Random Processes Spectral Characteristics

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Introduction

- You should be familiar with F.T. by integration or by using tables.
  - $x(t) \leftrightarrow X(\omega)$
  - $X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$
  - $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{j\omega t}d\omega$
  - For deterministic signals we can sketch the magnitude and phase of $X(\omega)$
- We do not need full pdf to find power. R or S are enough.

Power Density Spectrum & Properties

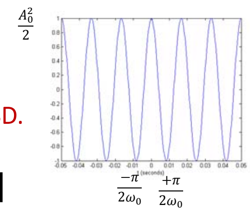
- The power density spectrum (power spectral density) PSD.
- Parsaval's Theorem:
- $P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)]dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$
- $S_{XX}(\omega) = \lim_{T\to\infty} \frac{E[|X_T(\omega)|^2]}{2T}$
- $P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega$
- The $\frac{1}{2\pi}$ factor is there because we are using ω.
- Power is related to the time average of the second moment $A\{E[X^2(t)]\}$, for w.s.s. $= \overline{X^2}$

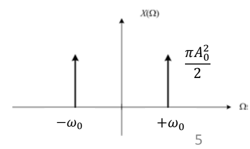
Example

- Find the power in the following random process $X(t) = A_0 \cos(\omega_0 t + \theta)$, $A_0 \& \omega_0$ *are constant* θ *is* $U\left(0, \frac{\pi}{2}\right)$
- $E[X^2(t)] = E[A_0^2 \cos^2(\omega_0 t + \theta)] = E\left[\frac{A_0^2}{2} + \frac{A_0^2}{2}\cos(2\omega_0 t + 2\theta)\right] = \frac{A_0^2}{2} + \frac{A_0^2}{2}\int_0^{\frac{\pi}{2}} \frac{2}{\pi}\cos(2\omega_0 t + 2\theta)d\theta = \frac{A_0^2}{2} - \frac{A_0^2}{\pi}\sin(2\omega_0 t)$. Not W.S.S.
- $P_{XX} = A\{E[X^2(t)]\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E[X^2(t)]dt = \frac{A_0^2}{2}$
- Alternatively,
- $X_T(\omega) \to |X_T(\omega)|^2 = X_T(\omega_0) X_T^*(\omega_0)$
- $S_{XX} = \frac{E[|X_T(\omega)|^2]}{2T}$, $P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = \frac{A_0^2}{2}$
- Try it yourself!

Relationship between Power Spectrum and Autocorrelation Function

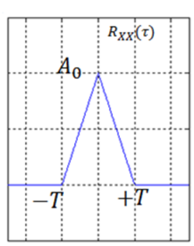
- $R_{XX}(\tau) \leftrightarrow S_{XX}(\omega)$
- $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = A[R_{XX}(t, t+\tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)]e^{-j\omega\tau}d\tau$
- **For W.S.S. $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$, we ge t the Wiener-Khintchine relations**
- $R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\, e^{-j\omega\tau}d\tau$
- Example I: $R_{XX}(\tau) = \left(\frac{A_0^2}{2}\right)\cos(\omega_0\tau)$ find the PSD.
- $S_{XX}(\omega) = \frac{A_0^2\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
- Explain what happens through RC filters

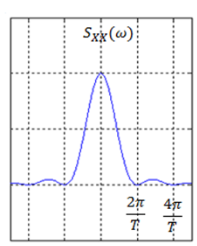




Example II:

- $R_{XX}(\tau) = \begin{cases} A_0\left[1 - \left(\frac{|\tau|}{T}\right)\right] & -T \le \tau \le T \\ 0 & elsewhere \end{cases}$
- $T > 0$ & A_0 $are\ constants, A_0 > 0,$. Find the PSD.
- $S_{XX}(\omega) =$ $\int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = A_0\int_{-T}^{0}\left[1 + \left(\frac{\tau}{T}\right)\right]e^{-j\omega\tau}d\tau + A_0\int_{0}^{+T}\left[1 - \left(\frac{\tau}{T}\right)\right]e^{-j\omega\tau}d\tau$
- $= A_0 T sinc^2\left(\frac{\omega T}{2}\right)$





Properties of the PSD

- *Note: some books use* $\wp$ for power spectral density. Others use S…etc

1. $S_{XX}(\omega) \geq 0$
2. $S_{XX}(-\omega) = S_{XX}(\omega)$, for real $X(t)$.
3. $S_{XX}(\omega)$ is real
4. $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\{E[X^2(t)]\}$
5. $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
   - recall taking the derivative in frequency domain is equivalent to multiplying by $j\omega$
6. PSD and time average of the auto-correlation form a Fourier transform pair
   - $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = A[R_{XX}(t, t+\tau)]$
   - $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)]e^{-j\omega\tau}d\tau$
   - **For W.S.S.** $\boldsymbol{A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)}$
   - $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega = R_{XX}(\tau)$
   - $S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)\, e^{-j\omega\tau}d\tau$

Experiment with Matlab

Practice 1

- Determine which of the following functions can and cannot be valid power density spectrums. For those that are not, explain why.

- $\frac{\cos(3\omega)}{1+\omega^2}$
- $\frac{1}{(1+\omega^2)^2}$
- $\frac{|\omega|}{1+2\omega+\omega^2}$
- $\frac{1}{\sqrt{1-3\omega^2}}$

Practice 2

A random process has the autocorrelation function

$$R_{XX}(\tau) = Bcos^2(\omega_0\tau)\exp(-W|\tau|)$$

Where B, ω_0 and W are positive constants

(a) Find and sketch the power spectrum of X(t)

(b) Compute the average power in the lowpass part of the power spectrum.

(c) Repeat for bandpass case.

In each case assume $\omega_0 \gg W$.

Bandwidth of Power Density Spectrum for baseband processes

- **Low pass or Baseband**: most of the spectral components are around $\omega = 0$, and the magnitude decreases at higher frequencies.
- The PSD is similar to the PDF for being real and positive but the area need not be 1.
- The rms bandwidth ,W_{rms}, is defined like the standard deviation of the normalized PSD to be:

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega)d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega}$$

Bandwidth of Power Density Spectrum for Bandpass Processes

- **Bandpass process**: the significant components cluster near some frequency $\bar{\omega}_0$ and $-\bar{\omega}_0$. (assume real processes)
- **Mean frequency**:

$$\bar{\omega}_0 = \frac{\int_0^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_0^{+\infty} S_{XX}(\omega) d\omega}$$

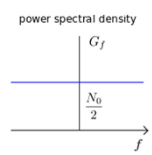
- rms bandwidth of bandpass (real) processes:

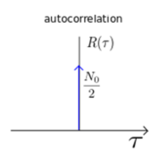
$$W_{rms}^2 = 4 \frac{\int_0^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_0^{+\infty} S_{XX}(\omega) d\omega}$$

- Why there is a factor of 4?

White Noise

$$S_{NN}(\omega) = \frac{N_0}{2}$$





$$R_{NN}(\tau) = \left(\frac{N_0}{2}\right) \delta(\tau)$$

- Unrealizable because of the infinity average power:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{NN}(\omega) d\omega = \infty$$

- Thermal noise generated be electrons in any electrical conductor has a PSD that is constant up to a very high frequency and then decreases.
- A resistor at temperature T Kelvin produces noise across the open circuited terminal having a power spectrum

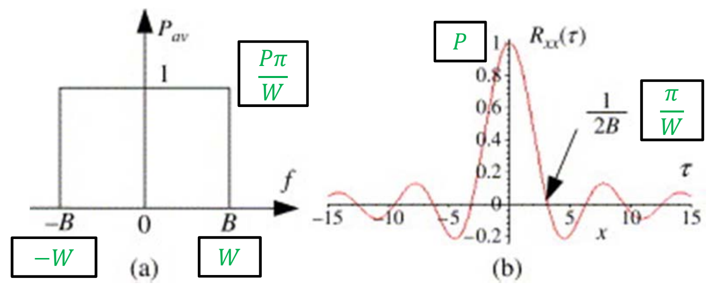
$$S_{NN}(\omega) = \frac{\frac{\mathbf{N_0}}{\mathbf{2}} \left(\frac{\alpha|\omega|}{T}\right)}{e^{\frac{\alpha|\omega|}{T}} - 1}$$

- where $\alpha = 7.64(10^{-12}) kelvin - seconds$, $T = 290\ K$.
- Remains above $0.9 \frac{N_0}{2}$ for very high frequencies around 1000GHz .

Bandlimited White Noise

$$S_{NN}(\omega) = \begin{cases} \frac{P\pi}{W} & -W < \omega < +W \\ 0 & elsewhere \end{cases}$$

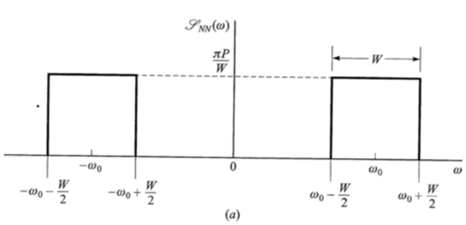
$$R_{NN}(\tau) = P\frac{\sin(W\tau)}{W\tau}$$

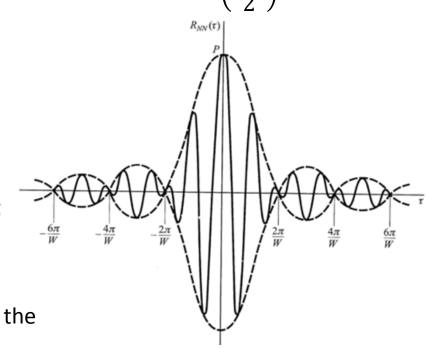


Bandpass Bandlimited White Noise

$$S_{NN}(\omega) = \begin{cases} \frac{P\pi}{W} & \omega_0 - \left(\frac{W}{2}\right) < |\omega| < \omega_0 + \left(\frac{W}{2}\right) \\ 0 & elsewhere \end{cases}$$

$$R_{NN}(\tau) = P\frac{\sin\left(\frac{W\tau}{2}\right)}{\left(\frac{W\tau}{2}\right)}\cos(\omega_0\tau)$$





where ω_0 and W are constants and P is the power in the noise.

Practice

A wide-sense stationary noise process $N(t)$ has an autocorrelation

$$R_{NN}(\tau) = Pe^{-3|\tau|}$$

Find and sketch the PSD

Answer $S_{NN}(\omega) = \frac{6P}{9+\omega^2}$

Using the FT def.