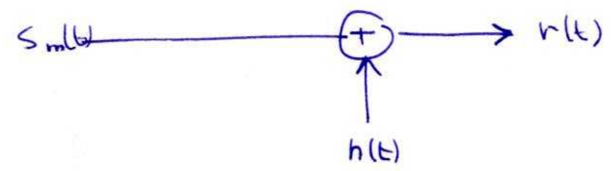
The additive White Gaussian noise Ch.



$r(t) = s_m(t) + n(t)$ $\qquad m = 1, 2, \ldots, M$

$s_m(t)$ is the transmitted signal.

$n(t)$ is sample waveform of zero-mean white Gaussian process with PSD $N_0/2$

$r(t)$ is the received form.

How do you decide based on the observed $r(t)$ $0 \leq t \leq T$
which waveform was transmitted

How did we suggest to solve this previously?

- Find a basis $\varphi_1(t), \ldots, \varphi_N(t)$
- Project $r(t)$ & find representation of $r(t)$ in this space.
- Check the closest point s_m to r. $(\hat{m})$
-

Is this an optimal decision?

Decision rule that results in ~~the~~

- min probability of error
  $P_e = P[\hat{m} \neq m]$
- min prob. of disagreement bet TX message & received message.

Optimal Detection for the Vector AWGN Channel

Additive AWGN is the vector equivalent to the waveform AWGN ch.

- Components of the noise vector are iid zero-mean Gaussian r.v.'s with var. $\frac{N_0}{2}$

- Joint pdf of noise vector is

$$p(n) = \left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{\sum_{j=1}^{N} n_j^2}{N_0}}$$

$$= \left(\frac{1}{\sqrt{\pi N_0}}\right)^N e^{-\frac{\|n\|^2}{N_0}}$$

MAP detector:

$$\hat{m} = \underset{1 \le m \le M}{\arg\max} \left[P_m \, p(r|s_m) \right]$$

$$= \arg\max P_m \, P_n(r - s_m)$$

$$= \arg\max P_m \, e^{-\frac{\|r - s_m\|^2}{N_0}}$$

$$= \arg\max \ln P_m - \frac{\|r - s_m\|^2}{N_0}$$

$$= \arg\max \left[\frac{N_0}{2} \ln P_m - \frac{1}{2} \|r - s_m\|^2 \right]$$

$$= \arg\max \frac{N_0}{2} \ln P_m - \frac{1}{2} \left(\|r\|^2 + \|s_m\|^2 - 2 r \cdot s_m \right)$$

$$= \text{''} \ \underbrace{\frac{N_0}{2} \ln P_m - \frac{1}{2} \mathcal{E}_m}_{\eta_m} + r \cdot s_m$$

MAP rule

$$\hat{m} = \underset{1 \le m \le M}{\arg\max} \left[\eta_m + r \cdot s_m \right]$$

Special cases:

1) When messages are equiprobable $P_m = 1/M$

$$\hat{m} = \arg\max \left[- \|r - s_m\|^2 \right]$$

$$= \underset{1 \le m \le M}{\arg\min} \|r - s_m\|^2$$

find closest s_m using std Euclidean dist.

nearest neighbor/ min dist. detector

Boundaries of D_m & $D_{m'}$ are set of points equi-distant from s_m & $s_{m'}$ (perpendicular bisector of line connecting 2 pts)

2) When signals are equi probable & of equal energy

$$\eta_m = \frac{N_0}{2} \ln P_m - \frac{1}{2} \mathcal{E}_m$$

is indep. of m

Opt. detection rule

$$\hat{m} = \underset{1 \le m \le M}{\arg\max} \; r \cdot s_m$$

(signal most highly correlated with r)

Decision region

$$D_m = \{ r \in \mathbb{R}^N : \eta_m + r \cdot s_m > \eta_{m'} + r \cdot s_{m'} \quad m' \neq m \}$$

Opti.

Optimal detection for binary antipodal signaling

$s_1(t) = s(t)$ with prob. p

$s_2(t) = -s(t)$ " " $1-p$

$$\varphi(t) = \frac{s(t)}{\sqrt{\mathcal{E}_s}}$$

$s_1(t) = \sqrt{\mathcal{E}_s}\,\varphi(t) \longrightarrow s_1 = \sqrt{\mathcal{E}_s}$

$s_2(t) = -\sqrt{\mathcal{E}_s}\,\varphi(t) \longrightarrow s_2 = -\sqrt{\mathcal{E}_s}$

$N = 1$

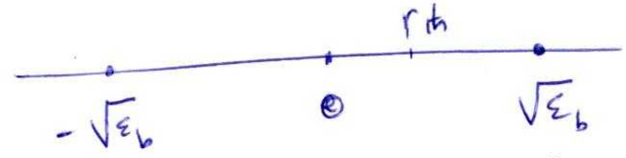
$\mathcal{E}_{avg} = \mathcal{E}_s = \mathcal{E}_b$

Decision region

$$D_1 = \{ r : r.s_1 + \eta_1 > r.s_2 + \eta_2 \}$$

$$= \{ r : r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln p - \cancel{\frac{\mathcal{E}_b}{2}} > -r\sqrt{\mathcal{E}_b} + \frac{N_0}{2}\ln(1-p) - \cancel{\frac{\mathcal{E}_b}{2}} \}$$

$$= \{ r : r > \underbrace{\frac{N_0}{4\sqrt{\mathcal{E}_b}} \ln\frac{(1-p)}{p}}_{r_{th}} \}$$



r_{th} closer to the least probable signal.

$$P_e = P_{e|s_1} P(s_1) + P_{e|s_2} P(s_2)$$

$$= p \underbrace{\int_{D_2} p(r|s_1)\,dr}_{\sqrt{\mathcal{E}_b}+n} + (1-p) \int_{D_1} p(r|s_2)\,dr$$

$$= p \int_{-\infty}^{r_{th}} \mathcal{N}\left(\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right) + (1-p) \int_{r_{th}}^{\infty} \mathcal{N}\left(-\sqrt{\mathcal{E}_b}, \frac{N_0}{2}\right)$$

$$= p\left(1 - Q\left(\frac{r_{th} - \sqrt{\mathcal{E}_b}}{N_0/2}\right)\right) + (1-p)\, Q\left(\frac{r_{th} + \sqrt{\mathcal{E}_b}}{N_0/2}\right)$$

$$= p\, Q\left(\frac{\sqrt{\mathcal{E}_b} - r_{th}}{\sqrt{N_0/2}}\right) + (1-p)\, Q\left(\frac{r_{th} + \sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right)$$

Assume messages are equiprobable, then

$$r_{th} = 0$$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$$

Each message corresponds to one bit.

So $P_b = P_e$.

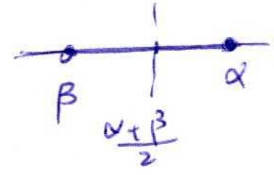
Error Prob. for 2 equiprobable signals

$s_1(t)$ & $s_2(t)$

Either we have one dim or 2 dimension

$s_1(t) = \alpha\, \varphi(t) \longrightarrow s_1 = \alpha$

$s_2(t) = \beta\, \varphi(t) \longrightarrow s_2 = \beta$



$$P_e|s_1 = \Pr\left(r < \frac{\alpha+\beta}{2} \,\middle|\, s_1\right) \tfrac{1}{2}$$

$$= \Pr\left(n + \alpha < \frac{\alpha+\beta}{2}\right) \tfrac{1}{2}$$

$$= \Pr\left(n < \frac{\beta-\alpha}{2}\right)$$

$$= \Pr\left(n > \frac{\alpha-\beta}{2}\right)$$

$$= Q\left(\frac{\frac{\alpha-\beta}{2}}{\sqrt{\frac{N_0}{2}}}\right)$$

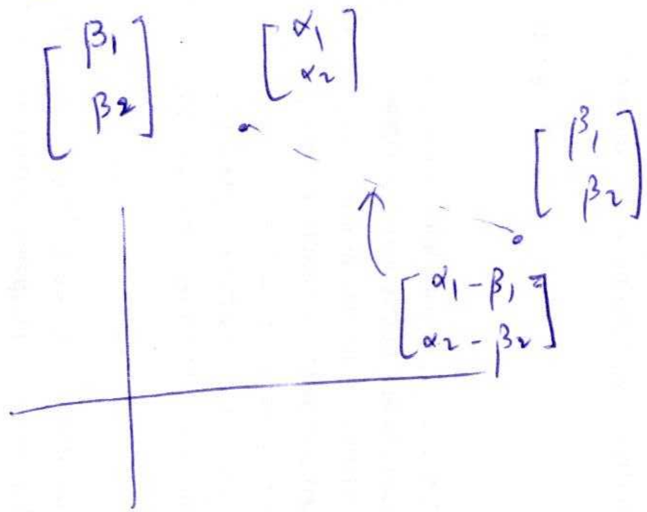
$$= Q\left(\sqrt{\frac{(\alpha-\beta)^2}{2N_0}}\right)$$

$$P_e|s_2 = P_e|s_1 = P_e = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

Two dimensions $N = 2$

$s_1(t) \longrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

$s_2(t) \longrightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$



n -

Opt. detection Binary Orthog. Signaling

orthog. signals satisfy

$$\int_{-\infty}^{\infty} s_i(t)\, s_j(t)\, dt = \begin{cases} \mathcal{E} & i = j \\ 0 & i \neq j \end{cases}$$

$M = 2$

$\mathcal{E}_{avg} = \mathcal{E}_b = \mathcal{E}$

Choose $\varphi_1(t) = \dfrac{s_1(t)}{\sqrt{\mathcal{E}_b}}$ $\cancel{\Rightarrow s_1 = \sqrt{\mathcal{E}_b}}$ $\Rightarrow s_1 = \begin{bmatrix} \sqrt{\mathcal{E}_b} \\ 0 \end{bmatrix}$

$\varphi_2(t) = \dfrac{s_2(t)}{\sqrt{\mathcal{E}_b}}$ $\quad s_2 = \begin{bmatrix} 0 \\ \sqrt{\mathcal{E}_b} \end{bmatrix}$

$$d = \sqrt{(\sqrt{\mathcal{E}_b} - 0)^2 + (0 - \sqrt{\mathcal{E}_b})^2}$$
$$= \sqrt{2\mathcal{E}_b}$$

So $P_e = Q\left(\sqrt{\dfrac{d^2}{2N_0}}\right)$

$$= Q\left(\sqrt{\dfrac{\mathcal{E}_b}{N_0}}\right)$$

Compare with error prob. of antipodal signaling

Antipodal with avg energy of $\mathcal{E}_b$ yields

$$P_{e\,\text{antipodal}} = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{2\gamma_b}\right)$$

$$P_{e\,\text{orthog}} = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{\gamma_b}\right)$$

~~bi~~binary orthog. signaling underperforms binary antipodal by a factor of 2 (or 3dB)

The term $\gamma_b = \frac{\mathcal{E}_b}{N_0}$ is the signal energy per bit, or SNR per bit of Comm. sys.