Discrete-Time Linear Time-Invariant Systems

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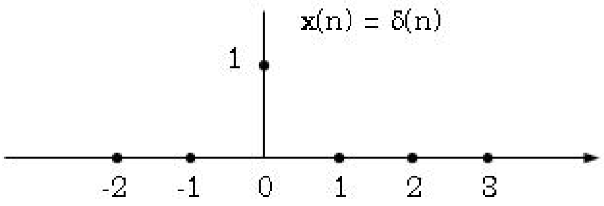
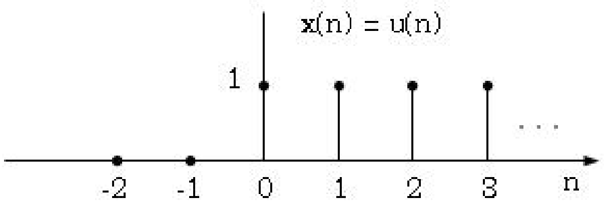
Summary of Important Concepts : Discrete Time Signals and Systems (Ch9)

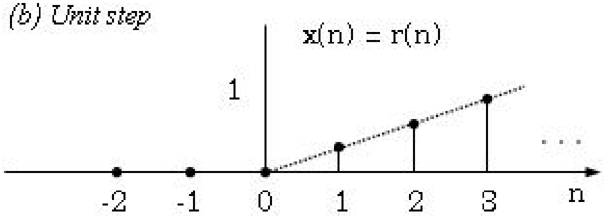
- $f(t) \rightarrow f(nT_s) = f(t)|_{t=nT_s} = f[n] \neq f(t)|_{t=n}$
- Unit Step & Unit Impulse Functions
- $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n \end{cases}$
- $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
- $r[n] = nu[n]$
- $\delta[n] = u[n] - u[n-1]$



(a) Unit sample (Unit impulse)



(b) Unit step



(c) Unit ramp

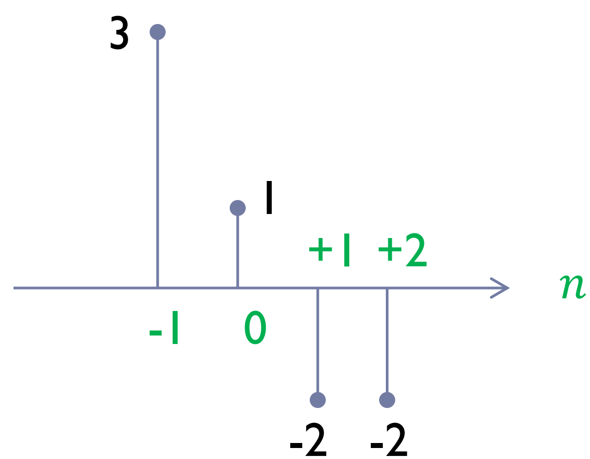
Equivalent Operations in Discrete Domain

- $\int_{-\infty}^{t} x(\tau)d\tau$ $\quad$ $\sum_{k=-\infty}^{n} x[k$
- $\frac{d}{dt}x(t)$ $\quad$ $x[n]-x[n-1]$
- $x(t)\delta(t)=x(0)\delta(t)$ $\quad$ $x[n]\delta[n]=x[0]\delta[0]$
- $\delta(t)=\frac{du(t)}{dt}$ $\quad$ $\delta[n]=u[n]-u[n-1]$
- $u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$ $\quad$ $u[n]=\sum_{k=-\infty}^{n}\delta[k]$

Similar Time and Amplitude Operations like Continuous Time Signals

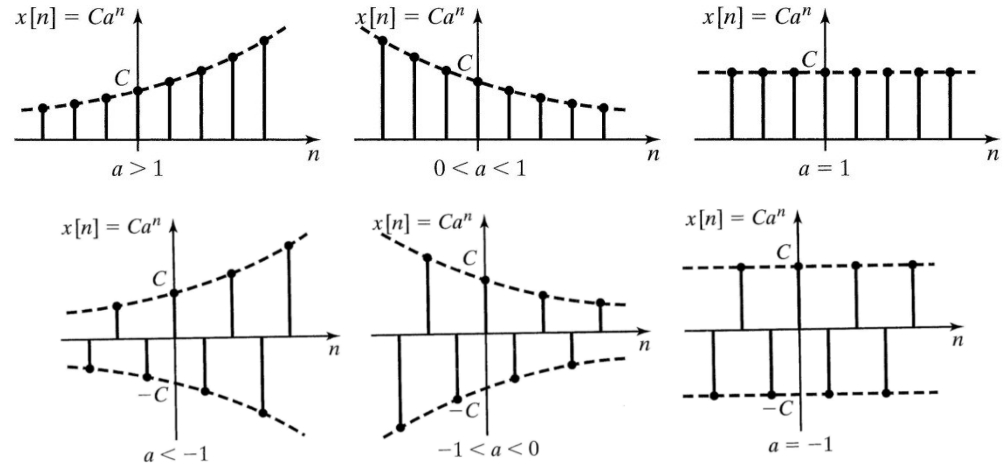
- $x[-n]$
- $x[an]$
- $x[n-n_0]$
- $-x[n]$
- $|A|x[n]$
- $x[n]+B$

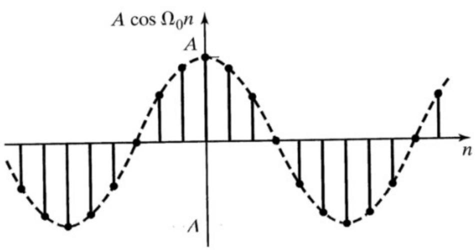
- Example: Sketch $y[n]$ given that $x[n]$ is shown in the figure



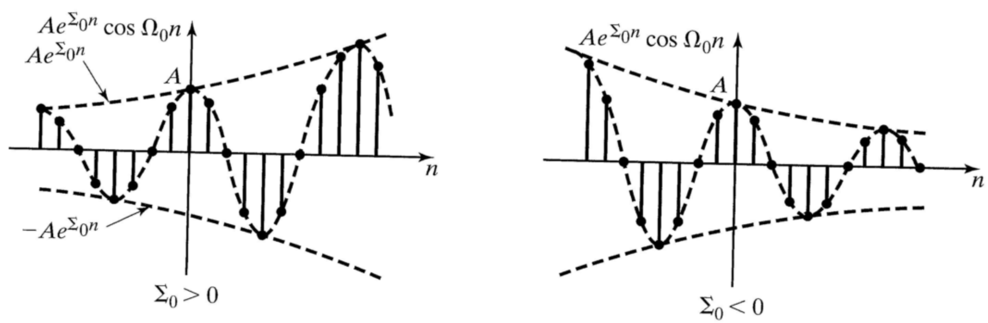
Common Discrete Time Signals

- $x[n] = Ca^n$
- Case I: C and a are real



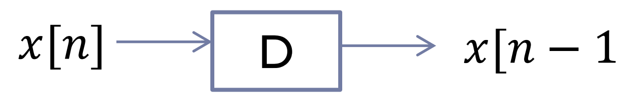


- Case II: C and a are complex , a unity magnitude
- Case III: C and a are complex



Discrete Time System Properties

- One of the important blocks in discrete systems is *ideal delay.*



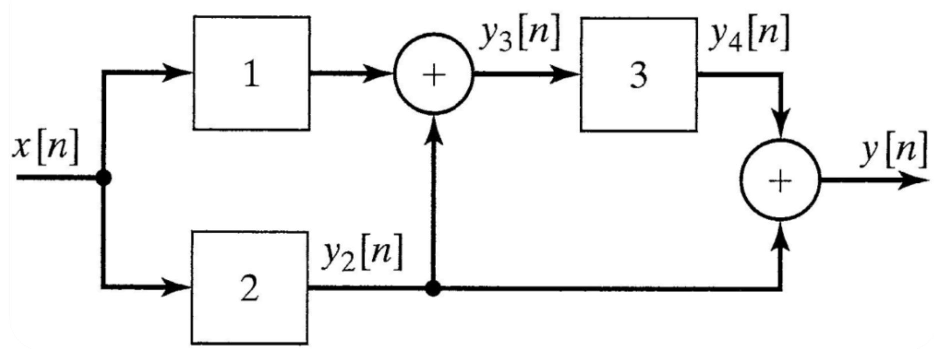
- Properties of Discrete Time Signals
  - Memroy :
    - e.g. $y[n] = 5x[n]$, $y[n] = \sum_{k=-\infty}^{n-1} x[k]$
    - No memory (static) , memory (Dynamic)
  - Invertiblility
    - e.g. $y[n] = |x[n]|$, non invertible
  - Causality
  - Stability
  - Time Invariance
  - Linearity

Introduction to LTI Discrete Systems (Ch10)

- **Comparing Discrete System with Continuous Time Systems**
  - Easier to analyze and design
  - Solving difference equations is easier than solving differential equations.
  - Characteristics are periodic in Frequency
- **Why LTI ?**
  - Many physical Systems can be modeled as LTI
  - Easier To solve
  - Available resources

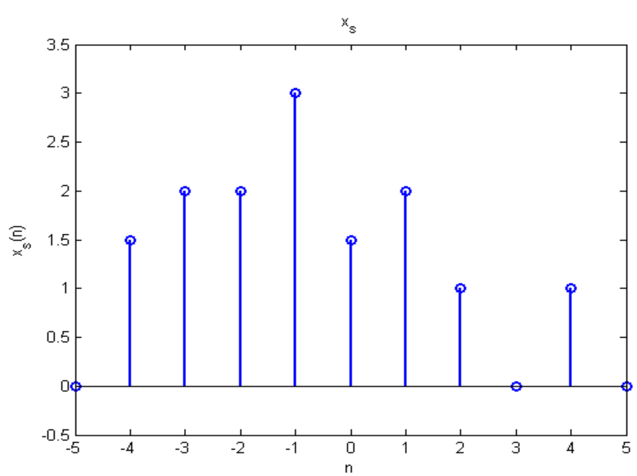
Introduction to LTI Discrete Systems

- Example of a system representation by block diagram
  - $y[n] = T_2(x[n]) + T_3(T_1(x[n]) + T_2(x[n]))$
- Recall that for time invariant
  - $x[n] \rightarrow y[n]$
  - $x[n - n_0] \rightarrow y[n - n_0]$
- For linearity
  - $x_1[n] \rightarrow y_1[n]$
  - $x_2[n] \rightarrow y_2[n]$
  - $a_1 x_1[n] + a_1 x_1[n] \rightarrow a_1 y_1[n] + a_2 y_2[n]$



Impulse Representation of Discrete Time signals

- We can represent signals as sum of scaled delta
- $\delta[n]$ unit sample function / unit impulse function
  - $x_{-1}[n] = x[n]\delta[n+1] = x[-1]\delta[n+1]$
  - $x_0[n] = x[n]\delta[n] = x[0]\delta[n]$
  - ….
- $x[n] = \cdots + x_{-1}[n] + x_0[n] + x_1[n] + \cdots$
- $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$



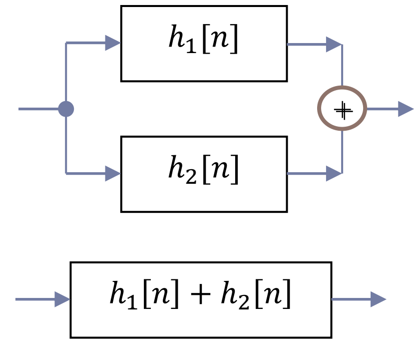
Convolution for Discrete-Time Systems

- For a discrete LTI system
  - $x[n] \rightarrow y[n]$
  - $x[n - n_0] \rightarrow y[n - n_0]$
  - $x[k]\delta[n - n_0] \rightarrow x[k]h[n - n_0]$
- Since the input can be represented as sum of deltas
  - $x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$
- then
  - $y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$
- since $x[n] * h[n] = h[n] * x[n]$ we can also write
  - $y[n] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$

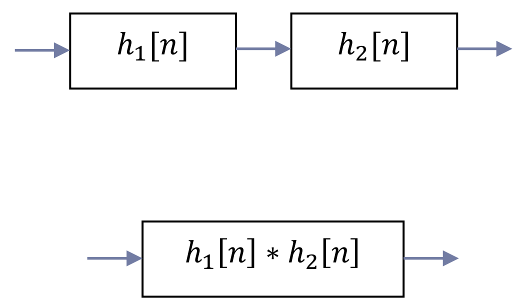
- $y[0] = \cdots + x[-2]h[2] + x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2] + \cdots$
- Sum of indices in each term equal to the sample of interest
-
- In general
- $y[n] = \cdots + x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$
-
- Recall the following properties,
- $\delta[n] * h[n-n_0] = h[n-n_0]$
- $\delta[n-n_0] * h[n] = h[n-n_0]$
- Do not confuse multiplication with convolution
- $\delta[n]g[n-n_0] = g[-n_0]\delta[n]$
- $\delta[n-n_0]g[n] = g[n_0]\delta[n-n_0]$

Properties of Convolution

- Commutative property
  $x[n] * h[n] = h[n] * x[n]$



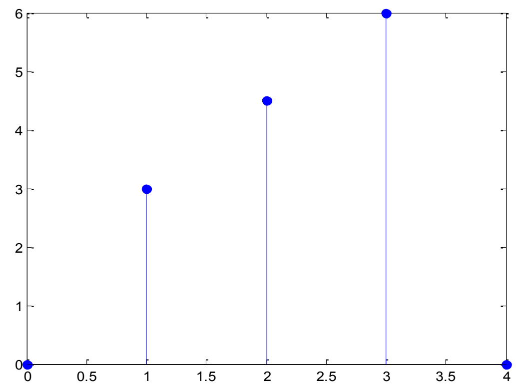
- Associative property
- $(f[n] * g[n]) * h[n] = f[n] * (g[n] * h[n]) = (h[n] * f[n]) * g[n]$
-

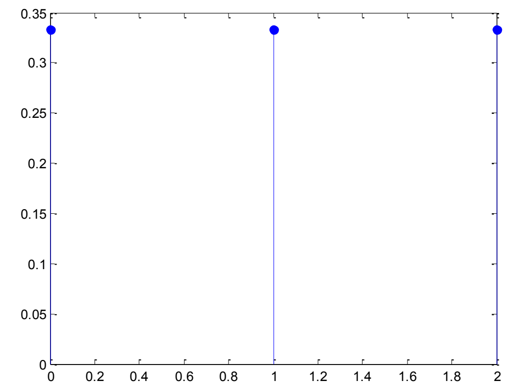


- Distributive property
- $x[n] * h_1[n] + x[n] * h_2[n] = x[n] * (h_1[n] + h_2[n])$

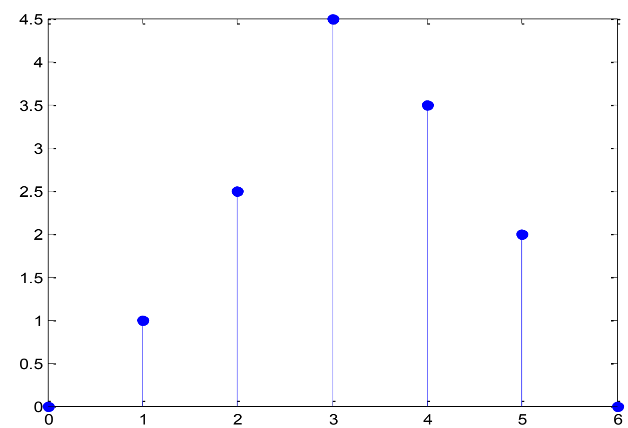
Example: System Response by Convolution

- use table to perform discrete convolution!
- http://www.jhu.edu/signals/discreteconv2/index.html
- Total number of points $= sum - 1$
- Matlab Code
- n=0:6;
- x=[0 3 4.5 6 0];
- h=[1/3 1/3 1/3];
- y=conv(x,h)
- stem(n,y,'fill')
-



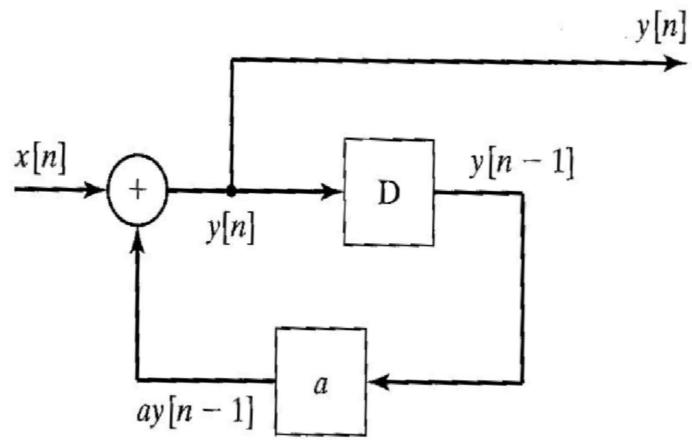


-2	-2	-1	0	1	2	3
			1/3	1/3	1/3	
6	4.5	3	0			0
	6	4.5	3	0		1
		6	4.5	3	0	2.5
			6	4.5	3	4.5
				6	4.5	3.5
					6	2
						0



Example II: Calculation of the impulse response of a discrete system

- $y[n] = ay[n-1] + x[n]$



- To find the impulse response we make $x[n] = \delta[n]$, then $y[n] = h[n]$
- $h[0] = ah[-1] + \delta[0] = a(0) + 1 = 1$
- $h[1] = ah[0] + \delta[1] = a(1) + 0 = a$
- $h[2] = ah[1] + \delta[2] = a(a) + 0 = a^2$
- $h[3] = ah[2] + \delta[3] = a(a^2) + 0 = a^3$

- $h[n] = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} = a^n u[n]$
- The unit impulse response consists of an unbounded number of terms; this system is called an *infinite impulse response* (IIR) system.

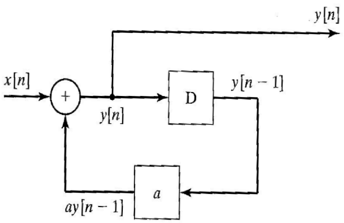
Continue the example Solution

- In the previous example, the impulse response contained a finite number of nonzero terms. This kind of system is called *finite impulse response* (FIR) systems.
- The impulse response is seldom used directly, instead we give the z-transform (to be introduced)
- The alternatives for representing a system are:
  - Impulse response
  - Difference equation
  - Block diagram
  - z-transform

0
1
2.5
4.5
3.5
2
0

Example III: Step response of a discrete system

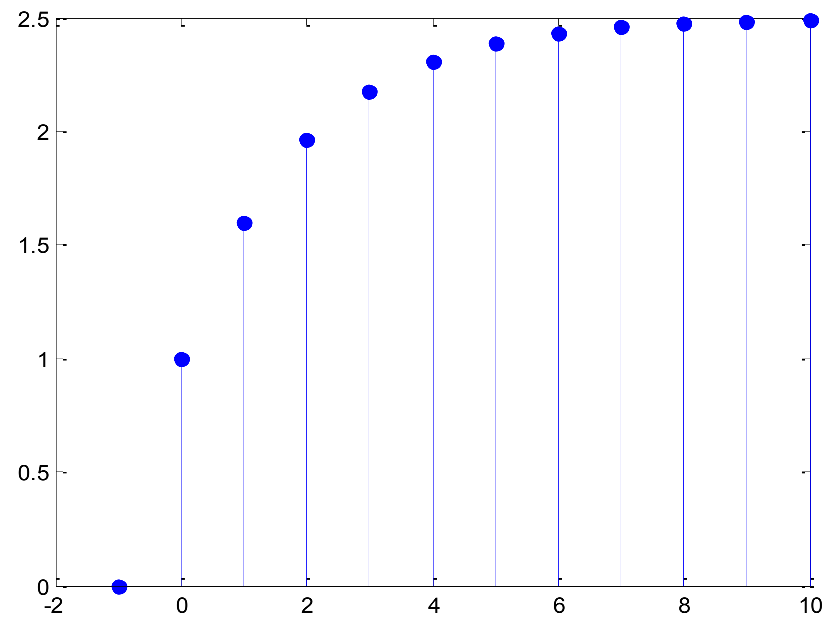
- Let $a = 0.6$, for the above system the impulse resp $h[n] = (0.6)^n u[n]$, Find the step response



- We can write the output as
- $y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = \sum_{k=-\infty}^{\infty} u[n-k](0.6)^k u[k] = \sum_{k=0}^{n} (0.6)^k$
- Using Appendix C
- $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$
- If $a = 1$ then the summation is $n + 1$ for $a = 1$ (we cannot use the formula above), otherwise
- $y[n] = \sum_{k=0}^{n} (0.6)^k = \frac{1-0.6^{n+1}}{1-0.6} = 2.5[1 - (0.6)^{n+1}], \quad n \geq 0$

Continue example 3

- The calculation of y yields
- $y[0] = 1$
- $y[1] = 1.6$
- $y[2] = 1.96$
- $\ldots$
- $y[\infty] = 2.5$



- Matlab Code
  - n=-1:10;
  - y=2.5*(1-(0.6.^(n+1)));
  - stem(n,y,'fill')

3 Properties of Discrete-Time LTI System

- The input-output relation for LTI systems
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$
- **Memory**
  - $y[n] = \cdots + x[n+2]h[-2] + x[n+1]h[-1] + x[h]h[0] + x[n-1]h[1] + \cdots . = h[0]x[n]$
  - For a memoryless system $h[n] = k\delta[n]$
  - A memoryless LTI system is then a pure gain.
- **Invertibility**
  - $h[n] * h_i[n] = \delta[n]$
  - z-transform (to be discussed) is one way to find the inverse system.
  - Example: If the system is $\sin[\frac{\pi n}{2}]$ which is zero for n even, then the system us not invertible.

Continue properties

- **Causality**
  - A signal that is zero for $n < 0$ is called a *causal signal*
  - For a causal system
  - $h[n] = 0 \; for \; n < 0$
  - We can write the convolution equation as
  - $y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} x[n-k]h[k]$
- **Stability**
  - A system is BIBO stable if
  - $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
  - For an LTI casual system, this condition reduces to
  - $\sum_{k=0}^{\infty} |h[k]| < \infty$

Example: Stability of LTI discrete systems

- Study the memory, causality, and stability characteristics of the following :
  - *a)* $h[n] = \left(\frac{1}{2}\right)^n u[n]$
  - *b)* $h[n] = (2)^n u[n]$
  - *c)* $h[n] = \left(\frac{1}{2}\right)^n u[n+1]$
- a) has memory (dynamic) since $h[n] \neq K\delta[n]$
  - Causal $h[n] = 0 \; for \; n < 0$
  - Stable (Appendix C)
  - $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$
- b) is similar but unstable
  - $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} (2)^n = 1 + 2 + 4 + 8 + \cdots$
- c) The system has memory , not causal $h[-1] = 2 \neq 0$
  - The system is stable =2+$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2 + \frac{1}{1-\frac{1}{2}} = 4$

Unit Step Response

- The unit step response is denoted
  - $s[n] = \sum_{k=-\infty}^{\infty} u[n-k]h[k] = \sum_{k=-\infty}^{n} h[k]$
  - This is because $u[n-k] = 0\ for\ (n-k) < 0\ or\ for\ k > n$
- Also we can form a difference equation
  - $s[n] - s[n-1] = \sum_{k=-\infty}^{n} h[k] - \sum_{k=-\infty}^{n-1} h[k] = h[n]$
- The unit step response completely describes the input output characteristics of a system
-
- *See examples 10.6 & Example 10.7*

Example: Step response from the impulse response

- The system $h[n] = 0.6^n u[n]$ is dynamic , causal , and stable
- The step response is $s[n] = \sum_{k=-\infty}^{+\infty} h[k] = \sum_{k=0}^{n} 0.6^k$
  - $s[n] = \sum_{k=0}^{n} 0.6^k = \frac{1-0.6^{n+1}}{1-0.6} u[n] = 2.5(1-0.6^{n+1}) u[n]$
  - $u[n]$ is necessary, because $s[n] = 0\ for\ n < 0\ (causal)$
- To verify
  - $h[n] = s[n] - s[n-1] = 2.5(1-0.6^{n+1})u[n] - 2.5(1-0.6^{n})u[n-1]$
  - For $n = 0, h[0] = 2.5(1-0.6) = 1$ (First term only)
  - For $n \geq 1, h[n] = 2.5(1-0.6^{n+1} - 1 + 0.6^n) = 2.5(0.6^n)(1-0.6) = 0.6^n$
  - $h[n] = 0.6^n u[n]$

Difference Equation Model

- LTI discrete-time systems are usually modeled by a linear difference equation with constant coefficients.
- Digital filters are important example
- Note the difference between the system model and the physical system
- $a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_{M-1} x[n-M+1] + b_M x[n-M]$
- $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k], \quad a_0 \neq 0$
- N^{th} order equation : the max shift of the dependent variable.
- Example: $y[n] = 0.6y[n-1] + x[n]$, first order
- Example: $y[n] = \frac{x[n]+x[n-1]+x[n-2]}{3}$, zeros order