Continuous-Time Linear Time-Invariant (LTI) Systems

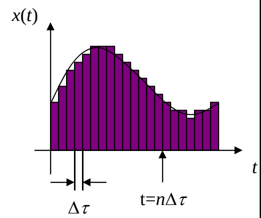
Dr. Ali Hussein Muqaibel

Introduction

- Why LTI?
  - Many physical systems.
  - Easy to solve mathematically
  - Available information about analysis and design.
  - We can apply superposition

Impulse representation of Continuous Time Systems

- A signal can be represented as a sum of deltas (impulses).



$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\Delta\tau)\ \mathrm{rect}\left(\frac{t-n\Delta\tau}{\Delta\tau}\right)$$

$$x(t) = \lim_{\Delta\tau\to 0} \sum_{n=-\infty}^{\infty} x(n\Delta\tau)\ \mathrm{rect}\left(\frac{t-n\Delta\tau}{\Delta\tau}\right) = \int_{-\infty}^{\infty} x(\tau)\ \delta(t-\tau)\ d\tau$$

- To verify $x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$
- $x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(-(\tau-t))d\tau$
- $x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(\tau-t)d\tau$

$\delta(t) = \delta(-t)$

The well know sifting (sampling) property

The Impulse Response

- **Impulse response** of a system is response of the system to an input that is a unit impulse (i.e., a Dirac delta function in continuous time)
- Therefore, we know how to calculate the system output for any input, *x*(*t*)

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\ \delta(t-\tau)\ d\tau \Rightarrow \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau = y(t)$$

- This operation is called **convolution**

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

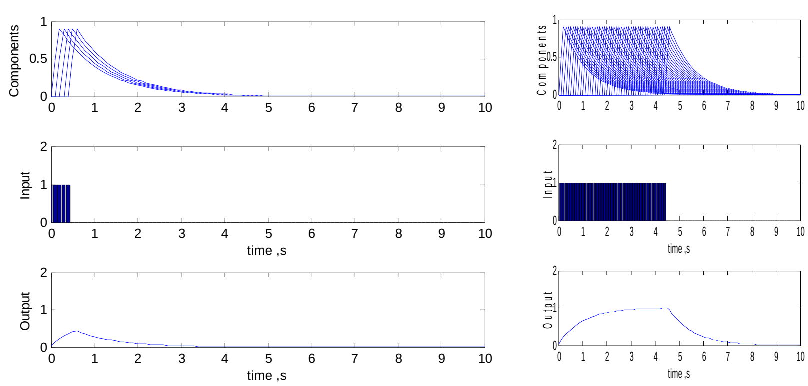
Convolution Concept

- $x(t) = u(t)$
- $h(t) = e^{-t}u(t)$
- What is y(t)?
- $x(t) \approx \sum_{k=0}^{\infty} \delta(t - \Delta k)$
- $y(t) \approx \sum_{k=0}^{\infty} e^{-(t-\Delta k)} u(t - \Delta k)$
- If we take the limit as $\Delta \rightarrow \mathbf{0}$ we get $y(t)$

```
% Dr.Ali Hussein Mqaibel
clear all
close all
clc
t_step=0.1;
t=0:t_step:10;
h=exp(-t).*Heaviside(t);
subplot(3,1,1)
plot(t,h)
xlabel ('time ,s')
ylabel ('The impulse response')
axis([0 10 0 2])
y=h;
x=[1 zeros(1,length(h)-1)].*Heaviside(t);

for i=1:length(t);
   subplot(3,1,1)

   plot(t,[zeros(1,i) h(1:end-i)])
   y=y+[zeros(1,i) h(1:end-i)];
   x(1,i)=1;
   hold on
   ylabel ('Components')
   subplot (3,1,2)
bar(t,x)
xlabel ('time ,s')
ylabel ('Input')
axis([0 10 0 2])

subplot(3,1,3)
   plot(t,y*t_step)
xlabel ('time ,s')
ylabel ('Output')
axis([0 10 0 2])
   pause
end
```

Dr. Ali Hussein Muqaibel

Convolution for Continuous Time LTI Systems



Dr. Ali Hussein Muqaibel

Some Convolution Properties

- $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$
- $y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = h(t) * x(t)$
- What is the output if the input is impulse ?
  - $y(t) = \delta(t) * h(t) = h(t)$
- For LTI system
  - $y(t-t_0) = \delta(t-t_0) * h(t) = h(t-t_0)$
- Recall some delta properties (no convolution)
  - $\delta(t-t_0)g(t) = g(t_0)\delta(t-t_0)$
  - $g(t-t_0)\delta(t) = g(-t_0)\delta(t)$
- $h(t)$ contains a complete input-output description .i.e. if the impulse response is known , the system response to any input can be found

Dr. Ali Hussein Muqaibel

Example: Impulse Response of an Integrator

- $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$
- What is $h(t)$?
  - $h(t) = y(t)|_{x(t)=\delta(t)} = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$
- What is the output if the input is ramp? "ramp response"
  - $y(t) = x(t) * h(t) = tu(t) * u(t) = \int_{-\infty}^{+\infty} \tau u(\tau)u(t-\tau)d\tau$
  - $y(t) = \begin{cases} \int_0^{+\infty} \tau u(t-\tau)d\tau & t>0 \\ 0 & t<0 \end{cases} = \begin{cases} \int_0^{t} \tau d\tau & t>0 \\ 0 & t<0 \end{cases} = \begin{cases} \frac{t^2}{2} & t>0 \\ 0 & t<0 \end{cases} = \frac{t^2}{2}u(t)$
  - Verify from the system equation
  - $y(t) = \int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} \tau u(\tau)d\tau$
  - $y(t) = \begin{cases} \int_{-\infty}^{t} \tau d\tau & t>0 \\ 0 & t<0 \end{cases} = \frac{t^2}{2}u(t)$
- Common mistake to forget $u(t)$
- Before you start graphical convolution visit: (The Joy of Convolution)
- http://www.jhu.edu/~signals/convolve/index.html

Dr. Ali Hussein Muqaibel

Graphical Convolution Methods

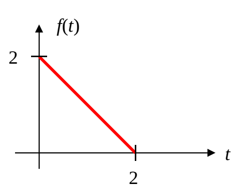
- From the convolution integral, convolution is equivalent to

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

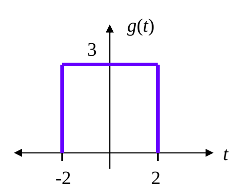
  - Rotating one of the functions about the y axis
  - Shifting it by *t*
  - Multiplying this flipped, shifted function with the other function
  - Calculating the area under this product
  - Assigning this value to $f_1(t) * f_2(t)$ at *t*

Graphical Convolution Example

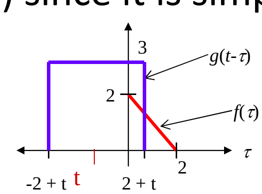
- Convolve the following two functions:



*



- Replace *t* with τ in *f*(*t*) and *g*(*t*)
- Choose to flip and slide $g(\tau)$ since it is simpler and symmetric
- Functions overlap like this:

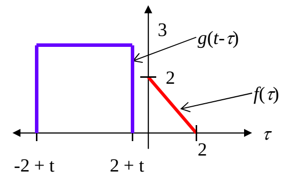


Graphical Convolution Example

- Convolution can be divided into 5 parts

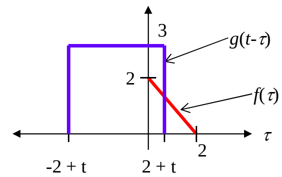
I. $t < -2$

- Two functions do not overlap
- Area under the product of the functions is zero



II. $-2 \le t < 0$

- Part of $g(t)$ overlaps part of $f(t)$
- Area under the product of the functions is

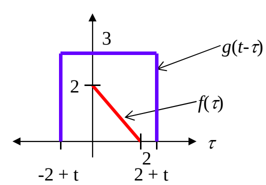


$$\int_0^{2+t} 3(-\tau+2)\,d\tau = 3\left(-\frac{\tau^2}{2}+2\tau\right)\Bigg|_0^{2+t} = -\frac{3(2+t)^2}{2}+6(2+t) = -\frac{3t^2}{2}+6$$

Graphical Convolution Example

III. $0 \le t < 2$

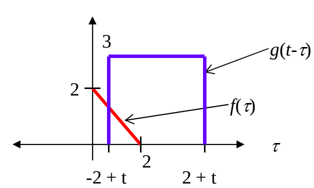
- Here, $g(t)$ completely overlaps $f(t)$
- Area under the product is just



$$\int_0^{2} 3(-\tau+2)\,d\tau = 3\left(-\frac{\tau^2}{2}+2\tau\right)\Bigg|_0^{2} = 6$$

IV. $2 \le t < 4$

- Part of $g(t)$ and $f(t)$ overlap
- Calculated similarly to $-2 \le t < 0$



V. $t \ge 4$

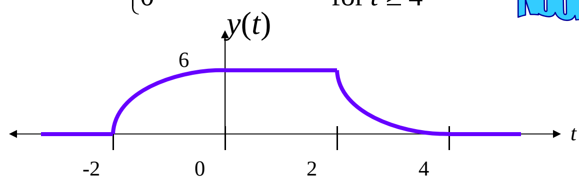
- $g(t)$ and $f(t)$ do not overlap
- Area under their product is zero

Graphical Convolution Example

- Result of convolution (5 intervals of interest):

$$y(t) = f(t) * g(t) = \begin{cases} 0 & \text{for } t < -2 \\ -\frac{3}{2}t^2 + 6 & \text{for } -2 \le t < 0 \\ 6 & \text{for } 0 \le t < 2 \\ \frac{3}{2}t^2 - 12t + 24 & \text{for } 2 \le t < 4 \\ 0 & \text{for } t \ge 4 \end{cases}$$

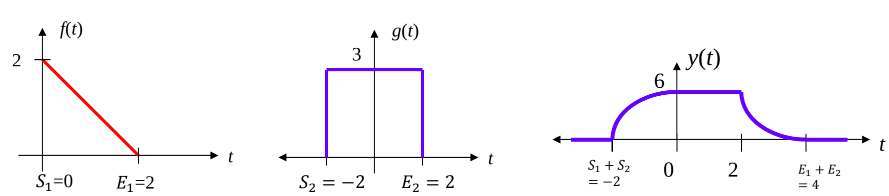
No Overlap
Partial Overlap
Complete Overlap
Partial Overlap
No Overlap



Dr. Ali Hussein Muqaibel

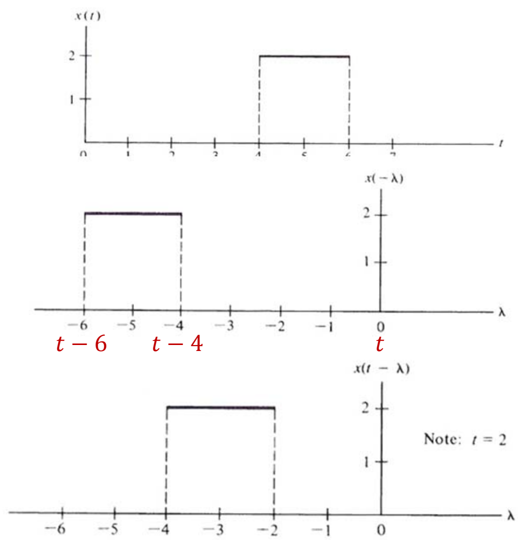
How to Check the Convolution

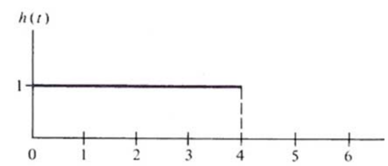
- Start and end points: Start point equal to the sum of the starting points of the two signals and the end point equals to the sum of the end points.
- Area under the curve=product of the individual areas.



Dr. Ali Hussein Muqaibel

Example 2: Graphical Convolution



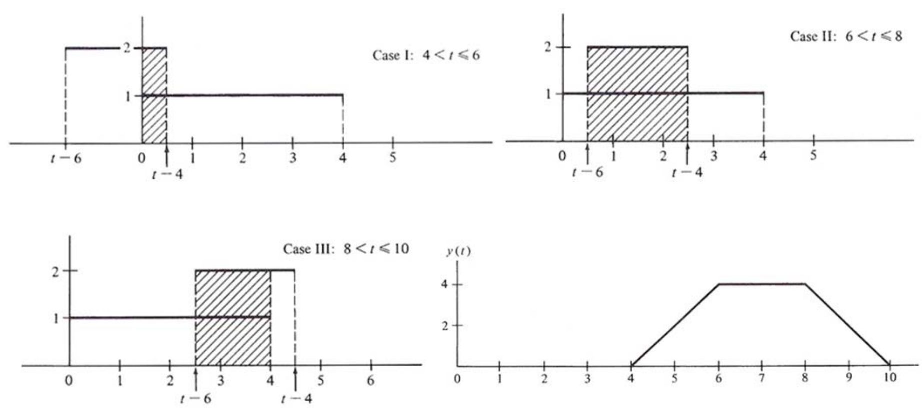


$$x(t) = 2rect\left(\frac{t-5}{2}\right)$$

$$h(t) = rect\left(\frac{t-2}{4}\right)$$

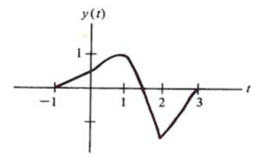
Find the convolution between $x(t)$ and $h(t)$.

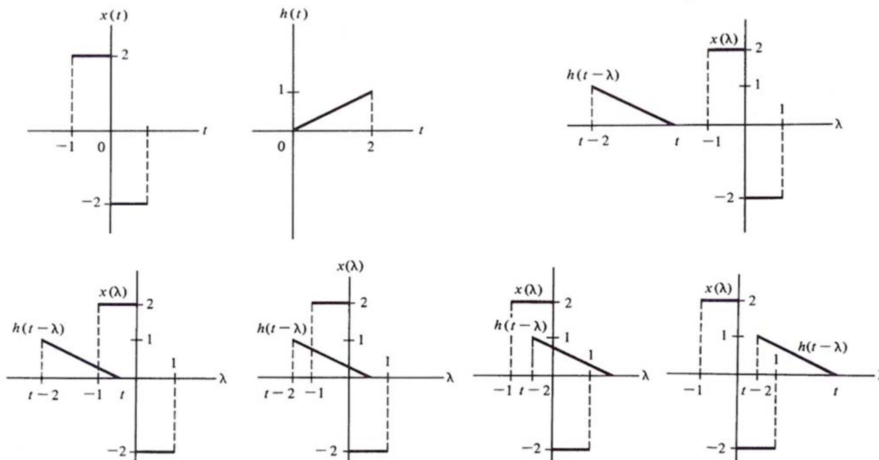
Continue Example 2:



If we are interested in the output only at a specific time, we do not have to do full convolution

Example 3:

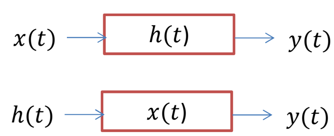


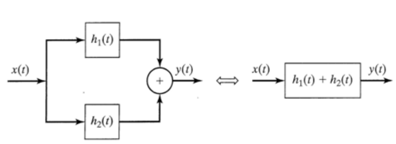


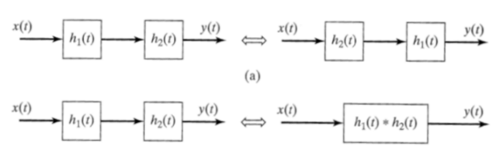
See additional Examples in the book .. You must write by your hand!

Properties of Convolution

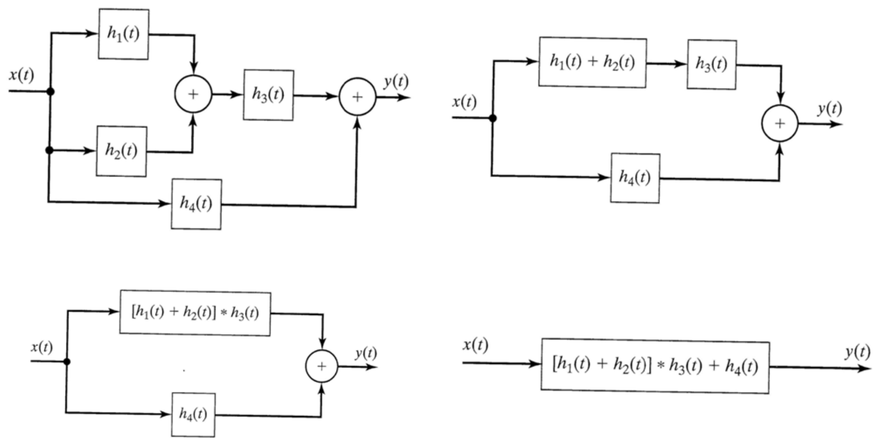
- Commutative property
  - $x(t) * h(t) = h(t) * x(t)$
- Associative Property
  - $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$
- Distributive property
  - $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$







Example: Impulse Response of Interconnected System



Dr. Ali Hussein Muqaibel

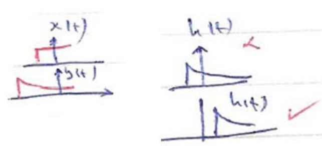
Properties of Continuous-time LTI Systems

- The input-output characteristics of a continuous-time LTI system are completely described by its impulse response $h(t)$
- $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$
- Memory:
  - In general LTI are dynamic because of integration over time.
  - The only way to get the integration out is if
  - $h(t) = k\delta(t) \Rightarrow y(t) = kx(t)$ ideal amplifier
  - An LTI system is memoryless if and only if $h(t) = k\delta(t)$
- Invertability
  - $x(t) * h(t) * h_i(t) = x(t)$
  - $x(t) * \delta(t) = x(t)$
  - $h(t) * h_i(t) = \delta(t)$
  - An LTI system is invertible only if we can find $h_i(t)$ such that $h(t) * h_i(t) = \delta(t)$
  - No procedure is given to find $h_i(t)$

Dr. Ali Hussein Muqaibel

Continue.. Properties of Continuous-time LTI Systems

- Causality
  - A signal that is zero for $t < 0$ is called a causal signal
  - Because $\delta(t)$occurs at $t = 0 \Rightarrow h(t) = 0 \, , t < 0.$
  - $y(t) = \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau = \int_{0}^{+\infty} x(t-\tau)h(\tau)d\tau$
  - Alternatively
  - $h(t-\tau)$=0 for $t - \tau \Rightarrow \tau > t$
  - $y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(t-\tau)h(\tau)d\tau$



Only for causal LTI

Continue..2.. Properties of Continuous-time LTI Systems

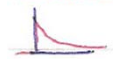
- Stability
- Since $| \int_{-\infty}^{+\infty} x_1(t)x_2(t)dt| \leq \int_{-\infty}^{+\infty} |x_1(t)x_2(t)|dt$
- For a bounded input $|x(t)| < M$ for all t , the output is
- $|y(t)| = | \int_{-\infty}^{+\infty} x(t-\tau)h(\tau)d\tau| \leq \int_{-\infty}^{+\infty} |x(t-\tau)h(\tau)|d\tau$
- $= \int_{-\infty}^{+\infty} |x(t-\tau)||h(\tau)|d\tau$
- $\leq \int_{-\infty}^{+\infty} M|h(\tau)|d\tau$
- $= M \int_{-\infty}^{+\infty} |h(\tau)|d\tau$
- i.e $h(t)$ is absolutely integrable
- $\int_{-\infty}^{+\infty} |h(t)|dt < \infty$

BIBO Stability test

$$\int_{-\infty}^{+\infty} |h(t)|dt < \infty$$

Stability Examples

- Example: Is the following system BIBO Stable?
- $h(t) = e^{-3t}u(t)$



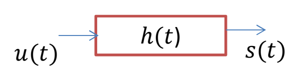
  - $\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{-3t} dt = \frac{1}{3} < \infty$
- $h(t) = e^{+3t}u(t)$.... Unstable
- $h(t) = u(t)$

  - Integrator $y(\mathrm{t}) = \int_{-\infty}^{t} x(\tau) d\tau$
  - $\int_{0}^{+\infty} |h(t)| dt = \int_{0}^{+\infty} dt = \rightarrow \infty$
  - Unstable
  - Can be done by inspection of the graph of $h(t)$

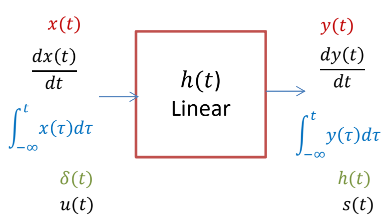
BIBO Stability test
$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty$$

Dr. Ali Hussein Muqaibel

Unit Step Response



- If the input is $u(t)$, the output is noted as $s(t)$ (some books they use $a(t)$)
- Relation between $s(t)$ and $h(t)$
  - $s(t) = \int_{-\infty}^{+\infty} u(\tau) h(t-\tau) d\tau = \int_{0}^{\infty} h(t-\tau) d\tau$
  - $s(t) = \int_{-\infty}^{+\infty} h(\tau) u(t-\tau) d\tau = \int_{-\infty}^{t} h(t-\tau) d\tau = \int_{0}^{t} h(\tau) d\tau$
  - $h(t) = \frac{ds(t)}{dt}$
- Note do not confuse $\delta(t)$ with $s(t)$ in writing !
- The unit step response completely describes the input-output characteristics of an LTI system.



Dr. Ali Hussein Muqaibel

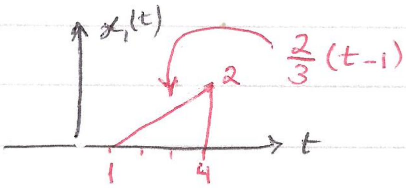
Example: Step Response from the Impulse Response

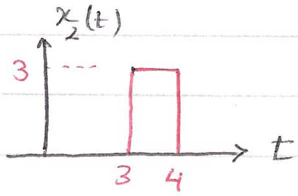
- $h(t) = e^{-3t}u(t)$, find the step response.
- $s(t) = \int_0^t h(\tau)d\tau$
- $= \int_0^t e^{-3\tau}d\tau$
- $= \frac{e^{-3t}}{-3}\Big|_0^t = \frac{1}{3}(1 - e^{-3t})u(t)$
- Verify:
  - $h(t) = \frac{ds(t)}{dt} = e^{-3t}u(t) + \frac{1}{3}(1 - e^{-3t})\delta(t) = e^{-3t}u(t)$

Sifting property 0

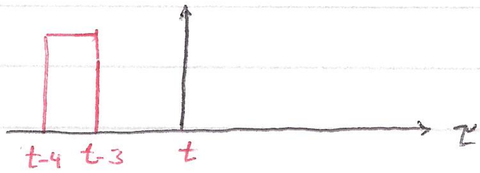
Practice more convolution examples. Consider both Graphical and analytical (See old exams and quizzes)

Graphical Convolution Example:

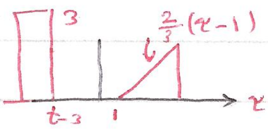




$y(t) = x_1(t) * x_2(t)$



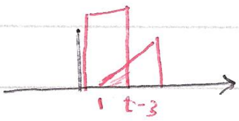
Case I no overlap



$t-3 \leq 1 \Rightarrow t \leq 4$

$y(t) = 0$

Case II partial overlap

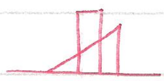


$$\int_1^{t-3} \frac{2}{3}(\tau-1)(3)\,d\tau = 2\left[\frac{\tau^2}{2}-\tau\right]_1^{t-3} = 2\left[\frac{(t-3)^2}{2}-(t-3)-\left(\frac{1}{2}-1\right)\right]$$

$$= 2\left[\frac{t^2-6t+9}{2}-(t-3)+\frac{1}{2}\right] = t^2-8t+16 \qquad 5 \geq t \geq 4$$

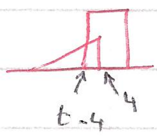
$t-4 \leq 1 \Rightarrow t \leq 5$

Case III Full overlap



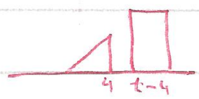
$$\int_{t-4}^{t-3} \frac{2}{3}(\tau-1)(3)\,d\tau = 2\left[\frac{\tau^2}{2}-\tau\right]_{t-4}^{t-3} = 2t-9$$

Case IV $t-3 \geq 4 \Rightarrow t = 7$ partial

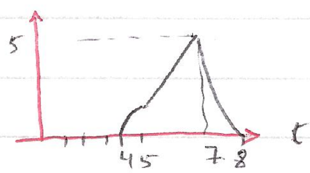


$$\int_{t-4}^{4} \ldots = -t^2+10t-16$$

Case V $t-4 > 4$ $t \geq 8$ no overlap $y(t) = 0$



$$y(t) = \begin{cases} 0 & t \leq 4 \\ t^2-8t+16 & 5 \geq t > 4 \\ 2t-9 & 7 \geq t > 5 \\ -t^2+10t-16 & 8 \geq t > 7 \\ 0 & t \geq 8 \end{cases}$$



check start = 1+3 = 4 ✓

end = 4+4 = 8 ✓

Area, 3 × 4 = 12 ok.