KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

ELECTRICAL ENGINEERING DEPARTMENT

Probabilistic Methods in Electrical Engineering
EE 315

SECOND MAJOR

DATE: December 7, 2011

TIME: 5:30-7:30 pm

Name: _______KEY__________________

ID :   _________________________

Section # : ______________________

QUESTION	MARK
1	**/20**
2	**/20**
3	**/30**
4	**10**
TOTAL	**/80**

Problem #1 (20)

The following information is known about two jointly Gaussian random variables X and Y :

$$E[X]=0, \quad E[Y]=-1, \quad E[X^2]=4, \quad E[Y^2]=9, \text{ and } R_{XY}=-4$$

Two new random variables W and U are defined as

$$W=3X+Y$$

$$U=-X-2Y$$

a) Find $E[W]$, $E[U]$, $E[W^2]$, and $E[U^2]$.

b) Find the variances σ_X^2 , σ_Y^2, σ_U^2 and σ_W^2

c) Find the correlation R_{WU}

d) Are W and U uncorrelated? Justify your answer.

e) Determine the joint pdf of W and U .

Solution

(a) $E[W] = E[3X+Y] = 3E[X] + E[Y] = 3(0) + (-1) = -1$

$E[U] = E[-X-2Y] = -E[X] - 2E[Y] = 0 - 2(-1) = 2$

$E[W^2] = E[(3X+Y)^2] = E[9X^2+6XY+Y^2] = E[9X^2] + 6E[XY] + E[Y^2] = 9(4)+6(-4)+9 = 21$

$E[U^2] = E[(-X-2Y)^2] = E[X^2+4XY+4Y^2] = E[X^2] + 4E[XY] + 4E[Y^2] = 4 + 4(-4) + 4(9) = 24$

(b) $\sigma_X^2 = E[X^2] - (E[X])^2 = 4 - (0)^2 = 4$

$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = 9 - (-1)^2 = 8$

$\sigma_U^2 = E[U^2] - (E[U])^2 = 24 - (2)^2 = 20$

$\sigma_W^2 = E[W^2] - (E[W])^2 = 21 - (-1)^2 = 20$

(c) $R_{WU} = E[WU] = E[(3X+Y)(-X-2Y)] = -E[(3X+Y)(X+2Y)] = -3E[X^2] - 7E[XY] - 2E[Y^2]$

$= -3(4) - 7(-4) - 2(9) = -2$

(d) $R_{WU} = E[WU] \overset{?}{=} E[W]E[U]$

$R_{WU} = E[WU] = -2 \quad E[W] = -1 \quad E[U] = 2 \Rightarrow E[WU] = E[W]E[U] \Rightarrow$ W and U are uncorrelated

$$(e)\ f_{W,U}(w,u)=f_{W,U}(w,u)=\frac{1}{2\pi\sigma_W\sigma_U\sqrt{1-\rho_{WU}{}^2}}\exp\left\{\frac{-1}{2(1-\rho_{WU}{}^2)}\left[\frac{(w-\overline{W})^2}{\sigma_W^2}-\frac{2\rho_{WU}(w-\overline{W})(u-\overline{U})}{\sigma_W\sigma_U}+\frac{(u-\overline{U})^2}{\sigma_U^2}\right]\right\}$$

Since W and U are uncorrelated $\Rightarrow \rho_{WU}=\dfrac{\mathrm{R_{WU}}-\mathrm{E[W]E[U]}}{\sigma_W\sigma_U}=\dfrac{0}{\sigma_W\sigma_U}=0$

$$\Rightarrow f_{W,U}(w,u)=f_{W,U}(w,u)=\frac{1}{2\pi\sigma_W\sigma_U}\exp\left\{\frac{-1}{2}\left[\frac{(w-\overline{W})^2}{\sigma_W^2}+\frac{(u-\overline{U})^2}{\sigma_U^2}\right]\right\}$$

$$f_{W,U}(w,u)=\frac{1}{2\pi\sqrt{20}\sqrt{20}}\exp\left\{\frac{-1}{2}\left[\frac{(w+1)^2}{20}+\frac{(u-2)^2}{20}\right]\right\}$$

$$f_{W,U}(w,u)=\frac{1}{40\pi}\exp\left\{\frac{-1}{2}\left[\frac{(w+1)^2+(u-2)^2}{20}\right]\right\}$$

Another Solution

$$\text{(e) } f_{X,Y}(x,y)=\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}{}^2}}\exp\left\{\frac{-1}{2(1-\rho_{XY}{}^2)}\left[\frac{(x-\overline{X})^2}{\sigma_X^2}-\frac{2\rho_{XY}(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y}+\frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]\right\}$$

$$\rho_{XY}=\frac{\mathrm{R_{XY}}-\overline{X}\,\overline{Y}}{\sigma_W\sigma_U}=\frac{-4-(0)(-1)}{\sqrt{4}\sqrt{8}}=\frac{-1}{\sqrt{2}}$$

$$f_{X,Y}(x,y)=\frac{1}{2\pi\sqrt{4}\sqrt{8}\sqrt{1-\left(\frac{-1}{\sqrt{2}}\right)^2}}\exp\left\{\frac{-1}{2\left(1-\left(\frac{-1}{\sqrt{2}}\right)^2\right)}\left[\frac{(x-0)^2}{4}-\frac{2\left(\frac{-1}{\sqrt{2}}\right)(x-0)(y+1)}{\sqrt{4}\sqrt{8}}+\frac{(y+1)^2}{8}\right]\right\}$$

$$f_{X,Y}(x,y)=\frac{1}{8\pi}\exp\left\{-\left[\frac{x^2}{4}+\frac{x(y+1)}{4}+\frac{(y+1)^2}{8}\right]\right\}$$

$$\left.\begin{aligned}W&=3X+Y\\U&=X-Y\end{aligned}\right\}\Rightarrow\begin{aligned}X&=\frac{2}{5}W+\frac{1}{5}U\\Y&=\frac{-1}{5}W-\frac{3}{5}U\end{aligned}\Rightarrow \mathrm{J}=\begin{bmatrix}\frac{\partial \mathrm{X}}{\partial \mathrm{W}}&\frac{\partial \mathrm{X}}{\partial \mathrm{U}}\\\frac{\partial \mathrm{Y}}{\partial \mathrm{W}}&\frac{\partial \mathrm{Y}}{\partial \mathrm{U}}\end{bmatrix}=\begin{bmatrix}\frac{2}{5}&\frac{1}{5}\\\frac{-1}{5}&\frac{-3}{5}\end{bmatrix}\Rightarrow|\mathrm{J}|=\left|\left(\frac{2}{5}\right)\left(\frac{-3}{5}\right)-\left(\frac{1}{5}\right)\left(\frac{-1}{5}\right)\right|=\frac{1}{5}$$

$$f_{W,U}(w,u)=\frac{1}{8\pi}\exp\left\{-\left[\frac{x^2}{4}+\frac{x(y+1)}{4}+\frac{(y+1)^2}{8}\right]_{\substack{X=\frac{2}{5}W+\frac{1}{5}U\\Y=\frac{-1}{5}W-\frac{3}{5}U}}\right\}|\mathrm{J}|$$

$$=\frac{1}{40\pi}\exp\left\{\frac{-1}{2}\left[\frac{(w+1)^2+(u-2)^2}{20}\right]\right\}$$

Problem #2 (20)

A random variable X that is exponentially distributed with pdf

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

is transformed into another random variable Y using the transformation

$$Y = 2(X-2)^2 - 6$$

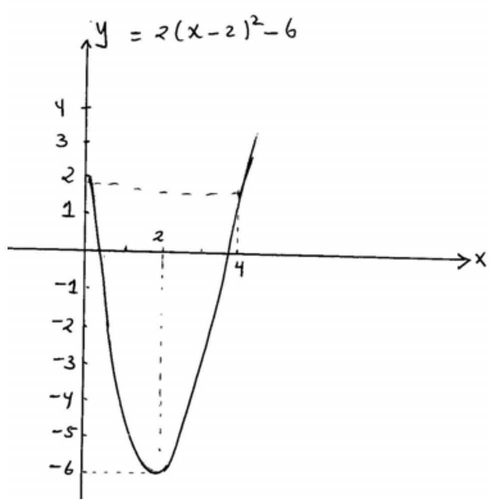
Find the pdf of the random variable Y, that is $f_Y(y)$. ?

Solution

$$Y = 2(X-2)^2 - 6 \Rightarrow 2(X-2)^2 = Y + 6 \Rightarrow (X-2)^2 = \frac{Y}{2} + 3 \Rightarrow \pm\sqrt{\frac{Y}{2}+3}$$

$$\Rightarrow x_1 = 2 + \sqrt{\frac{Y}{2}+3} \qquad x_2 = 2 - \sqrt{\frac{Y}{2}+3}$$

$$\frac{dy}{dx} = 4(x-2)$$



For $-6 \le y \le 2$ we have two roots x_1 and x_2

$$f_Y(y) = \frac{f_X\left(x = x_1 = 2 - \sqrt{\frac{Y}{2}+3}\right)}{\left|\frac{dy}{dx}\right|_{x = x_1 = 2-\sqrt{\frac{Y}{2}+3}}} + \frac{f_X\left(x = x_1 = 2 + \sqrt{\frac{Y}{2}+3}\right)}{\left|\frac{dy}{dx}\right|_{x = x_1 = 2+\sqrt{\frac{Y}{2}+3}}}$$

$$\left|\frac{dy}{dx}\right|_{x = x_1 = 2-\sqrt{\frac{Y}{2}+3}} = \left|4(x-2)\right|_{x = x_1 = 2-\sqrt{\frac{Y}{2}+3}} = \left|4\left(2 - \sqrt{\frac{Y}{2}+3} - 2\right)\right| = 4\sqrt{\frac{Y}{2}+3}$$

$$\left|\frac{dy}{dx}\right|_{x = x_2 = 2+\sqrt{\frac{Y}{2}+3}} = \left|4(x-2)\right|_{x = x_2 = 2+\sqrt{\frac{Y}{2}+3}} = \left|4\left(2 + \sqrt{\frac{Y}{2}+3} - 2\right)\right| = 4\sqrt{\frac{Y}{2}+3}$$

$$\Rightarrow \quad \frac{e^{-2-\sqrt{\frac{Y}{2}+3}}}{4\sqrt{\frac{Y}{2}+3}} + \frac{e^{-2+\sqrt{\frac{Y}{2}+3}}}{4\sqrt{\frac{Y}{2}+3}} = \frac{e^{-2}[e^{-\sqrt{\frac{Y}{2}+3}} + e^{\sqrt{\frac{Y}{2}+3}}]}{4\sqrt{\frac{Y}{2}+3}}$$

For $y \ge 2$ we have one root x_2

$$f_Y(y) = \frac{f_X\left(x = x_1 = 2 + \sqrt{\frac{Y}{2}+3}\right)}{\left|\frac{dy}{dx}\right|_{x = x_1 = 2+\sqrt{\frac{Y}{2}+3}}} = \frac{e^{-2+\sqrt{\frac{Y}{2}+3}}}{4\sqrt{\frac{Y}{2}+3}}$$

Therefore

$$f_Y(y) = \begin{cases} \dfrac{e^{-2}[e^{-\sqrt{\frac{Y}{2}+3}} + e^{\sqrt{\frac{Y}{2}+3}}]}{4\sqrt{\frac{Y}{2}+3}} & -6 \le y \le 2 \\[2ex] \dfrac{e^{-2-\sqrt{\frac{Y}{2}+3}}}{4\sqrt{\frac{Y}{2}+3}} & y \ge 2 \\[2ex] 0 & y \le -6 \end{cases}$$

Problem #3 (30)

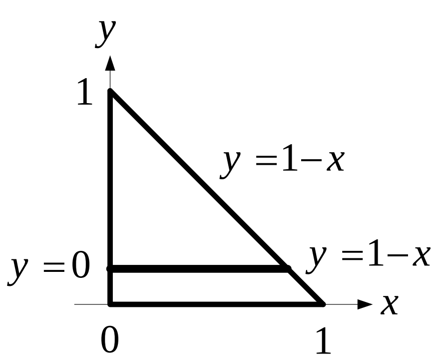
Let X and Y be two random variables with joint probability density

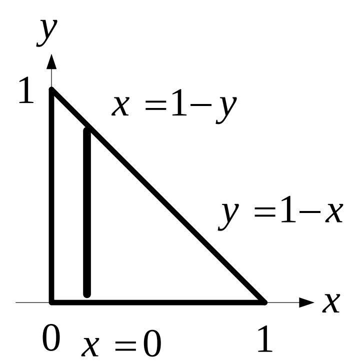
as shown below function

$$f_{X,Y}(x,y)=\begin{cases}24xy & x\ge 0 \quad y\ge 0 \quad x+y\le 1\\ \\ 0 & else\ where\end{cases}$$

(a) Are the random variables X and Y independent ? *Explain* ?
(b) Are the random variables X and Y correlated ? Explain ?
(c) Are the random variables X and Y orthogonal ? Explain ?
(d) Find $E[X \mid Y=0.5]$.
(e) Find the probability $P\left\{\frac{Y}{2}\le X \le Y\right\}$?

Solution





$$(a)\ f_{X,Y}(x,y)\overset{?}{=}f_X(x)f_Y(y) \quad \text{for independence}$$

$$f_X(x)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)dy=\int_{y=0}^{y=1-x}24xydy = 24x\left.\frac{y^2}{2}\right|_{y=0}^{y=1-x}=12x(1-x)^2 \qquad 0<x<1$$

$$f_Y(y)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)dx=\int_{x=0}^{x=1-y}24xydx = 24y\left.\frac{x^2}{2}\right|_{x=0}^{x=1-y}=12y(1-y)^2 \qquad 0<y<1$$

Since $f_{X,Y}(x,y)\neq f_X(x)f_Y(y)$ X and Y are not independent (*b*) $E[XY] \overset{?}{=} E[X]E[Y]$ for correlation uncorrelation

$$E[XY]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} 24x^2y^2\,dy\,dx = \int_{x=0}^{x=1} 24x^2 \int_{y=0}^{y=1-x} y^2\,dy\,dx$$

$$= \int_{x=0}^{x=1} 24x^2 \left.\frac{y^3}{3}\right|_{y=0}^{y=1-x} dx = \int_{x=0}^{x=1} 8x^2(1-x)^3\,dx = 8\int_{x=0}^{x=1}(x^2-3x^3+3x^4-x^5)\,dx = \frac{2}{15} = 0.1333$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\,12x(1-x)^2\,dx = 12\int_0^1 x^2(1-2x+x^2)\,dx = \frac{2}{5} = 0.4$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^1 y\,12y(1-y)^2\,dy = 12\int_0^1 y^2(1-2y+y^2)\,dy = \frac{2}{5} = 0.4$$

Since $E[XY] \neq E[X]E[Y]$ X and Y are correlated

(c) $E[XY] \overset{?}{=} 0$ for orthognal , Since $E[XY] = \frac{2}{15} \neq 0 \Rightarrow$ X and Y are not orthognal

(d) $E[X|Y=0.5] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y=0.5)\,dx$

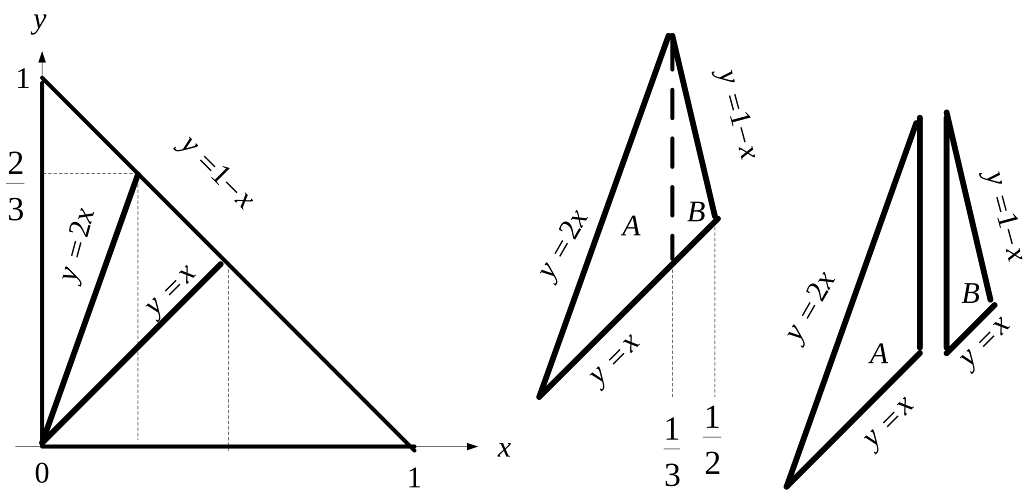
$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{24xy}{12y(1-y)^2} = \frac{2x}{(1-y)^2} \qquad 0 \le x \le 1-y$$

$$f_{X|Y}(x \mid y=0.5) = \frac{2x}{(1-0.5)^2} = 8x \qquad 0 \le x \le 0.5$$

$$\Rightarrow E[X|Y=0.5] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y=0.5)\,dx = \int_0^{0.5} x\,8x\,dx = \frac{1}{3} = 0.333$$

(e)

$$\left\{\frac{Y}{2} \le X \le Y\right\}$$



$$P\left\{\frac{Y}{2} \le X \le Y\right\} = \text{P(A)} + \text{P(B)}$$

$$\text{P(A)} = \int_{x=0}^{x=1/3} \int_{y=x}^{y=2x} 24xy\,dy\,dx = 24 \int_{x=0}^{x=1/3} x \left.\frac{\text{y}^2}{2}\right|_{y=x}^{y=2x} dx = 12 \int_{x=0}^{x=1/3} x\ (4x^2 - x$$

$$\text{P(B)} = \int_{x=1/3}^{x=1/2} \int_{y=x}^{y=1-x} 24xy\,dy\,dx = 24 \int_{x=1/3}^{x=1/2} x \left.\frac{\text{y}^2}{2}\right|_{y=x}^{y=1-x} dx = 12 \int_{x=1/3}^{x=1/2} x\ ((1-x)$$

$$\Rightarrow P\left\{\frac{Y}{2} \le X \le Y\right\} = \frac{1}{9} + \frac{7}{54} = \frac{13}{54} = 0.240$$

Problem #4 (10)

Let X and Y be two jointly Gaussian random variables , and

let W and U be two new random variables defined as

$$W = X + aY$$
$$U = X - aY$$

Find a in terms of the moments of X and Y such that W and U are orthogonal ?

Solution

$$E[WU]=E[(X+aY)(X-aY)] = E[X^2 - aXY + aXY - a^2Y^2] = E[X^2 - a^2Y^2]$$
$$= E[X^2] - a^2E[Y^2] = 0$$

$$\Rightarrow \ a^2 = \frac{E[X^2]}{E[Y^2]} \quad \Rightarrow \quad a = \pm\sqrt{\frac{E[X^2]}{E[Y^2]}}$$