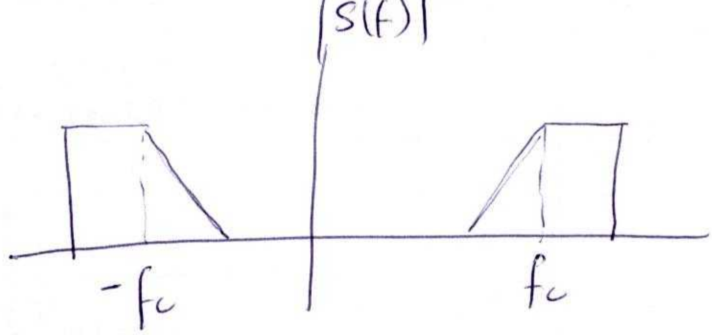
Representation of Band-Pass Signals

Consider $s(t)$ which is narrowband around f_c



Construct a signal $S_+(f)$ that only has the +ve frequencies in $s(t)$

$$S_+(f) = 2u(f)\,S(f)$$

$$\Rightarrow \quad s_+(t) = \int_{-\infty}^{\infty} S_+(f)\, e^{j2\pi f_c t}\, df$$

$$= \mathcal{F}^{-1}\left[2u(f)\right] * \mathcal{F}^{-1}\left[S(f)\right]$$

$$= \left[\delta(t) + \frac{j}{\pi t}\right] * s(t)$$

$$= s(t) + j\,\frac{1}{\pi t} * s(t)$$

$$= s(t) + j\,\hat{s}(t)$$

$$\boxed{\hat{s}(t) = \frac{1}{\pi t} * s(t)}$$

(Hilbert Transform of $s(t)$)

$$\boxed{h(t) = \frac{1}{\pi t} \qquad -\infty < t < \infty}$$

Hilbert Transformer

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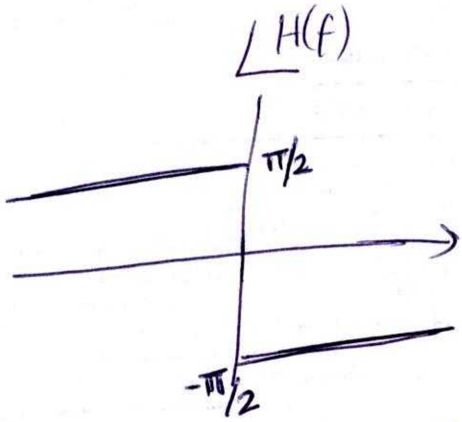
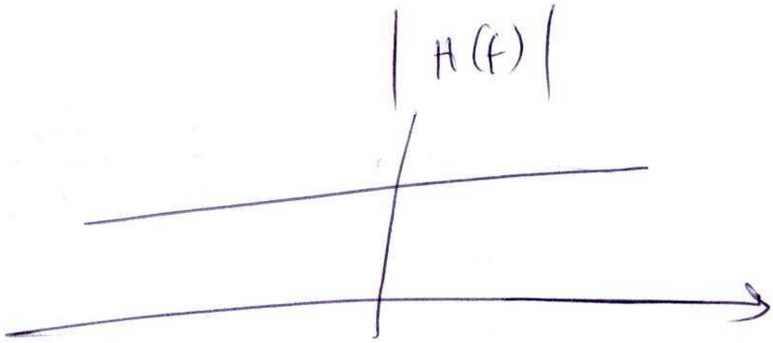
$$H(f) = \begin{cases} -j & f > 0 \\ j & f < 0 \end{cases}$$





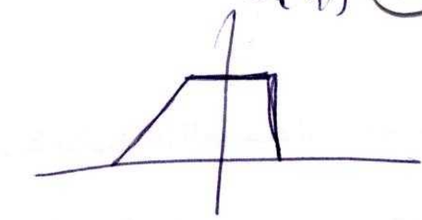
Ex i.e. **Hilbert transform shifts phase at each freq. by $-\pi/2$.**

Ex What is the Hilbert transform of

1. $\cos 2\pi f_c t \longrightarrow \cos(2\pi f_c t - \pi/2) = \sin 2\pi f_c t$
2. $\cos 2\pi f_{c_1} t + \cos 2\pi f_{c_2} t \longrightarrow \sin 2\pi f_{c_1} t + \sin 2\pi f_{c_2} t$.

Having represented the +ve freq's of the signal by $S^+(f)$, let's shift the content to zero to get

$S_\ell(f)$



$$S_\ell(f) = S_+(f + f_c)$$

$$\Rightarrow \quad s_\ell(t) = s_+(t)\, e^{-j 2\pi f_c t}$$

$$= \left[s(t) + j\,\hat{s}(t) \right] e^{-j 2\pi f_c t}$$

$$\Rightarrow \quad \boxed{s(t) + j\,\hat{s}(t) = s_\ell(t)\, e^{j 2\pi f_c t}}$$

$s_\ell(t)$ is complex valued

$s_\ell(t) = x(t) + j y(t)$

Relation bet. time domain signals

so

$$s(t) = \mathrm{Re}\left\{ s_\ell(t)\, e^{j 2\pi f_c t} \right\}$$

$$= x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t$$

$$\hat{s}(t) = x(t) \sin 2\pi f_c t + y(t) \cos 2\pi f_c t$$

This formula shows that $x(t)$ & $y(t)$ are low pass representations of $s(t)$.

$x(t)$ in phase

$y(t)$ quadrature phase

Relationship between the spectra

$$S(f) = \int_{-\infty}^{\infty} s(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \mathrm{Re}\left[s_l(t) e^{j2\pi f_c t} \right] e^{-j2\pi f t}\, dt$$

Use $\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$

$$\Rightarrow \quad S(f) = \frac{1}{2} \int_{-\infty}^{\infty} \left(s_l(t) e^{j2\pi f_c t} + s_l^*(t) e^{-j2\pi f_c t} \right) e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2} \mathcal{F}\left[s_l(t) e^{j2\pi f_c t} \right] + \frac{1}{2} \mathcal{F}\left[s_l^*(t) e^{-j2\pi f_c t} \right]$$

$$= \frac{1}{2} S_l(f - f_c) + \frac{1}{2} S_l^*(-f - f_c)$$

Relation bet. the energies :

$$\mathcal{E} = \int_{-\infty}^{\infty} s^2(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left\{ \mathrm{Re}\left[s_l(t) e^{j2\pi f_c t} \right] \right\}^2 dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} |s_l(t)|^2\, dt + \frac{1}{2} \int_{-\infty}^{\infty} |s_l(t)|^2 \cos\left[4\pi f_c t + 2\theta(t) \right]$$

0

$\tan^{-1} \frac{y(t)}{x(t)}$

varies slowly with respect to $2f_c$ as $s(t)$ is narrow band

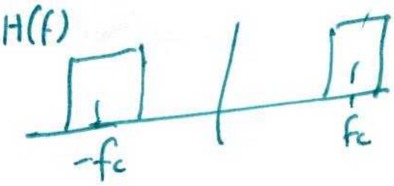
$$\Rightarrow \quad \boxed{\mathcal{E} = \frac{1}{2} \int_{-\infty}^{\infty} |s_l(t)|^2\, dt}$$

Representation of Linear Band-Pass Systems

System described by impulse response or freq. response

$h(t)$

$H(f)$



$h(t)$ is real $\Rightarrow$ conjugate symmetric

$$H^*(-f) = H(f)$$

Define $H_l(f-f_c)$ as

$$H_l(f-f_c) = \begin{cases} H(f) & f>0 \\ 0 & f<0 \end{cases}$$

$$\Rightarrow H_l(-f-f_c) = \ldots$$

$$\Rightarrow H_l^*(-f-f_c) = \begin{cases} 0 & f>0 \\ H^*(-f) & f<0 \end{cases}$$

Add both sides to get

$$H_l(f-f_c) + H_l^*(-f-f_c) = \begin{cases} H(f) & f>0 \\ H^*(-f) & f<0 \end{cases}$$

$$\Rightarrow \boxed{H_l(f-f_c) + H_l^*(-f-f_c) = H(f)}$$

Resembles the relation for signal

In time domain

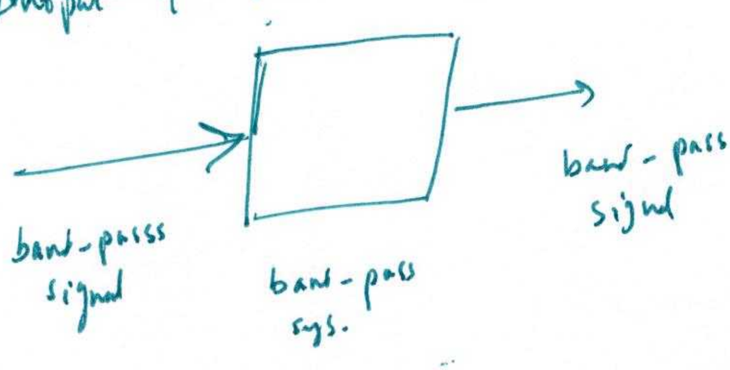
$$h(t) = h_\ell(t)\, e^{j2\pi f_c t} + h_\ell^*(t)\, e^{-j2\pi f_c t}$$
$$= 2\,Re\left[h_\ell(t)\, e^{j2\pi f_c t}\right]$$

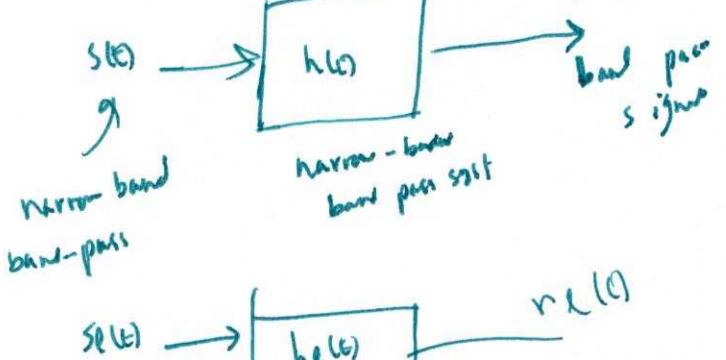
$h_\ell(t)$ is the equivalent low pass system of $h(t)$.

Narrowband band-pass signal represented by eqv. low pass signal

Narrowband band-pass sys represented by eq. low pass system

~~E output of band~~





o/p is band-pass signal

we can write

$\Rightarrow$ $r(t) = Re\{ r_l(t)\, e^{j2\pi f_c t} \}$

time domain

$\rightarrow$ $r(t)$ is related to $s(t)$ & $h(t)$. by

$$r(t) = \int_{-\infty}^{\infty} s(\tau)\, h(t-\tau)\, d\tau$$

freq. $\rightarrow$ domain

$$R(f) = H(f)\, S(f)$$

$$R(f) = \frac{1}{2}\left[S_l(f-f_c) + S_l^*(-f-f_c) \right] \left[H_l(f-f_c) + H_l^*(-f-f_c) \right]$$

$s(t)$ is narrow band $\Rightarrow$ $S_l(f-f_c) \simeq 0$ for $f<0$

$h(t)$ is impulse resp. $\Rightarrow$ $H_l(f-f_c) \simeq$ for $f<0$

Narrow band condition $\Rightarrow$

$$S_l(f-f_c)\, H_l^*(-f-f_c) = 0$$

$$S_l^*(-f-f_c)\, H_l(f-f_c) = 0$$

$$\Rightarrow \quad R(f) = \frac{1}{2}\left[S_l(f-f_c)\, H_l(f-f_c) + S_l^*(-f-f_c)\, H_l^*(-f-f_c) \right]$$

$$= \frac{1}{2}\left[R_l(f-f_c) + R_l^*(-f-f_c) \right]$$

where $\boxed{R_l(f) = S_l(f)\, H_l(f)}$

$$r_l(t) = \int_{-\infty}^{\infty} s_l(\tau)\, h_l(t-\tau)$$