Introduction to Discrete-Time Signals and Systems
Digital to Analog Conversion (D2A)

Reconstruction of Signals from Sample Data

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Class Objective and Outcomes

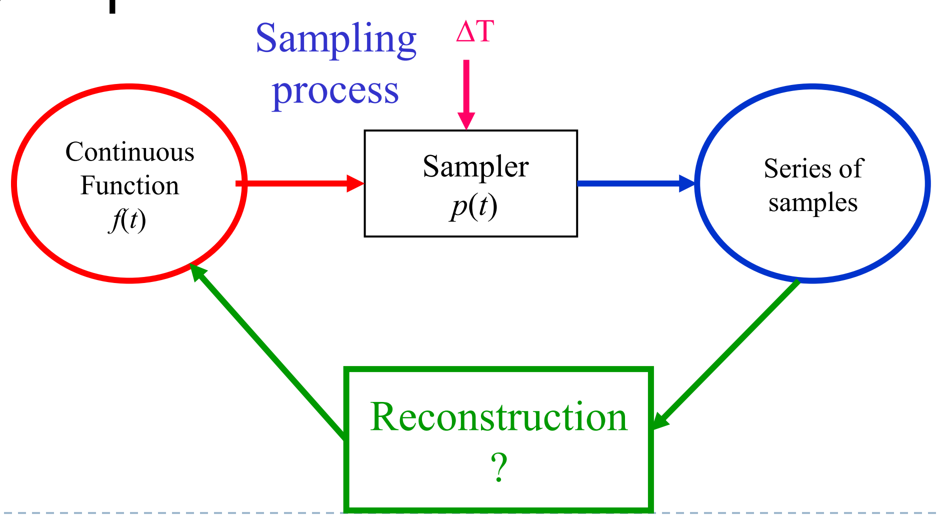
- Introduction to digital to analog (D/A) reconstruction
- Reconstruction/ Interpolation
  - Reconstruction in the frequency domain: Low-pass filter
  - Interpolation in time domain: *sinc* function
  - Interpolation in time domain: sample and hold
  - Practical reconstruction (e.g. using RC circuit)
  - Understand aliasing effects for signal reconstruction

Class outcomes

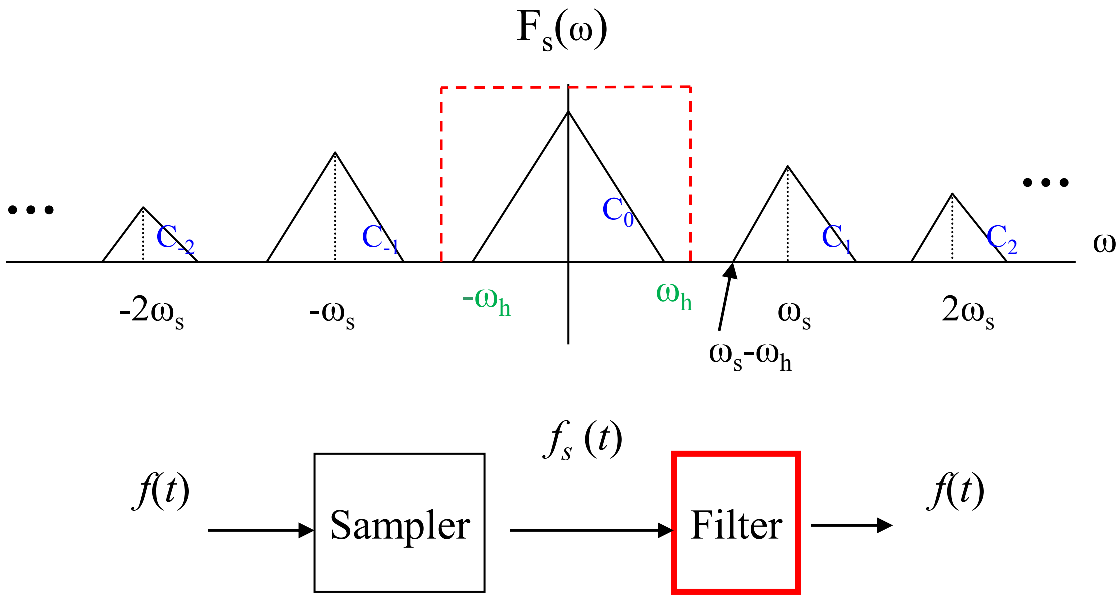
- After finishing this lecture you should be able to:
  - Specify the different characteristics of the reconstruction low-pass filter.
  - Perform reconstruction in the time-domain utilizing the *sinc* interpolation function.
  - Perform reconstruction using *sample and data-hold* in time and frequency.
  - Understand the effect of practical reconstruction (e.g. using RC circuits)
  - Describe the aliasing effects and the condition required to avoid aliasing.

Introduction to Signal Reconstruction

- The continuous signal can be reconstructed from its samples.
- Recall that analog to digital conversion is important as it allows us to process the signal in the digital domain. After the processing is over one has to reconstruct the continuous signal.
- Reconstruction is important as the final signal (audio/video/…) is usually required in its continuous form.



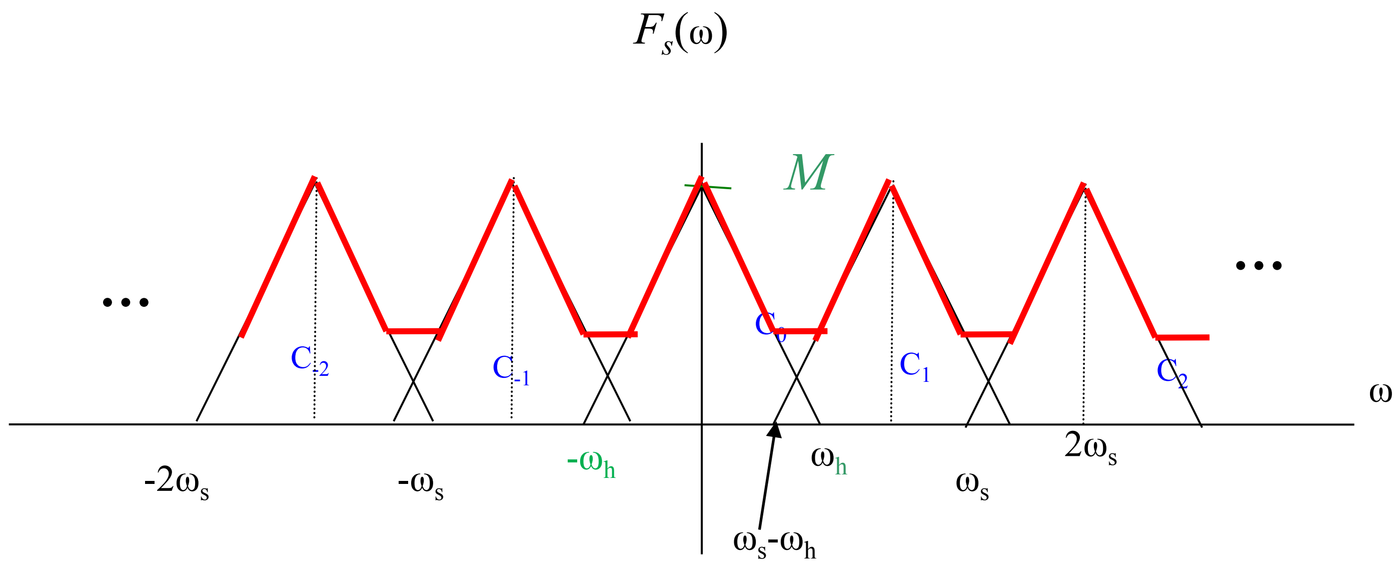
Reconstruction: Digital to Analog Conversion



- Recall that the spectrum of the sampled continuous times-signal $x(t)$ is composed of the spectrum of $x(t)$ plus the spectrum of $x(t)$ translated to each harmonic of the sampling frequency.
- The original signal can be perfectly reconstructed using a low-pass filter with cut-off frequency equals to $f_s/2$ provided that the original signal was sampled at a frequency above $2f_h$.

Aliasing

- If the original signal was sampled at a rate less than twice the highest frequency then the translated spectrums will overlap and the original signal will **not** be reconstructed properly.
- This effect is known as aliasing and it is illustrated in the figure below



Ideal Reconstruction Filter

- An ideal low-pass filter can be used to reconstruct the data. It has the following transfer function

$$H(\omega) = \begin{cases} T_s & |\omega| < 0.5\omega_s \\ 0 & otherwise \end{cases}$$

- $= T_s rect\left(\frac{\omega}{\omega_s}\right)$
- Using Inverse Fourier Transform

$$h(t) = \frac{T_s}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} e^{j\omega t} d\omega = \frac{2\pi}{\omega_s j 2\pi t}\left(e^{j\frac{\omega_s}{2}t} - e^{-j\frac{\omega_s}{2}t} \right)$$

$$h(t) = \frac{\sin\left(\frac{\omega_s}{2}t\right)}{\frac{\omega_s}{2}t} = \text{sinc}(\omega_s t)$$

- Note: that the impulse response is not time limited and non-causal.
- Using the convolutional integral, we may write the constructed signal as

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}(\omega_s(t - nT_s)/2)$$

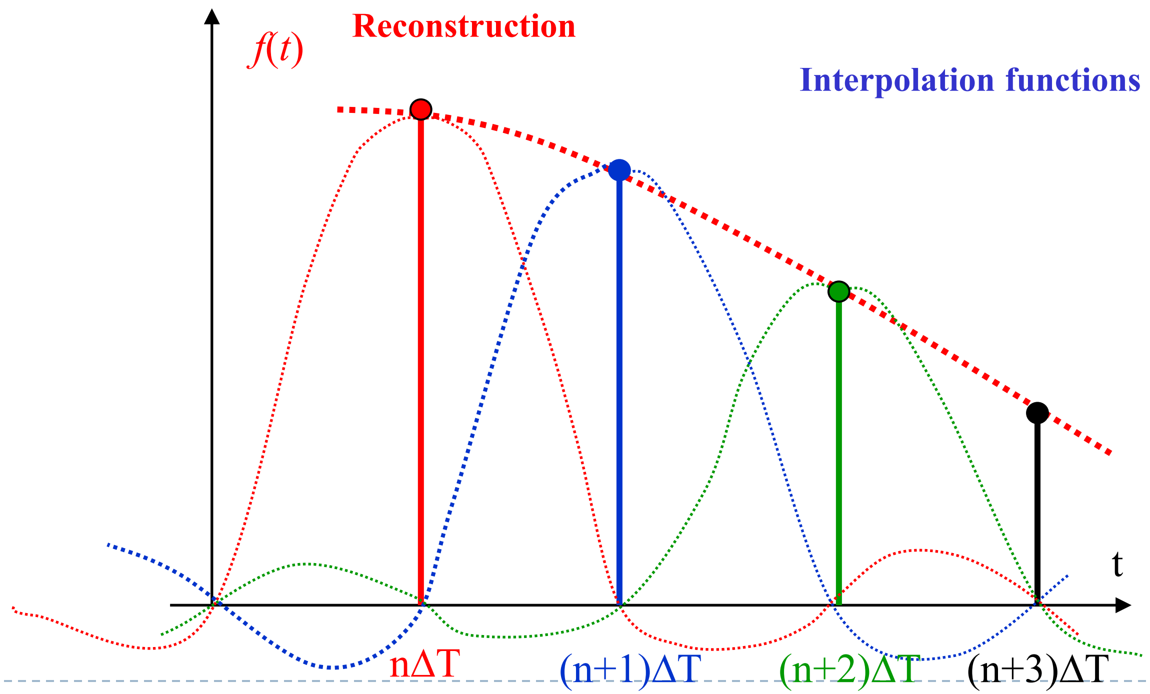
- Since we cannot take infinite number of samples, approximately

$$x(t) = \sum_{n=k-l+1}^{k+l} x(nT_s)\,\text{sinc}(\omega_s(t - nT_s)/2)$$

Interpolation: viewing reconstruction in the time-domain

- This equation suggests that original signal can be reconstructed by weighting each sample by a *sinc* function centered at the sample time and summing. This operation is illustrated in the figure below

$$x(t) = \sum_{n=k-l+1}^{k+l} x(nT_s)\,\mathrm{sinc}\left(\omega_s(t - nT_s)/2\right)$$



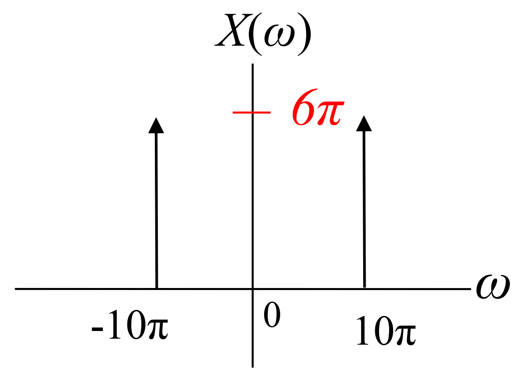
Example

- Given the signal $x(t) = 6\cos(10\pi t) = 6\cos(2\pi(5)t)$
- To illustrate the idea of proper reconstruction and aliasing, two different sampling frequencies are considered $(f_{s1} = 7\ Hz, and\ f_{s2} = 14\ Hz)$.
- The highest frequency of the signal under consideration – and the only frequency- is 5 Hz.
- The objective is to see the effect of sampling a signal at both a frequency less and greater than twice the highest frequency.

Example Solution

- By Fourier transform

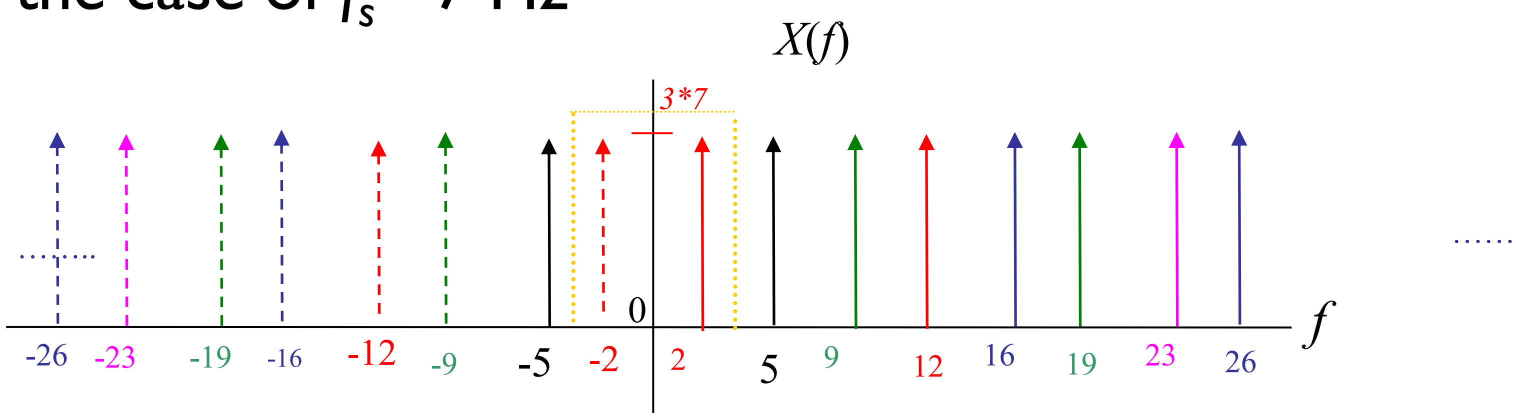
$$X(\omega) = 6\pi\left(\delta(\omega - 10\pi) + \delta(\omega + 10\pi)\right)$$



- From which, the spectrum of the sampled signal can be easily found

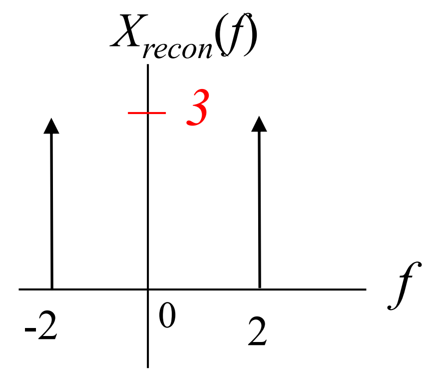
$$X_s(\omega) = \frac{6\pi}{T_s}\sum_{n=-\infty}^{\infty}\left[\delta(\omega - 10\pi - n\omega_s) + \delta(\omega + 10\pi - n\omega_s)\right]$$

- For the case of f_s= 7 Hz



Solution : sampling below Nyquist rate

A low-pass filter with cut-off frequency $=f_s/2=7/2=3.5$ Hz is used. The amplitude of the filter in the low-pass region should be $1/f_s=1/7$.
The reconstructed spectrum is shown



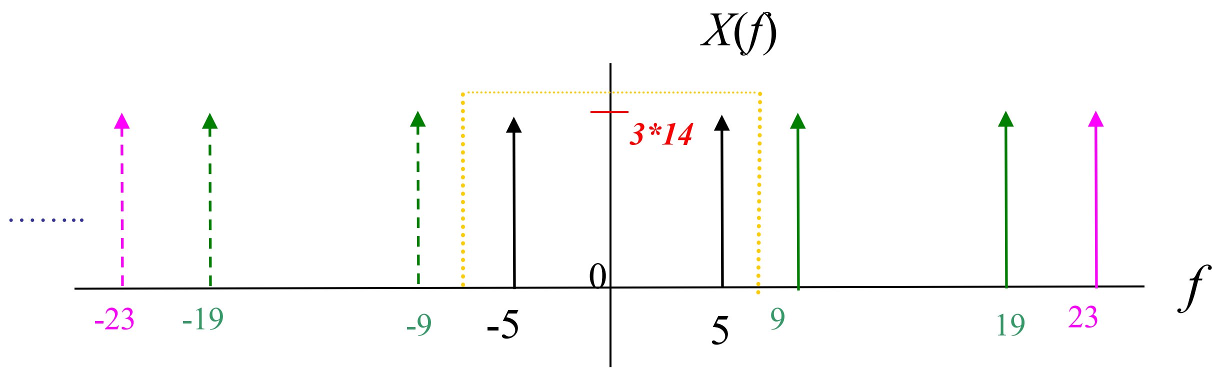
This is equivalent in the time domain to

$$x(t) = 6\cos(4\pi t) = 6\cos(2\pi(2)t)$$

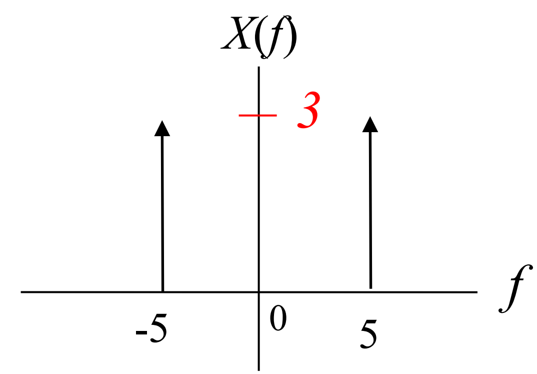
- Because the original signal was sampled below Nyquist rate it could not be reconstructed properly.
- Note that the reconstructed signal is similar to the original one with lower frequency as a result of aliasing.

Example Solution : Above Nyquist rate

Now, let the sampling frequency be $14Hz$ which above Nyquist rate. The spectrum of the sampled signal becomes

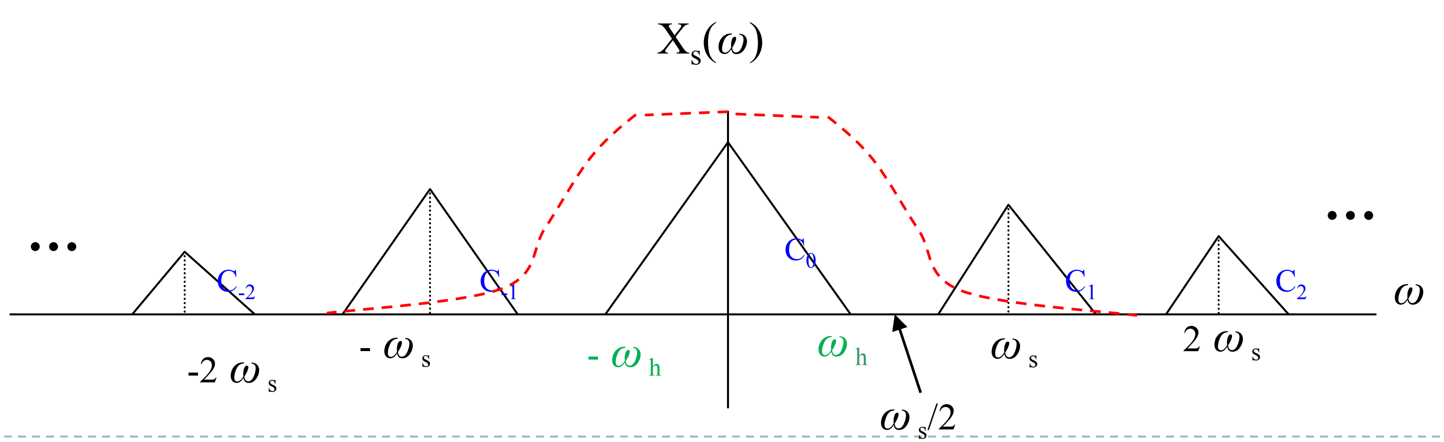


Now, a low-pass filter with cut-off frequency $= f_s/2 = 7/2 = 7\ Hz$. The amplitude of the filter in the low-pass region should be $1/f_s = 1/14$. The reconstructed spectrum is exactly like the original signal.

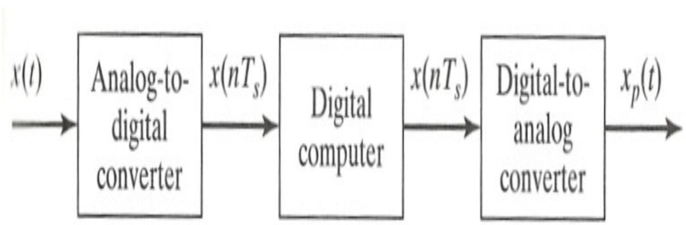


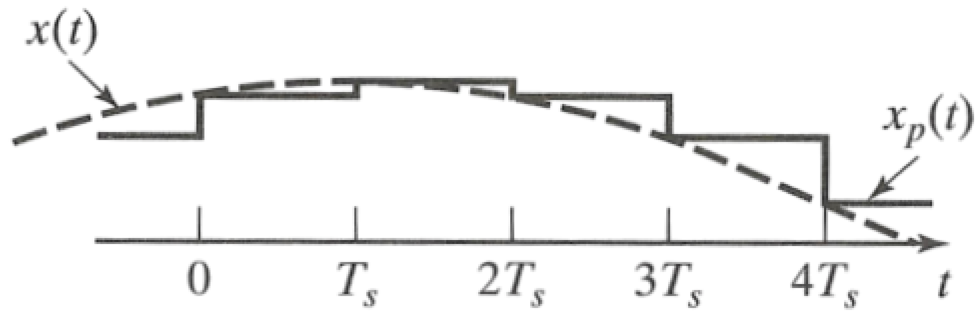
Practical Reconstruction

- There are other different methods to reconstruct the signals which are not exact:
  - In the time-domain one may use linear interpolation between the points. Other averaging techniques are also possible.
  - In frequency-domain, RC circuit might be used to approximate low-pass filter.
    - As shown in the figure below the reconstructed spectrum may suffer from variation in the amplitude in the pass-band region in addition to non-zero amplitude in the stop-band region.



Digital to Analog Conversion (Sample and Hold)





- At the output of the sample and hold D2A conversion circuit we can write

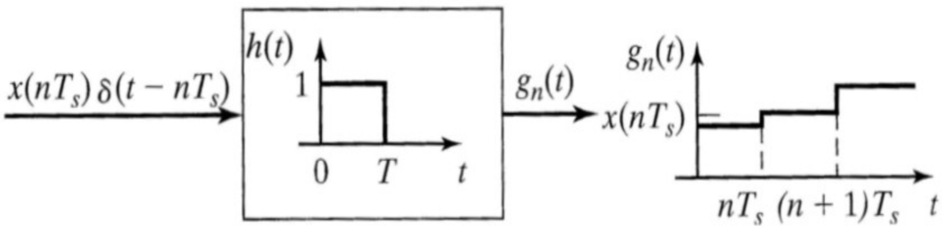
$$x_p(t) = \cdots + x(0)[u(t) - u(t - T_s)] + x(T_s)[u(t - T_s) - u(t - 2T_s)] + x(2T_s)[u(t - 2T_s) - u(t - 3T_s)]$$

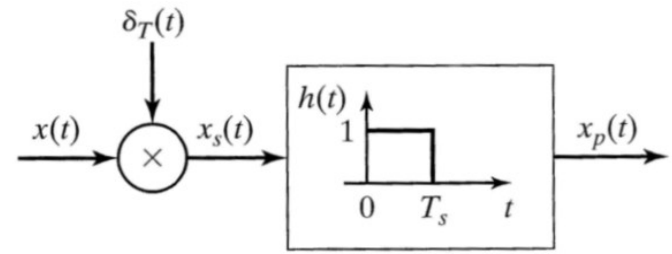
- Which can be written in summation form as

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)[u(t - nT_s) - u(t - (n+1)T_s)]$$

- This can be realized by a system with the following impulse response
- $h(t) = u(t) - u(t - T_s)$

Zero Order Data-Hold





- As in the figure, recall that the input signal is the sampled data signal
- $x_s(t) = \sum_{n=-\infty}^{+\infty} x(nT_s)\delta(t - nT_s)$
- This reconstruction is known as *zero-order hold*

Frequency Domain : Zero Hold

- In frequency domain

$$\delta(t-t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega t_0}$$

- The F.T. of the sampled signal is

$$X_s(\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s)e^{-jnT_s\omega}$$

- The transfer function for the sample and hold system is the F.T. of the impulse response given above

$$H(\omega) = \frac{1-e^{-jT_s\omega}}{j\omega}$$

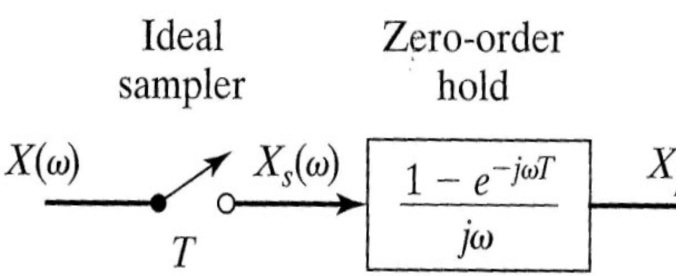
- Using Euler's relation and algebraic manipulation, we can rewrite the transfer function as

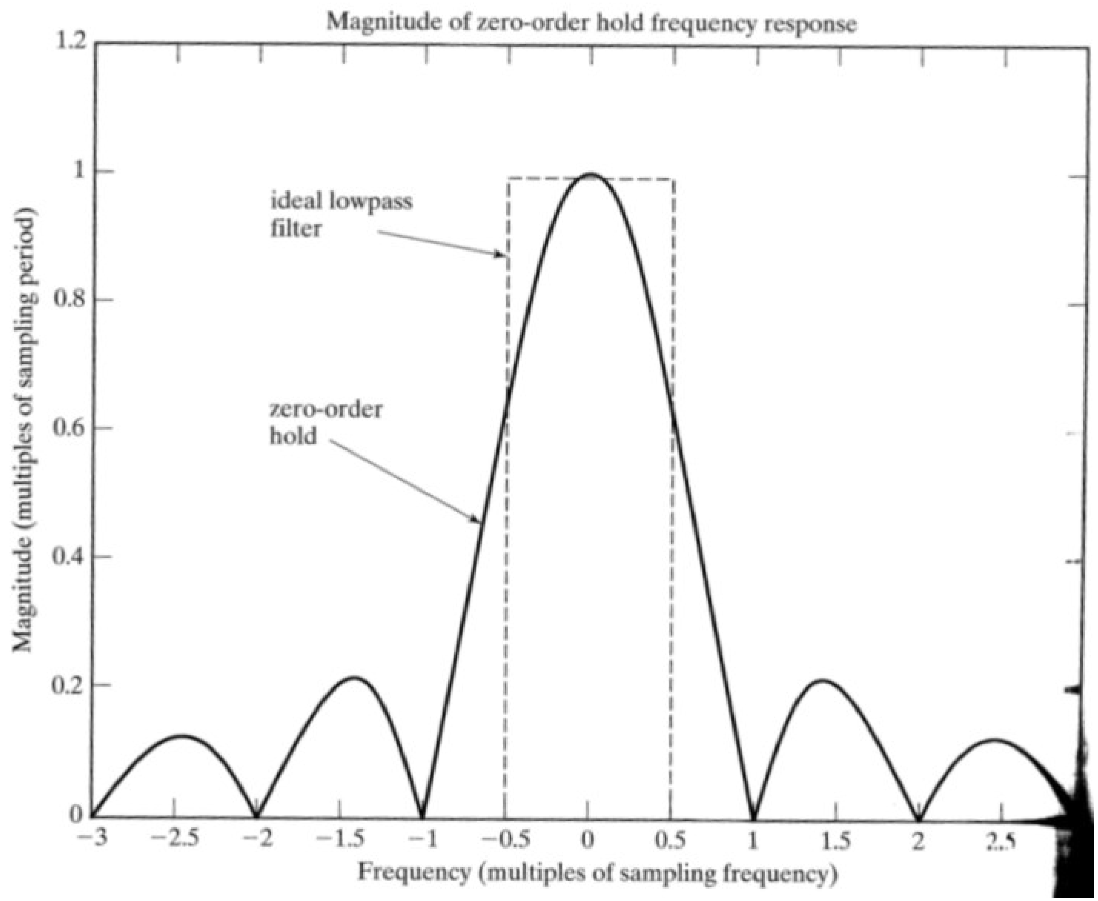
$$H(\omega) = T_s sinc\,(\frac{\pi\omega}{\omega_s})e^{-\frac{j\pi\omega}{\omega_s}}$$

- The output of the sample and data-hold is given by

$$X_p(\omega) = X_s(\omega)H(\omega) = \sum_{n=-\infty}^{+\infty} x(nT_s)e^{-jnT_s\omega} \cdot T_s sinc\,\left(\frac{\pi\omega}{\omega_s}\right)e^{-\frac{j\pi\omega}{\omega_s}}$$

Frequency Response: Zero Order Hold





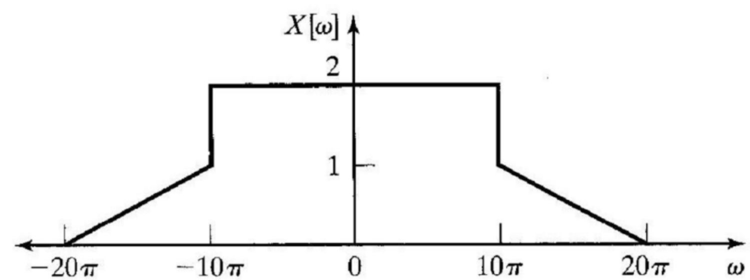
Self Test:

- Consider the following signal, $x(t) = 4\cos(8\pi t) + 6\cos(6\pi t)$
- What is the minimum required sampling frequency to avoid aliasing?
- If the signal is sampled at a rate of 10 samples/second, what are the possible bandwidths of the low-pass filter required to reconstruct $x(t)$ from $x_s(t)$?

- **Answer:**
- Greater than twice the highest frequency=2*4=8 Hz.
- If we sketch the spectrum of the sampled signal. It is easy to see that the bandwidth should be between 4 & 6 Hz.
- You re encouraged to sketch the spectrum of $x(t)$ and the spectrum of $x_s(t)$

Practice

- Find the minimum sampling frequency. Assume ideal components
  - $y(t) = \cos(300\pi t)$
  - $z(t) = \sin(100\pi t) - \cos(300\pi t)$
  - $x(t) = sinc(100\pi t)$
  - $w(t) = sinc^2(100\pi t)$



- The signal with the amplitude frequency shown in the figure is to be sampled with an ideal sampler.
  - Sketch the spectrum of the resulting signal for $|\omega| \leq 120\pi$ rad/s when sampling periods of $40, 50, and\ 100ms$ are used.
  - Which of the sampling frequencies is acceptable for use if the signal is to be reconstructed with an ideal low-pass filter?