Evaluating the Bit Error Rate of

Square M-QAM

over the AWGN Channel

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In partial completion of the requirements for

EE242: Digital Communications and Coding

Submitted to:

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Submitted: December 12, 2009

Abstract

In this paper, a review and discussion of another paper entitled, "Bit Error Probability of M-ary Quadrature Amplitude Modulation" [4] *is presented. The mentioned paper above derived a general expression for the bit error rate (BER) of an M-ary square QAM. The derivation of the BER expression uses the advantages of the Gray Code bitmapping and assumed the presence of Additive White Gaussian Noise or AWGN (zero mean with variance $N_0/2$)*

1. Introduction

Error rates are usually presented by either using the computation of symbol error rates or by estimation using union bounds. One of the reasons behind the lack of expressions of the bit error rate (BER) is that the exact analysis of the BER is complicated and it is usually not expressed in closed-form solutions. Also, the transformation of the symbol error rate to BER is not straightforward.

So why do we need to modulation efficiency in terms of the BER? It is because the estimation using union bounds does not guarantee accuracy and at the end of the day we want to differentiate the efficiencies of the different types modulations in terms of their very basic unit of representation, bits

Signal waveforms corresponding QAM can be expressed as

$$s_m(t) = A_{mi}\, g(t) \cos 2\pi f_c t - A_{mq} \sin 2\pi f_c t \qquad (1)$$

where *g(t)* is the signal pulse and A_{mi} and A_{mq} (in-phase and quadrature phase components) are the signal amplitudes of the quadrature carriers which takes the assumed set of discrete values

$$\{(2m - 1 - M), m = 1, 2, \ldots, M\} \qquad (2)$$
$$where\ M = 2^k \quad , \quad k = bits / symbol$$

This set of values represents a QAM that has a rectangular diagram or constellation. A special case of a rectangular QAM is an M-ary square QAM. An M-ary square QAM (i.e., $M = 4, 16, 64, 256, \ldots$) has its signal amplitudes take on the values of $\pm d, \pm 3d, \ldots, \pm(\sqrt{M} - 1)d$ on both directions where d is the Euclidean distance between adjacent points in the constellation. Given equations (3), (4), (5), and (6), we can compute the distance d in terms of ε_{bavg}, the average energy per bit which will be very useful in the computation of BER.

$$d = \sqrt{2\varepsilon_g} \qquad (3)$$

$$\varepsilon_m = \frac{\varepsilon_g}{2}(A_{mi}^2 + A_{mq}^2) \qquad (4),$$

$$\varepsilon_{avg} = \frac{M - 1}{3}\varepsilon_g \qquad (5),$$

$$\varepsilon_{bavg} = \frac{M - 1}{3\log_2 M}\varepsilon_g \qquad (6).$$

Using the equations above, we obtain the new expression of d in terms of the average energy per bit,

$$d = \sqrt{\frac{6\log_2 M}{M - 1}\varepsilon_{bavg}} \qquad (7).$$

As with the original paper, assumptions should be stated before proceeding with the computation. It is assumed that all symbols are equiprobable and that a perfect 2 dimensional Gray Coding method is used. Also, a zero-mean additive white Gaussian noise is assumed with variance $N_0/2$. Lastly, we assume that for simplicity, perfect carrier recovery and symbol synchronization is attained.

2. BER Computations of Square QAM.

2.1. BER of 4-QAM

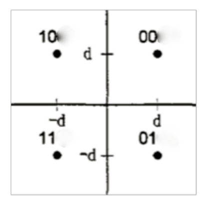

Figure 1: 4-QAM Constellation.

We first analyze the simplest square QAM constellation, 4-QAM, as illustrated in Figure 1 above. The bit assignment uses the Gray Code mapping method. The vertical axis is the Q-channel and the horizontal is the I-channel. Bit assignment is in the order of $i_1 q_2$. We can see that if $I \geq 0, i_1 = 0$ and if $I \leq 0, i_1 = 1$. The same goes when $Q \geq 0$ and $Q \leq 0$ would result to $q_1 = 0$ and $q_1 = 1$ respectively. We can also observe that an error would occur if the noise is greater d when $i_1 = 1$ and if the noise is less that d when $i_1 = 1$. The next set of computations presents the computation of $P_b(k)$, the probability that the *kth* bit of the in-phase and quadrature phase components are in error in terms of η. We first solve for the probability of bit error given the first in-phase component, $P_b|i_1$ and then solve for $P_b|q_1$.

$$P_{b|i_1=1} = \Pr\{n > d\} = Q\left(\frac{d}{\sqrt{\frac{N_0}{2}}}\right)). \qquad (8)$$

$$P_{b|i_1=0} = \Pr\{n < -d\} = Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) \qquad (9)$$

We can now get $P_{b|i1}$,

$$P_{b|i_1} = \frac{1}{2}(P_{b|i_1=0} + P_{b|i_1=1}) = Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) \qquad (10)$$

Solving for the above equation in terms of the SNR per bit $\eta = \frac{\varepsilon_b}{N_0}$ and the erfc($\cdot$) function where

$$Q(x) = \frac{1}{2} erfc(\frac{x}{\sqrt{2}}) \qquad (11),$$

we get,

$$P_{b|i_1} = \frac{1}{2} erfc(\frac{d}{\sqrt{N_0}}) \qquad (12).$$

We then substitute the value of d from (7) with $M = 4,$ we get,

$$P_{b|i_1} = \frac{1}{2} erfc(2\sqrt{2\eta}) \qquad (13).$$

After getting the probability of bit error given the in-phase component, we need to solve the probability of bit error given the quadrature component. But it easily observed that if we rotate the constellation, we would be getting the same probability, thus,

$$P_{b|q_1} = \frac{1}{2} erfc(2\sqrt{2\eta}) \qquad (14).$$

Solving for $P_b(1)$ is as simple as averaging the probability of bit error of the 1st bit of the in-phase and quadrature components. This would give us,

$$P_b(1) = \frac{1}{2} erfc(2\sqrt{2\eta})$$
$$P_b = P_b(1) \qquad (15).$$

2.2. BER of 16-QAM

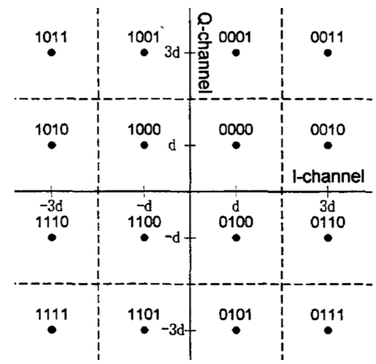

Figure 2: 16-QAM Constellation.

We start by analyzing the bit assignment ($i_1q_1i_2q_2$) of the constellation points. We can see that if $I \geq 0, i_1 = 0$ and if $I \leq 0, i_1 = 1$. The same goes when $Q \geq 0$ and $Q \leq 0$ would result to $q_1 = 0$ and $q_1 = 1$ respectively. With this we could separate the constellation above into 4 regions the same as a 4-QAM constellation. In this case, an error would occur when the noise is either greater than d or $3d$. The next set of computations presents the computation of $P_b(1)$, the probability that the *1st* bit of the in-phase and quadrature phase components are in error in terms of η. We first solve for the probability of bit error given the first in-phase component, $P_b|i_1$.

$$P_{b|i_1=1} = \frac{1}{2}\Pr\{n > d\} + \frac{1}{2}\Pr\{n > 3d\}$$
$$= \frac{1}{2}(Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{3d}{\sqrt{\frac{N_0}{2}}})). \qquad (16)$$

$$P_{b|i_1=0} = \frac{1}{2}\Pr\{n < -d\} + \frac{1}{2}\Pr\{n < -3d\}$$
$$= \frac{1}{2}(Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{3d}{\sqrt{\frac{N_0}{2}}})) \qquad (17)$$

$$P_{b|i_1} = \frac{1}{2}(P_{b|i_1=0} + P_{b|i_1=1})$$
$$= \frac{1}{2}((Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{3d}{\sqrt{\frac{N_0}{2}}}))) \qquad (18)$$

Solving for the above equation in terms of the SNR per bit $\eta = \frac{\varepsilon_b}{N_0}$ and the erfc($\cdot$) function, we get,

$$P_{b|i_1} = \frac{1}{2}(\frac{1}{2} erfc(\frac{d}{\sqrt{N_0}}) + \frac{1}{2} erfc(\frac{3d}{\sqrt{N_0}})) \qquad (19).$$

We then substitute the value of d from (7) with $M = 16,$ we obtain,

$$P_{bi_1} = \frac{1}{2}\left(\frac{1}{2}erfc(2\sqrt{\frac{2\eta}{5}}) + \frac{1}{2}erfc(6\sqrt{\frac{2\eta}{5}})\right)$$
$$P_{bi_1} = \frac{1}{4}\left(erfc(2\sqrt{\frac{2\eta}{5}}) + erfc(6\sqrt{\frac{2\eta}{5}})\right) \qquad (20).$$

After getting the probability of bit error given the in-phase component, we need to solve the probability of bit error given the quadrature component. But it is also easily observed that if we rotate the constellation, we would be getting the same probability, thus,

$$P_{bq_1} = \frac{1}{4}\left(erfc(2\sqrt{\frac{2\eta}{5}}) + erfc(6\sqrt{\frac{2\eta}{5}})\right) \qquad (21).$$

Solving for $P_b(1)$ is as simple as averaging the probability of bit error of the 1st bit of the in-phase and quadrature components. This would give us,

$$P_b(1) = \frac{1}{4}\left(erfc(2\sqrt{\frac{2\eta}{5}}) + erfc(6\sqrt{\frac{2\eta}{5}})\right) \qquad (22).$$

We now shift our focus to the last two bits (i_2q_2) of the symbols. We can see that that the bounds for the decision regions are the lines $I = \pm 2d$ and $Q = \pm 2d$. From the regions formed, the bit error cases could now be distinguished. It is easily noted that If $I,Q < -2d$ then $i_2,q_2 = 1$. If $-2d < I,Q < 2d$ then $i_2,q_2 = 0$. And If $I,Q > 2d$ then $i_2,q_2 = 1$.

Solving for $P_b|i_2$,

$$P_{b|i_2=1} = \frac{1}{2}\Pr\{d < n < 5d\} + \frac{1}{2}\Pr\{-5d < n < -d\}$$
$$= \left(Q\left(\frac{d}{\sqrt{\frac{N_0}{2}}}\right) - Q\left(\frac{5d}{\sqrt{\frac{N_0}{2}}}\right)\right) \qquad (23)$$
$$P_{b|i_2=0} = \frac{1}{2}\Pr\{-d < n < 3d\} + \frac{1}{2}\Pr\{-3d < n < d\}$$
$$= \left(Q\left(\frac{d}{\sqrt{\frac{N_0}{2}}}\right) + Q\left(\frac{3d}{\sqrt{\frac{N_0}{2}}}\right)\right) \qquad (24)$$
$$P_{bi_2} = \frac{1}{2}\left(P_{b|i_2=0} + P_{b|i2=1}\right)$$
$$= \frac{1}{2}\left(\left(2Q\left(\frac{d}{\sqrt{\frac{N_0}{2}}}\right) + Q\left(\frac{3d}{\sqrt{\frac{N_0}{2}}}\right) - Q\left(\frac{5d}{\sqrt{\frac{N_0}{2}}}\right)\right)\right) \qquad (25)$$

Solving for the above equation in terms of the SNR per bit $\eta = \frac{\varepsilon_b}{N_0}$ and in the erfc($\cdot$), we get,

$$P_{bi_2} = \frac{1}{2}\left(\frac{1}{2}\cdot 2erfc\left(\frac{d}{\sqrt{N_0}}\right) + \frac{1}{2}erfc\left(\frac{3d}{\sqrt{N_0}}\right) - \frac{1}{2}erfc\left(\frac{5d}{\sqrt{N_0}}\right)\right)$$
$$P_{bi_2} = \frac{1}{2}\left(\frac{1}{2}erfc(2\sqrt{\frac{2\eta}{5}}) + \frac{1}{2}erfc(6\sqrt{\frac{2\eta}{5}}) - \frac{1}{2}erfc(10\sqrt{\frac{2\eta}{5}})\right)$$
$$P_{bi_2} = \frac{1}{4}\left(erfc(2\sqrt{\frac{2\eta}{5}}) + erfc(6\sqrt{\frac{2\eta}{5}}) - erfc(10\sqrt{\frac{2\eta}{5}})\right) \qquad (26).$$

Since it is obvious, as with $P_b(1)$, the quadrature phase component of $P_b(2)$ is just equal to the in-phase component and therefore averaging it to get $P_b(2)$ yields,

$$P_b(2) = \frac{1}{4}\left(erfc(2\sqrt{\frac{2\eta}{5}}) + erfc(6\sqrt{\frac{2\eta}{5}}) - erfc(10\sqrt{\frac{2\eta}{5}})\right) \qquad (27).$$

We have just now computed the probability of bit error for a 16-QAM constellation. It is computed by averaging the conditional probabilities in Equations (15) and (21). It would result to,

$$P_b = \frac{1}{2}\sum_{k=1}^{2} P_b(k) \qquad (28).$$

2.3. BER of 64-QAM

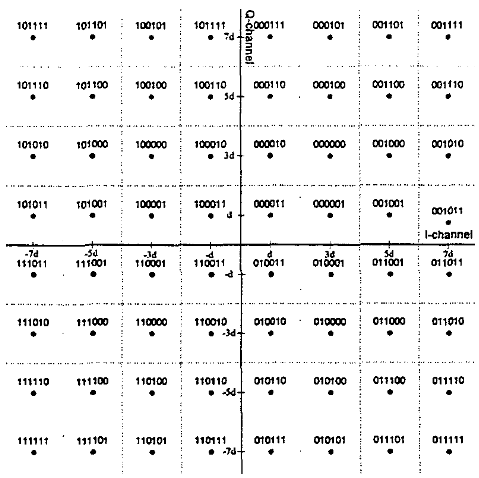

Figure 3: 64-QAM Constellation.

In the figure above, each constellation point is represented by a 6-bit symbol composed of three bits each from of the in-phase and the quadrature components, ($i_1q_1i_2q_2i_3q_3$).

We consider first the first two bits, i_1 and q_1, which are the first bit of each component. We can see that this case is just the same as the first case for 16-QAM where for $I \geq 0, i_1 = 0$ and for $I \leq 0, i_1 = 1$. The same goes when

$Q \geq 0$ and $Q \leq 0$ would result to $q_1 = 0$ and $q_1 = 1$ respectively.

$$P_{bk_1=1} = \frac{1}{4}[\Pr\{n > d\} + \Pr\{n > 3d\} + \Pr\{n > 5d\} + \Pr\{n > 7d\}]$$
$$= \frac{1}{4}(Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{3d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{5d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{7d}{\sqrt{\frac{N_0}{2}}})). \quad (28)$$

$$P_{bk_1=0} = \frac{1}{4}[\Pr\{n < -d\} + \Pr\{n < -3d\} + \Pr\{n < -5d\} + \Pr\{n < -7d\}]$$
$$= \frac{1}{4}(Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{3d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{5d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{7d}{\sqrt{\frac{N_0}{2}}})). \quad (29)$$

$$P_{bi_1} = \frac{1}{2}(P_{bk_1=0} + P_{bk_1=1})$$
$$= \frac{1}{2}(\frac{1}{2}[(Q(\frac{d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{3d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{5d}{\sqrt{\frac{N_0}{2}}}) + Q(\frac{7d}{\sqrt{\frac{N_0}{2}}})])) \quad (30)$$

Substituting d (using Equation (7) with $M=64$) and solving the above equation in terms of the SNR per bit $\eta = \frac{\varepsilon_b}{N_0}$ in the erfc($\cdot$) function we get,

$$P_{bi_1} = \frac{1}{8}(erfc(2\sqrt{\frac{2\eta}{7}}) + erfc(6\sqrt{\frac{2\eta}{7}}) + erfc(10\sqrt{\frac{2\eta}{7}}) + erfc(14\sqrt{\frac{2\eta}{7}})) \quad (31).$$

Similarly,

$$P_{bq_1} = \frac{1}{8}(erfc(2\sqrt{\frac{2\eta}{7}}) + erfc(6\sqrt{\frac{2\eta}{7}}) + erfc(10\sqrt{\frac{2\eta}{7}}) + erfc(14\sqrt{\frac{2\eta}{7}})) \quad (32).$$

Solving for $P_b(1)$,

$$P_b(1) = \frac{1}{8}(erfc(2\sqrt{\frac{2\eta}{7}}) + erfc(6\sqrt{\frac{2\eta}{7}}) + erfc(10\sqrt{\frac{2\eta}{7}}) + erfc(14\sqrt{\frac{2\eta}{7}})) \quad (33).$$

For the second case, we consider the next two bits, i_2 and q_2. Observing the constellation, we can see that the regions are determined by the bounds $I = \pm 4d$ and $Q = \pm 4d$. Specifically, if $I, Q < -4d$ or $I, Q > 4d$ then i_2, $q_2 = 1$. If $-4d < I, Q < 4d$ then i_2, $q_2 = 0$. We could also see that each quadrant of the constellation is composed of 16-QAM where we could use this to get $P_b(2)$, the probability that the *2nd* bit of the in-phase and quadrature phase components are in error in terms of η. Using the same procedure as with the first case, we get,

$$P_b(2) = \frac{1}{8}(2erfc(2\sqrt{\frac{2\eta}{7}}) + 2erfc(6\sqrt{\frac{2\eta}{7}}) + erfc(10\sqrt{\frac{2\eta}{7}}) + erfc(14\sqrt{\frac{2\eta}{7}})$$
$$- erfc(18\sqrt{\frac{2\eta}{7}}) - erfc(22\sqrt{\frac{2\eta}{7}})) \quad (34).$$

For the last case, we consider the last two bits, i_3 and q_3. The decision regions are bounded by the lines $I = \pm 2d$ and $Q = \pm 6d$. In this case, we can see that each quadrant of the 64-QAM is composed of 4 sets of 4-QAM (each on the quadrants of the 16-QAM described in the second case). Using these facts and the previous procedures, we get $P_b(3)$, the probability that the *3rd* bit of the in-phase and quadrature phase components are in error in terms of η as,

$$P_b(3) = \frac{1}{8}(4erfc(2\sqrt{\frac{2\eta}{7}}) + 3erfc(6\sqrt{\frac{2\eta}{7}}) - 3erfc(10\sqrt{\frac{2\eta}{7}}) - 2erfc(14\sqrt{\frac{2\eta}{7}})$$
$$+ 2erfc(18\sqrt{\frac{2\eta}{7}}) + erfc(22\sqrt{\frac{2\eta}{7}}) - erfc(26\sqrt{\frac{2\eta}{7}})) \quad (35).$$

We have just now computed the probability of bit error for a 16-QAM constellation. It is computed by averaging the conditional probabilities in Equations (15) and (21). It would result to,

$$P_b = \frac{1}{3}\sum_{k=1}^{3} P_b(k) \quad (36).$$

2.3. General BER Expression for M-ary Square QAM

If we continue the same procedure for M = 256, 1024, 4096… , we would see that the general expression for the M-ary Square QAM is

$$P_b = \frac{1}{\log_2 \sqrt{M}} \sum_{k=1}^{\log_2 \sqrt{M}} P_b(k) \quad (37).$$

3. Conclusion

In this paper, we have just presented a review of the general expression for the BER of an M-ary Gray-coded square QAM over an AWGN channel. From the computations above, we see that the derivations always take into consideration the fact that square QAM's are symmetrical and we don't need to complete the computations. It is also important to easily distinguish the decision regions specified by the bounds. With the right bounds, the probability of bit error would be determined correctly by the position of the bits and the noise associated to it.

References

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