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# The $z$-transform Part 2

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# The material to be covered in this lecture is as follows:

- Properties of the z-transform
  - Linearity
  - Initial and final value theorems
  - Time-delay
- z-transform table
- Inverse z-transform
- Application of *z*-transform to discrete-time systems

# After finishing this lecture you should be able to:

- Find the *z*-transform for a given signal utilizing the *z*-transform tables
- Utilize the z-transform properties like the initial and final value theorems
- Find the inverse *z*-transform.
- Utilize *z*-transform to perform convolution for discrete-time systems.

# Derivation of the *z*-Transform

- The z−transform is defined as follows: $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$
- The coefficient  denote the sample value and  denotes that the sample occurs *n* sample periods after the *t*=0 reference.
- Rather than starting form the given definition for the z-transform, we may build a table for the popular signals and another table for the z-transform properties.
- Like the Fourier and Laplace transform, we have two options either to start from the definition or we may utilize the tables to find the proper transform.
- The next slide illustrates a few *z*-transform pairs.
- Then we will investigate some of the *z*-transform properties:
  - Linearity
  - Time-shifting property
  - Initial and final value theorems

# Table of $z$-transform pairs

| $f[n]$ | $F(z)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $\delta(t-k\Delta T)$ | $z^{-k}$ |
| $u(t)$ | $\frac{z}{z-1}$ |
| $t$ | $\frac{\Delta T z}{(z-1)^2}$ |
| $t^2$ | $\frac{\Delta T^2 z(z+1)}{(z-1)^3}$ |
| $e^{-at}$ | $\frac{z}{z-e^{-a\Delta T}}$ |
| $te^{-at}$ | $\frac{\Delta T z e^{-a\Delta T}}{(z-e^{-a\Delta T})^2}$ |
| $a^n u[n]$ | $\frac{z}{z-a}$ |

**TABLE 11.2** *z*-Transforms

| $f[n], n \geq 0$ | $F(z)$ | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $\delta[n-n_0]$ | $z^{-n_0}$ | $z \neq 0$ |
| 3. $u[n]$ | $\frac{z}{z-1}$ | $\|z\| > 1$ |
| 4. $n$ | $\frac{z}{(z-1)^2}$ | $\|z\| > 1$ |
| 5. $n^2$ | $\frac{z(z+1)}{(z-1)^3}$ | $\|z\| > 1$ |
| 6. $a^n$ | $\frac{z}{z-a}$ | $\|z\| > \|a\|$ |
| 7. $na^n$ | $\frac{az}{(z-a)^2}$ | $\|z\| > \|a\|$ |
| 8. $n^2a^n$ | $\frac{az(z+a)}{(z-a)^3}$ | $\|z\| > \|a\|$ |
| 9. $\sin bn$ | $\frac{z \sin b}{z^2 - 2z\cos b + 1}$ | $\|z\| > 1$ |
| 10. $\cos bn$ | $\frac{z(z-\cos b)}{z^2 - 2z\cos b + 1}$ | $\|z\| > 1$ |
| 11. $a^n \sin bn$ | $\frac{az \sin b}{z^2 - 2az\cos b + a^2}$ | $\|z\| > \|a\|$ |
| 12. $a^n \cos bn$ | $\frac{z(z - a\cos b)}{z^2 - 2az\cos b + a^2}$ | $\|z\| > \|a\|$ |

# Linearity of the *z*-Transform

- If $x_1[n] \overset{z}{\leftrightarrow} X_1(z)$ with region of convergence, $ROC = R_1$.
- And $x_2[n] \overset{z}{\leftrightarrow} X_2(z)$ with region of convergence, $ROC = R_2$.
- Then If $a_1 x_1[n] + a_2 x_2[n] \overset{z}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z)$
- with $ROC = R_1 \cap R_2$
- This follows directly from the definition of the *z*-transform because the summation operator is linear.
- It is easily extended to a linear combination of an arbitrary number of signals.
- This property includes the multiplication by constant property which states that if the signal is scaled by a constant its *z*-transform will be scaled by the same constant.
- $a_1 x_1[n] \overset{z}{\leftrightarrow} a_1 X_1(z)$

# Time-Shifting property for the $z$-Transform

- If $x[n] \overset{z}{\leftrightarrow} X(z)$ with $ROC = R$
- Then $x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}\ X(z)$ with *ROC=R*
- <u>Proof</u>

$$Z\{x[n-1]\} = \sum_{n=-\infty}^{\infty} x[n-1]\,z^{-n}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x[n-1]\,z^{-(n-1)}$$

$$= z^{-1} \sum_{m=-\infty}^{\infty} x[m]\,z^{-m} = z^{-1} Z\{x[n]\}$$

- This property will be very important for producing the **z-transform transfer function of a difference equation** which uses the property:

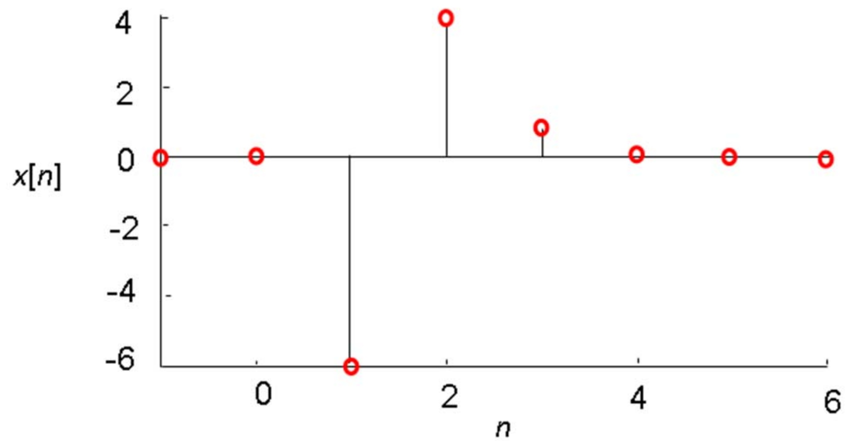
$$x[n-1] \overset{Z}{\leftrightarrow} z^{-1} X(z)$$

# Example 1: Properties of the $z$-transform

- Find the z-transform for the input signal

$$x[n] = 7(1/3)^{n-2}u[n-2] - 6(1/2)^{n-1}u[n-1]$$

- Solution:




- We know that $a^n u[n] \overset{Z}{\leftrightarrow} \dfrac{z}{z-a}$
- So

$$X(z) = 7z^{-2}\frac{z}{z-1/3} - 6z^{-1}\frac{z}{z-1/2}$$

$$= 7\frac{1}{z^2 - 1/3z} - 6\frac{1}{z-1/2}$$

# Initial and Final Value Theorems

- If $x[n]$ has a z-transform $X(z)$ and if $\lim X(z)\ as\ z \to \infty$ exists, then

$$\lim_{n \to 0} x[n] = x[0] = \lim_{z \to \infty} X(z)$$

- This theorem can be easily proven by the definition of the *z*-transform
- As we take the limit all terms will be zero except the first term

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = x[0] + x[1] z^{-1} + x[2] z^{-2} + \ldots$$

- The final value theorem which is given by
- $$\lim_{n \to \infty} x[n] = \lim_{z \to 1} \left( \left(1 - z^{-1}\right) X(z) \right)$$

# Example 2: Application of the initial and final value theorems

- Find the initial and final values for the following signal expressed in its z-transform

$$F(z)=\frac{0.792z^{2}}{(z-1)(z^{2}-0.416z+0.208)}$$

- Solution:
- Initial-value

$$F(z\to\infty)=\frac{0.792z^{2}}{z^{3}}=0$$

- Final –value

$$f(n\to\infty)=\frac{0.792}{(1-0.416+0.208)}=1$$

- These answers can be justified by looking at the expansion of the given expression

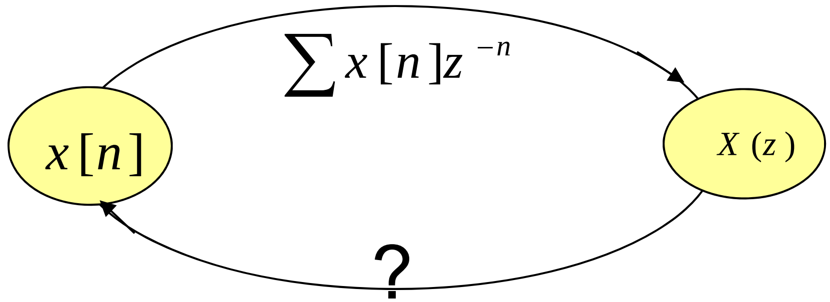
$$F(z)=0.792z^{-1}+1.12z^{-2}+1.091z^{-3}+1.01z^{-4}+0.983z^{-5}+0.989z^{-6}+0.99z^{-7}\ldots.$$

- The coefficient for  is zero which is the initial value.
- The coefficient converges to one as the negative power of z increases which corresponds to the final value.

# Tables of z-transform properties

| Name | Property |
|---|---|
| 1. Linearity, (11.8) | $\mathscr{Z}[a_1 f_1[n] + a_2 f_2[n]] = a_1 F_1(z) + a_2 F_2(z)$ |
| 2. Real shifting, (11.13) | $\mathscr{Z}[f[n - n_0]u[n - n_0]] = z^{-n_0} F(z), \quad n_0 \geqq 0$ |
| 3. Real shifting, (11.25) | $\mathscr{Z}[f[n + n_0]u[n]] = z^{n_0}[F(z) - \sum_{n=0}^{n_0-1} f[n] z^{-n}]$ |
| 4. Complex shifting, (11.23) | $\mathscr{Z}[a^n f[n]] = F(z/a)$ |
| 5. Multiplication by $n$ | $\mathscr{Z}[nf[n]] = -z \dfrac{dF(z)}{dz}$ |
| 6. Time scaling, (11.33) | $\mathscr{Z}[f[n/k]] = F(z^k)$, $k$ a positive integer |
| 7. Convolution, (11.38) | $\mathscr{Z}[x[n] * y[n]] = X(z)Y(z)$ |
| 8. Summation | $\mathscr{Z}[\sum_{k=0}^{n} f[k]] = \dfrac{z}{z-1} F(z)$ |
| 9. Initial value, (11.27) | $f[0] = \lim_{z \to \infty} F(z)$ |
| 10. Final value, (11.30) | $f[\infty] = \lim_{z \to 1} (z - 1)F(z)$, if $f[\infty]$ exists |

# Inverse z-Transform




Find the inverse z-transform for $\frac{z}{z-1}$ by long division .

- The inverse operation for the z-transform my be accomplished by:
  - Long division
  - Partial fraction expansion
- The z-transform of a sample sequence can be written as

$$X(z) = x(0) + x(T)z^{-1} + x(2T)z^{-2} + \ldots.$$

- If we can write *X*(z) into this form, the sample values can be determined by inspection.
- When *X*(z) is represented in a ratio of polynomials in z, this can be easily achieved by long division.
- Before carrying out the division, it is convenient to arrange both the numerator and the denominator in ascending powers of $z^{-1}$.

# Inverse *z*-transform using Partial Fraction Expansion

- Alternatively, we may avoid the long division by partial fraction expansion. The idea is similar to the method used for inverse Laplace transform.
- The objective is to manipulate $X(z)$ into a form that can be inverse *z*-transformed by using *z*-transform tables.
- Because of the forms of transforms,
  - it is usually best to perform partial fraction expansion of $H(z)/z$.
  - As an alternative $z^{-1}$ can be treated as the variable in the partial fraction expansion.
- Important: before doing partial-fraction expansion, make sure the *z*-transform is in proper rational function of !

# Example 3: Inverse $z$-Transform Using Partial Fraction Expansion

- Find the inverse z-transform using both partial fraction expansion and long division $X(z) = \frac{z^2}{(z-1)(z-0.2)}$
- Solution:
- If we treat $z^{-1}$ as the variable in the partial fraction expansion, we can write
  - $X(z) = \frac{1}{(1-z^{-1})(1-0.2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.2z^{-1}}$
  - Utilizing Heaviside's Expansion Method:
  - $A = (1 - z^{-1})X(z) = \frac{1}{1-0.2z^{-1}} = \frac{1}{0.8} = 1.25$
  - $B = (1 - 0.2z^{-1})X(z) = \frac{1}{1-z^{-1}} = -0.25$

$1-0.2z^{-1}=0 \ (z=0.2) \Rightarrow$

# Continue Solution of Example 3:

$$X(z) = \frac{1.25}{1 - z^{-1}} + \frac{-0.25}{1 - 0.2z^{-1}} \Rightarrow x(nT) = (1.25 - 0.25(0.2)^n)u[n]$$

- From which we may find that $x(0) = 1, x(T) = 1.2, x(2T) = 1.24, x(3T) = 1.248$
- We may get to the same answer using long division. $X(z)$ is written as
  - $X(z) = \frac{z^2}{z^2 - 1.2z + 0.2}$
- which is, after multiplying numerator and denominator by $z^{-2}$
  - $X(z) = \frac{1}{1 - 1.2z^{-1} + 0.2z^{-2}}$
- Now, it is left for you to show that the long division will result in the same answer given by
  - $X(z) = 1 + 1.2z^{-1} + 1.24z^{-2} + 1.248z^{-3} + \cdots$

# Difference Equation

- For continuous-time systems, differential equation may be solved using Laplace transform
- Similarly discrete-time systems result in ***Difference Equations*** which may be solved using *z*-transform
- Recall that discrete-time systems process a discrete-time input signal to produce a discrete-time output signal.
- The general symbolic notation for Discrete-Time System:
  - $y(nT) = H[x(nT)]$

# Transfer Function in the *z*-Domain

- There is a simple relationship for a signal **time-shift**
  - $x[n-1] \overset{z}{\leftrightarrow} \mathrm{z}^{-1}\mathrm{X(z)}$
- This is fundamental for deriving the transfer function of a difference equation which is expressed in terms of the input-output signal delays
- The **transfer function** of a discrete time LTI system is the *z*-transform of the system's impulse response
- The transfer function is a rational polynomial in the complex number z.
- Convolution is expressed as multiplication
  - $x[n] * h[n] \overset{z}{\leftrightarrow} X(z)H(z)$
- and this can be solved for particular signals and systems

# Example 4: Discrete-Time Convolution

- Calculate the output of a first order difference equation of a input signal $x[n] = 0.5^n u[n]$
- System transfer function (z-transform of the impulse response)
  - $y[n] - 0.8y[n-1] = x[n]$
- Taking the z-transform of the input signal
  - $0.5^n u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{z}{z-0.5}$
- Taking the z-transform of the difference equation
  - $Y(z) - 0.8z^{-1}Y(z) = X(z)$
  - $Y(z)(1 - 0.8z^{-1}) = X(z)$
  - $H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$
- The (z-transform of the) output is therefore the product:
- ROC $|z| > 0.8$

$$Y(z) = \frac{z^2}{(z-0.5)(z-0.8)}$$

$$= \frac{1}{0.3}\left(\frac{0.8z}{(z-0.5)} - \frac{0.5z}{(z-0.8)}\right)$$

$$y[n] = (0.8 * 0.5^n u[n] - 0.5 * 0.8^n u[n]) / 0.3$$

# Self Test

- Question 1:
  - If $Z\big(x(nT)\big) = \frac{1}{1+0.5z^{-1}}$, what's $Z(x(nT-2T))$?
  - Answer: $\frac{z^{-2}}{1+0.5z^{-1}}$
- Question 2:
  - Find the z-transform for $Y(z) = \frac{1}{z^2-1.2z+0.2}$
  - Answer:
  - $y(nT) = 5\delta[n] + (1.25 - 6.25(0.2)^n)u[n]$
  - If we compare with Example 3 we conclude that the answer should be
  - $y(nT) = (1.25 - 0.25(0.2)^{n-2})u[n-2]$
  - Which is the same . Try for $n = 0, 1,2,3, ..$ The values are $0,0,1, 1.2, \dots$
- Question 3:
  - Find $x(nT) * h(nT)$

Answer :

$$h(nT) = \left(\frac{1}{3}\right)^n u(n-5) = \left(\frac{1}{3}\right)^5 \left(\frac{1}{3}\right)^{n-5} u(n-5)$$

$$x(nT) = \left(\frac{1}{4}\right)^n u(n-3) = \left(\frac{1}{4}\right)^3 \left(\frac{1}{4}\right)^{n-3} u(n-3)$$

$$x(nT) * h(nT) = (\frac{1}{4})^3 (\frac{1}{3})^5 [4(\frac{1}{3})^{n-8} - 3(\frac{1}{4})^{n-8})] u(n-8)$$

# Continue Self Test

- Question 4 :
  - Calculate the step response to the system describe by the following difference equation
  - $6y[n] - 5y[n-1] + 1y[n-2] = x[n]$
- Answer

$$H(z) = \frac{1}{6 - 5z^{-1} + 1z^{-2}} = \frac{1}{(2 - z^{-1})(3 - z^{-1})}$$

$$u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{(3 - z^{-1})(2 - z^{-1})(1 - z^{-1})}$$

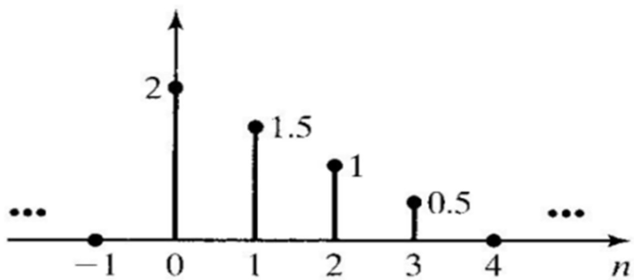
$$= 0.5\frac{1}{(3 - z^{-1})} - \frac{1}{(2 - z^{-1})} + 0.5\frac{1}{(1 - z^{-1})}$$

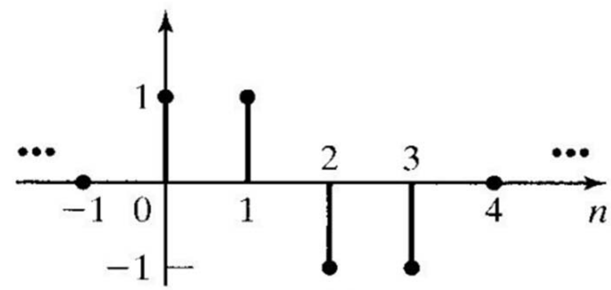
$$= 0.167\frac{1}{(1 - 1/3z^{-1})} - 0.5\frac{1}{(1 - 1/2z^{-1})} + 0.5\frac{1}{(1 - z^{-1})}$$

$$y[n] = (0.167(1/3)^n - 0.5(1/2)^n + 0.5)u[n]$$

# Continue Self Test

- Q5: Perform the following Convolution using z-transform and sketch the final answer




- $X(z) = 2 + 1.5z^{-1} + z^{-2} + 0.5z^{-3}$
- $H(z) = 1 + z^{-1} - z^{-2} - z^{-3}$
- $Y(z) = 2 + 3.5z^{-1} + 0.5z^{-2} - 2z^{-3} - 2z^{-4} - 1.5z^{-5} - 0.5z^{-6}$