha+\beta)\,d\beta\, d\alpha & 1\le x<\infty,\quad 1\le y<\infty \end{cases}$$

Since

$$\int_0^x\int_0^y(\alpha+\beta)\,d\beta\, d\alpha=\int_0^x\left[\alpha\beta+\frac{\beta^2}{2}\right]_{\beta=0}^{y}d\alpha=\int_0^x\left(\alpha y+\frac{y^2}{2}\right)d\alpha=\left[\frac{\alpha^2 y}{2}+\frac{\alpha y^2}{2}\right]_{\alpha=0}^{x}$$

$$=\left[\frac{x^2y}{2}+\frac{xy^2}{2}\right]$$

and

$$\int_0^x\int_0^1(\alpha+\beta)\,d\beta\, d\alpha=\left[\int_0^x\int_0^y(\alpha+\beta)\,d\beta\, d\alpha\right]_{y=1}$$

$$=\left[\frac{x^2y}{2}+\frac{xy^2}{2}\right]_{y=1}=\frac{x^2}{2}+\frac{x}{2}$$

and

$$\int_0^1\int_0^y(\alpha+\beta)\,d\beta\, d\alpha=\left[\int_0^x\int_0^y(\alpha+\beta)\,d\beta\, d\alpha\right]_{x=1}$$

$$=\left[\frac{x^2y}{2}+\frac{xy^2}{2}\right]_{x=1}=\frac{y}{2}+\frac{y^2}{2}$$

and

$$\int_0^1\int_0^1(\alpha+\beta)\,d\beta\, d\alpha=\left[\int_0^x\int_0^y(\alpha+\beta)\,d\beta\, d\alpha\right]_{x=1,\,y=1}$$

$$=\left[\frac{x^2y}{2}+\frac{xy^2}{2}\right]_{y=1}=\frac{1}{2}+\frac{1}{2}=1$$

So,

$$F_{X,Y}(x,y)=\begin{cases} 0, & -\infty<x<0, \quad -\infty<y<0 \\ 0, & -\infty<x<0, \quad 0\le y<1 \\ 0, & -\infty<x<0, \quad 1\le y<\infty \\ 0, & 0\le x<1, \quad -\infty<y<0 \\ 0, & 1\le x<\infty, \quad -\infty<y<0 \\ \dfrac{x^2y}{2}+\dfrac{xy^2}{2} & 0\le x<1, \quad 0\le y<1 \\ \dfrac{x^2}{2}+\dfrac{x}{2} & 0\le x<1, \quad 1\le y<\infty \\ \dfrac{y}{2}+\dfrac{y^2}{2} & 1\le x<\infty, \quad 0\le y<1 \\ 1 & 1\le x<\infty, \quad 1\le y<\infty \end{cases}$$

c) Find the marginal pdf of *Y*. (2 points)

$$f_Y(y)=\int_0^1(\alpha+y)\,d\alpha$$

$$=\left[\frac{\alpha^2}{2}+\alpha y\right]_{\alpha=0}^{1}$$

$$=\frac{1}{2}+y$$

d) Find the conditional pdf $f_X(x\,|\,Y=0.5)$. (5 points)

$$f_X(x\,|\,y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\begin{cases} \dfrac{0}{f_Y(y)} & -\infty<x<0 \\ \dfrac{f_{X,Y}(x,y)}{f_Y(y)} & 0\le x<1 \\ \dfrac{0}{f_Y(y)} & 1\le x<\infty \end{cases}$$

$$f_X(x\,|\,y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}=\begin{cases} 0 & -\infty<x<0 \\ \dfrac{x+y}{\frac{1}{2}+y} & 0\le x<1 \\ 0 & 1\le x<\infty \end{cases}$$

Now put $y = 0.5$ to get the final answer.

d) Find expectation $E\left[X \mid Y = 0.5\right]$. (6 points)

This can be evaluated using $f_X\left(x \mid y\right)$ found above

$$
\begin{aligned}
E\left[X \mid Y = 0.5\right] &= \int_{-\infty}^{\infty} x \cdot f_X(x \mid Y = 0.5)dx \\
&= \int_0^1 \left[x \cdot \frac{x+y}{\frac{1}{2}+y}\right]_{y=\frac{1}{2}} dx \\
&= \int_{-\infty}^{\infty} \left(x \cdot \frac{x+\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}}\right)dx \\
&= \int_0^1 \left(x^2 + \frac{x}{2}\right)dx \\
&= \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_{x=0}^1 \\
&= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}
\end{aligned}
$$

e) Are X and Y statistically independent? (6 points)

X and Y are not statistically independent (they are statistically dependent).

This is seen from the fact that

$$
\begin{aligned}
f_Y(y) &= \int_0^1 \left(\alpha + y\right)d\alpha \\
&= \left[\frac{\alpha^2}{2} + \alpha y\right]_{\alpha=0}^1 \\
&= \frac{1}{2} + y
\end{aligned}
$$

and

$$
\begin{aligned}
f_X(x) &= \int_0^1 \left(x + \beta\right)d\beta \\
&= \left[x\beta + \frac{\beta^2}{2}\right]_{\beta=0}^1 \\
&= x + \frac{1}{2}
\end{aligned}
$$

Clearly, $f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y)$

Q 2. The following information is known about two random variables X and Y :

$$E[X]=0, \qquad E[Y]=-1, \qquad E[X^2]=2, \qquad E[Y^2]=4, \quad and \quad R_{XY}=-2$$

Two new random variables W and Z are defined as

$$W = X + 2Y$$
$$Z = -X - 3Y$$

a) Find $E[W]$, $E[Z]$, $E[W^2]$, and $E[Z^2]$. (8 points)

The expectation operation $E[\cdot]$ is a linear operation

$$E[W]=E[X]+2E[Y]=0+2(-1)=-2$$

$$E[Z]=-E[X]-3E[Y]=0-3(-1)=3$$

$$E[W^2]=E\left[(X+2Y)^2\right]=E\left[X^2+4XY+4Y^2\right]=E[X^2]+4E[XY]+4E[Y^2]$$
$$=2+4(-2)+4(4)=10$$

$$E[Z^2]=E\left[(-X-3Y)^2\right]=E\left[X^2+6XY+9Y^2\right]=E[X^2]+6E[XY]+9E[Y^2]$$
$$=(2)+6(-2)+9(4)=26$$

b) Are W and Z uncorrelated? Orthogonal? Justify your answer. (5 + 2points)

To check if W and Z are uncorrelated, we have to check if $R_{WZ}=E[W]\cdot E[Z]$

$$E[W]=-2$$
$$E[Z]=3$$

$$E[WZ]=E\left[(X+2Y)(-X-3Y)\right]=E\left[-X^2-3XY-2XY-6Y^2\right]$$
$$=-E[X^2]-5E[XY]-6E[Y^2]$$
$$=-2-5(-2)-6(4)=-16$$

Since $\{R_{WZ}=E[WZ]=-16\}\neq\{E[W]E[Z]=-6\}$ ➔ W and Z are correlated to each other (not uncorrelated)

Also, since *E[W].E[Z]* = -6, and is not equal to zero, they are not orthogonal.