Linear Systems with Random Inputs

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Random Signal Response of Linear Systems

- Linear system fundamentals
- Consider: linear, stable, time-invariant system.
- $y(t) = \int_{-\infty}^{+\infty} h(\lambda)x(t-\lambda)d\lambda = \int_{-\infty}^{+\infty} x(\lambda)h(t-\lambda)d\lambda$
- $Y(t) = \int_{-\infty}^{+\infty} h(\lambda)X(t-\lambda)d\lambda$

Mean and Mean Squared Value of System Response

- Assume $X(t)$ is w.s.s.
- Assume integration and expectations are exchangeable
- **Mean:**
  - $E[Y(t)] = E\left[\int_{-\infty}^{+\infty} h(\lambda)X(t-\lambda)d\lambda\right] = \int_{-\infty}^{+\infty} h(\lambda)E[X(t-\lambda)]d\lambda = \bar{X} \int_{-\infty}^{+\infty} h(\lambda)d\lambda = \bar{Y}$ constant
- **Mean squared**
  - $E[Y^2(t)] = E\left[\int_{-\infty}^{+\infty} h(\lambda_1)X(t-\lambda_1)d\lambda_1 \int_{-\infty}^{+\infty} h(\lambda_2)X(t-\lambda_2)d\lambda_2\right]$
  - $= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[X(t-\lambda_1)X(t-\lambda_2)]h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$
  - $\overline{Y^2} = E[Y^2(t)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\lambda_1-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$
  - Not always easy to integrate!

Example:

Find $\overline{Y^2}$ if input is white noise

$$R_{XX}(\lambda_1-\lambda_2) = \left(\frac{N_0}{2}\right)\delta(\lambda_1-\lambda_2)$$

N_0 is a positive real constant.

$$\overline{Y^2} = E[Y^2(t)]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(\frac{N_0}{2}\right)\delta(\lambda_1-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$$

$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} h^2(\lambda_2)d\lambda_2$$

Output power is proportional to the area under the square of $h(t)$ in this case.

Autocorrelation Function of Response

Assume w.s.s.

$$R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau + \lambda_1 - \lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$$

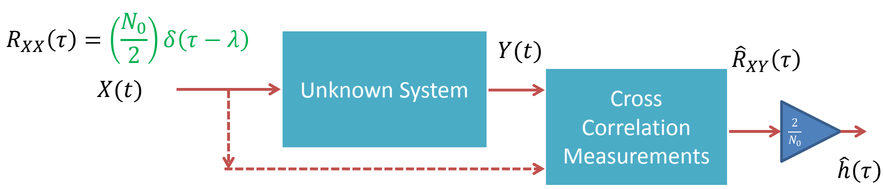
Two fold convolution

Cross Correlation Function of input and output

- $R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$
- $R_{YX}(\tau) = R_{XX}(\tau) * h(-\tau)$
- $R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{YX}(\tau) * h(\tau)$
- $X(t)$ & $Y(t)$ are jointly w.s.s. if $X(t)$ is w.s.s. because $Y(t)$ will be w.s.s.
- Example: For the same white noise example, find $R_{XY}(\tau)$ & $R_{YX}(\tau)$
- $R_{XY}(\tau) = \int_{-\infty}^{+\infty} \left(\frac{N_0}{2}\right) \delta(\tau - \lambda)h(\lambda)d\lambda = \frac{N_0}{2} h(\tau)$
- $R_{YX}(\tau) =$ $\int_{-\infty}^{+\infty} \left(\frac{N_0}{2}\right) \delta(\tau - \lambda)h(-\lambda)d\lambda = \frac{N_0}{2} h(-\tau) = R_{XY}(-\tau)$

System Evaluation Using Random Noise

- $R_{XY}(\tau) \approx \int_{-\infty}^{+\infty} \left(\frac{N_0}{2}\right) \delta(\tau - \lambda) h(\lambda) d\lambda = \left(\frac{N_0}{2}\right) h(\tau)$
- $h(\tau) \approx \left(\frac{2}{N_0}\right) R_{XY}(\tau)$
- $\hat{h}(\tau) = \left(\frac{2}{N_0}\right) \hat{R}_{XY}(\tau) \approx h(\tau)$



Spectral Characteristics of System Response

- To work with F.T. for systems, integration might be difficult.
- Assume $X(t)$ is w.s.s. => $Y(t)$ is w.s.s.=> $X(t)$& $Y(t)$ are jointly w.s.s.
- Power Density Spectrum Response:

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

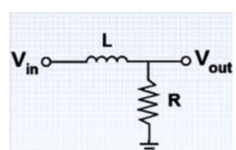
- $|H(\omega)|^2$ is the power transfer function
- Proof!

Proof of the PSD relation

- $S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$
- $S_{YY}(\omega) = \int_{-\infty}^{+\infty} R_{YY}(\tau)e^{-j\omega\tau}d\tau$
- Substitute in the above equation
- $R_{YY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau+\lambda_1-\lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$
- Then change variable $\lambda = \tau + \lambda_1 - \lambda_2,\, d\lambda = d\tau$
- $S_{YY}(\omega) =$
  $\int_{-\infty}^{+\infty} h(\lambda_1)e^{j\omega\lambda_1}d\lambda_1 \int_{-\infty}^{+\infty} h(\lambda_2)e^{-j\omega\lambda_2}d\lambda_2 \int_{-\infty}^{+\infty} R_{XX}(\lambda)e^{-j\omega\lambda}d\lambda$
- $S_{YY}(\omega) = H^*(\omega)H(\omega)S_{XX}(\omega) = S_{XX}(\omega)|H(\omega)|^2$
- The average power
- $P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)|H(\omega)|^2 d\omega$

Example

Find the power spectrum & average power of the response when the input $X(t)$ is white noise



$H(\omega) = \frac{1}{1+\frac{J\omega L}{R}}$, $|H(\omega)|^2 = \frac{1}{1+\left(\frac{\omega L}{R}\right)^2}$

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2 = \frac{N_0/2}{1+\left(\frac{\omega L}{R}\right)^2}$$

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{YY}(\omega)d\omega = \frac{N_0}{4\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{1+\left(\frac{\omega L}{R}\right)^2} = \frac{N_0 R}{4L}$$

Another check $H(\omega) = \frac{1}{1+\frac{J\omega L}{R}}$ and $h(t) = \frac{R}{L}e^{-\frac{Rt}{L}}u(t)$

For white noise from a previous example

$P_{YY} = \overline{Y^2} = \frac{N_0}{2}\int_{-\infty}^{+\infty} h^2(\lambda)d\lambda = \frac{N_0 R}{4L}$ same answer

Cross-Power Density Spectrum of Input & Output

$$S_{XY}(\omega) = S_{XX}(\omega)H(\omega)$$
$$S_{YX}(\omega) = S_{XX}(\omega)H(-\omega)$$

- Measurements of Power Density Spectrum

Practice 1

- A stationary random process $X(t)$, having an autocorrelation function

$$R_{XX}(\tau) = 2\exp(-4|\tau|)$$

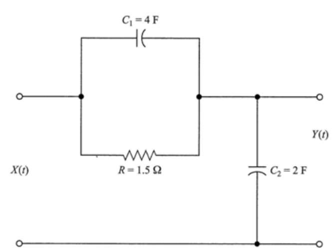
is applied to the network in the shown figure. Find

a) $S_{XX}(\omega)$
b) $|H(\omega)|^2$
c) $S_{YY}(\omega)$

a. $S_{XX}(\omega) = \frac{16}{16+\omega^2}$

b. $|H(\omega)|^2 = \frac{1+36\omega^2}{1+81\omega^2}$

c. $S_{YY}(\omega) = \frac{16(1+36\omega^2)}{(16+\omega^2)(1+81\omega^2)}$



Practice 2

- White noise, for which $R_{XX}(\tau) = 10^{-2}\delta(\tau)$, is applied to a network with impulse response.

$$h(t) = 3te^{(-4t)}u(t)$$

  - Find the network's noise output power (in a 1 Ohm resistor) using

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau + \lambda_1 - \lambda_2)h(\lambda_1)h(\lambda_2)d\lambda_1 d\lambda_2$$

  - Obtain an expression for the output power spectrum
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In Class Practice 3

- White noise with power density $\frac{N_0}{2} = 6(10^{-6})\frac{W}{Hz}$ is applied to an ideal filter (gain=1) with bandwidth W(rad/s). Find W so that the output's average noise power is 15 watts.