KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

ELECTRICAL ENGINEERING DEPARTMENT

EE 315

MAJOR 2

DATE: May 11, 2011

TIME: 7:00 PM-9:00 PM

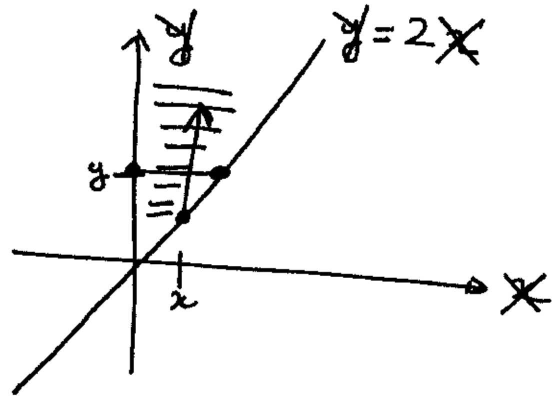
ID#	
Name	
Section#	

	Maximum	**Score**
Problem 1	**9**	
Problem 2	**10**	
Problem 3	**13**	
Problem 4	**8**	
TOTAL	**40**	

Pb 1

1.0

a. $k \int_{x=0}^{\infty} \int_{y=2x}^{\infty} e^{-x} e^{-y} \, dy \, dx = 1$



$k \int_0^{\infty} e^{-x} \frac{e^{-y}}{-1} \Big|_{2x}^{\infty} dx = k \int_0^{\infty} e^{-x} e^{-2x} dx$

2

$= k \frac{e^{-3x}}{-3} \Big|_0^{\infty} = \frac{k}{-3} [0 - 1] = \frac{k}{3} = 1 \rightarrow \boxed{k = 3}$.

1

b. for $f_X(x)$, we fix $X = x$

1.5

$f_X(x) = 3 \int_{y=2x}^{\infty} e^{-x} e^{-y} dy = 3 e^{-x} e^{-2x} = 3 e^{-3x}$ $x > 0$

1 3 and $\underline{0 \text{ for } x \leq 0}$ 0.5

$f_Y(y) = 3 e^{-y} \int_0^{\frac{y}{2}} e^{-x} dx = 3 e^{-y} [1 - e^{-\frac{y}{2}}]$ 1.5 $y > 0$

1 3 $\underline{0 \text{ for } y \leq 0}$ 0.5

c. $f_X(x) \cdot f_Y(y) = 9 e^{-3x} e^{-y} [1 - e^{-\frac{y}{2}}]$

0.5 $\neq f_{X,Y}(x,y)$ 1

hence X, Y not statistically independent.

0.5

Problem 2:

Two random variables X and Y have a joint pdf given by:

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{4}(2x+y), & 0\le x\le 1,\quad 0\le y\le 2\\ 0, & \text{elsewhere}\end{cases}$$

4.5 a) Find the joint CDF $F_{X,Y}(x,y)$ of X and Y over all $-\infty<x<\infty,\quad -\infty<y<\infty$.

3.0 b) Find the conditional pdf $f_X(x\mid Y=1)$.

2.5 c) Find the probability $P\{2X\le Y\}$.

(a)

- For $x<0$ or $y<0$ $\quad F_{X,Y}(x,y)=0$ $\quad$ 0.5

- For $x>1$ & $y\geq 2$ $\quad F_{X,Y}(x,y)=1$ $\quad$ 0.5

- For $0\le x\le 1$ & $0\le y\le 2$

$$F_{X,Y}(x,y)=\int_0^x\int_0^y \frac{1}{4}(2z_1+z_2)\,dz_2\,dz_1 \quad 0.5$$

$$=\frac{1}{4}\int_0^x\int_0^y 2z_1\,dz_2\,dz_1+\frac{1}{4}\int_0^x\int_0^y z_2\,dz_2\,dz_1 \quad 1$$

$$=\frac{1}{4}x^2y+\frac{1}{8}xy^2 \quad (*)$$

- For $0\le x\le 1$ & $y\geq 2$

$$F_{X,Y}(x,y)=\frac{1}{4}x^2y+\frac{1}{8}xy^2\Big|_{y=2} \quad \text{evaluate } (*) \text{ at } y=2$$

$$=\frac{1}{2}x+\frac{1}{2}x^2 \quad 1$$

- For $0\le y\le 2$ & $x\geq 1$

$$F_{X,Y}(x,y)=\frac{1}{4}x^2y+\frac{1}{8}xy^2\Big|_{x=1} \quad \text{evaluate } (*) \text{ at } x=1$$

$$=\frac{1}{4}y+\frac{1}{8}y^2 \quad 1$$

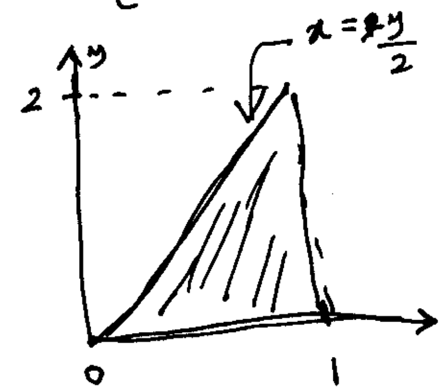
b) $$f_X(x\mid Y=1)=\frac{f_{X,Y}(x,y=1)}{f_Y(y=1)} \quad 0.5$$

$$f_Y(y)=\int_0^1 f_{X,Y}(x,y)\,dx \quad 0.5$$

So
$$f_X(x\mid Y=1)=\frac{\frac{1}{4}(2x+1)}{\frac{1}{4}+\frac{1}{4}\times 1}=\frac{2x+1}{2}$$

$$\Rightarrow f_X(x\mid Y=1)=\begin{cases}x+\frac{1}{2} & 0\le x\le 1\\ 0 & \text{otherwise}\end{cases} \quad 1$$

(c)



$$P\{2X\le Y\}=\int_0^2\int_0^{y/2}\frac{1}{4}(2x+y)\,dx\,dy \quad 1 \text{ (Correct limits)}$$

$$=\int_0^2 \frac{1}{4}x^2\Big|_0^{y/2}+\int_0^2\int_0^{y/2}\frac{1}{4}y\,dx\,dy$$

$$=\int_0^2\frac{1}{16}y^2\,dy+\int_0^2\frac{1}{8}y^2\,dy$$

$$=\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{3}=1/2. \quad 1.5$$

$$\frac{1}{4}\int^2 y^2\,dy \,/ +\frac{1}{8}\int_0^2 y^2\,dy$$

Problem 3:

The following information is known about two jointly Gaussian random variables X and Y:

$$E[X]=0, \quad E[Y]=-1, \quad E[X^2]=4, \quad E[Y^2]=9, \text{ and } R_{XY}=-4$$

Two new random variables W and U are defined as

$$W=3X+Y$$

$$U=-X-2Y$$

a) Find $E[W]$, $E[U]$, $E[W^2]$, and $E[U^2]$.

b) Find the variances σ_X^2, σ_Y^2, σ_U^2 and σ_W^2

c) Find the correlation R_{WU}

d) Are W and U uncorrelated? Justify your answer.

e) Determine the joint pdf of W and U.

4 a) $E[W] = E[3X+Y] = 3\bar{X} + \bar{Y} = -1$ 1

$E[U] = E[-X-2Y] = -\bar{X} - 2\bar{Y} = +2$ 1

$E[W^2] = \overline{W^2} = \overline{(3X+Y)^2} = 9\overline{X^2} + \overline{Y^2} + 6\overline{XY}$

$= 9\times 4 + 9 + 6(-4) = 45 - 24 = 21$ 1

$E[U^2] = \overline{(-X-2Y)^2} = \overline{X^2} + 4\overline{Y^2} + 4\overline{XY}$

$= 4 + 36 + 4(-4) = 40 - 16 = 24$ 1

4 b) $\sigma_X^2 = \overline{X^2} - \bar{X}^2 = 4$ 1

$\sigma_Y^2 = \overline{Y^2} - \bar{Y}^2 = 9 - 1 = 8$ 1

$\sigma_W^2 = \overline{W^2} - \bar{W}^2 = 21 - 1 = 20$ 1

$\sigma_U^2 = \overline{U^2} - \bar{U}^2 = 24 - 4 = 20$ 1

2 c) $R_{WU} = E[(3X+Y)(-X-2Y)] = E[-3X^2 - 7XY - 2Y^2]$

$= -3\overline{X^2} - 7\overline{XY} - 2\overline{Y^2} = -3(4) - 7(-4) - 2(9)$

$= -12 + 28 - 18 = -2$ 2

2 d) $C_{WU} = R_{WU} - \bar{W}\bar{U} = -2 - (-1)(2) = 0$

$\therefore$ W & U are uncorrelated. 2

1 e) W and U are Gaussian (linear transformation of X and Y)

with $\rho = 0$, $\sigma_W = \sqrt{20}$ $\sigma_Y = \sqrt{20} = 2\sqrt{5}$ $\bar{W} = -1$ $\bar{U} = 2$

$$f_{W,U}(w,u) = \frac{1}{2\pi\, 20} \exp\left\{-\frac{1}{2}\left[\frac{(w+1)^2}{20} + \frac{(u-2)^2}{20}\right]\right\}$$ 1

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Alternative Soln.

$$W = 3X + Y$$
$$U = -X - 2Y$$

$$\Rightarrow \begin{bmatrix} W \\ U \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

a)
$$\Rightarrow \begin{bmatrix} \bar{W} \\ \bar{U} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{W} \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 \\ +2 \end{bmatrix} \Rightarrow \boxed{\bar{W} = -1} \qquad \boxed{\bar{U} = +2}$$

b)
$$C_{W,U} = \begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix} C_{X,Y} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$$

$$C_{X,Y} = \begin{bmatrix} \sigma_X^2 & c_{X,Y} \\ c_{X,Y} & \sigma_Y^2 \end{bmatrix}.$$

$$\sigma_X^2 = E[X^2] - \bar{X}^2 = 4$$
$$\sigma_Y^2 = E[Y^2] - \bar{Y}^2 = 9 - 1 = 8$$
$$c_{X,Y} = R_{XY} - \bar{X}\bar{Y} = -4 - 0 = -4$$

$$\Rightarrow C_{W,U} = \begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 8 & -4 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix}$$

$$\Rightarrow \sigma_W^2 = 20$$
$$\sigma_U^2 = 20$$
$$C_{WU} = 0$$

$$\Rightarrow E[W^2] = \sigma_W^2 + \bar{W}^2 = 20 + (1)^2 = 21$$
$$E[U^2] = \sigma_U^2 + \bar{U}^2 = 20 + (2)^2 = 24$$

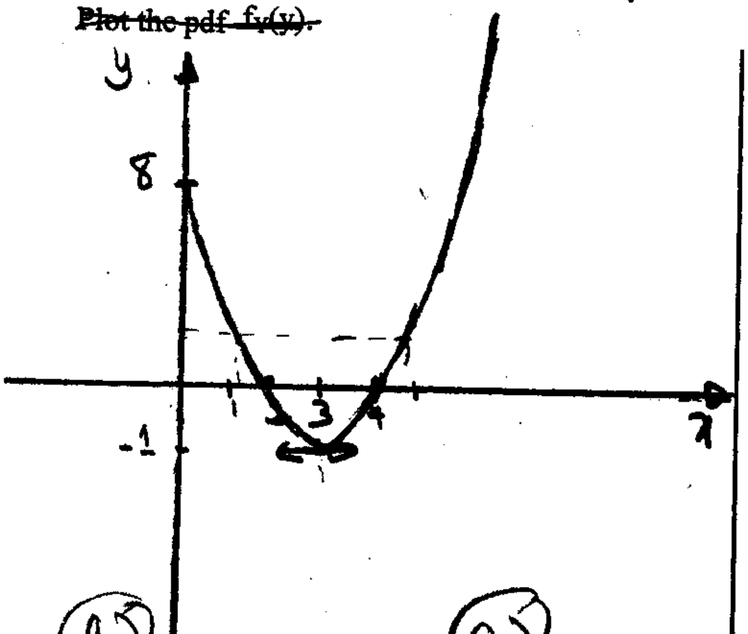
c)
$$R_{WU} = C_{WU} + \bar{W}\bar{U}$$
$$= 0 + (-1)(2) = -2$$

Problem 4:

A random variable X with the following probability density function:

$$f_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is transformed to a new random variable $Y = (X-3)^2 - 1$.
Find the probability density function of Y.
~~Plot the pdf $f_Y(y)$.~~



(0.5)
$* \; y < -1 \Rightarrow f_Y(y) = 0$ (0.5)

$* \; -1 \leq y \leq 8$: (0.5) 2 roots $\begin{cases} x_1 = -\sqrt{y+1} + 3 \; (0.5) \\ x_2 = \sqrt{y+1} + 3 \; (0.5) \end{cases}$

$$\left|\frac{dy}{dx}\right| = 2|x-3| \quad (0.5)$$

$$f_Y(y) = \frac{f_X(x_1)}{\left|\frac{dy}{dx}\right|_{x=x_1}} + \frac{f_X(x_2)}{\left|\frac{dy}{dx}\right|_{x=x_2}}$$

$$= \frac{2e^{2\sqrt{y+1}-6}}{2\left|-\sqrt{y+1}+3-3\right|} + \frac{2e^{-2\sqrt{y+1}-6}}{2\left|\sqrt{y+1}+3-3\right|}$$

$$= \frac{e^{-6}}{\sqrt{y+1}}\left[e^{2\sqrt{y+1}} + e^{-2\sqrt{y+1}}\right]$$

(0.5)
$* \; y > 8$: 1 root

$x_3 = \sqrt{y+1} + 3$ (0.5)

$$f_Y(y) = \frac{f_X(x_3)}{\left|\frac{dy}{dx}\right|_{x=x_3}} = \frac{e^{-6} e^{-2\sqrt{y+1}}}{\sqrt{y+1}} \quad (1)$$

$$\therefore f_Y(y) = \begin{cases} 0, & y < -1 \\ \frac{e^{-6}}{\sqrt{y+1}}\left[e^{2\sqrt{y+1}} + e^{-2\sqrt{y+1}}\right], & -1 \leq y \leq 8 \\ \frac{e^{-6}}{\sqrt{y+1}} e^{-2\sqrt{y+1}}, & y > 8 \end{cases}$$

(1)