A STATISTICAL RULE FOR ESTIMATING THE NUMBER AND PARAMETERS OF SINUSOIDS IN A BACKGROUND OF ADDITIVE WHITE GUAS'IAN NOISE(AWGN)

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ABSTRACT

A method is proposed for estimating the number and parameters of sinusoids in a background of AWGN. The method filters the sinusoids by exploiting their deterministic nature. Compared to others,the performance is more efficient and more reliable at low signal-to-noise ratio(SNR). The required assumptions are less stringent. The flexibility of the technique allows important modifications, such as, detecting the absence of sinusoids instead of their presence, and extending the technique to the multidimensional case. Simple microprocessor implementation is feasible.

Introduction

Estimating the number and parameters of sinusoids embedded in a background of AWGN finds applications in many areas; such as, Doppler estimation of radar and sonar returns time series analysis, system identification, antenna array processing, vibrational measurements, and geophysical data processing (1,2, 3,4).

Many techniques were proposed for sinusoidal estimation, each having its own advantages and disadvantages. In the following, some are presented. In (5), the parameters of a sharply tuned IIR filter are varied adaptively till the center frequency converges to that of the sinusoid. Convergence could also be achieved by iterative filtering (IFA) (6). In this method, the autoregressive parameters are estimated and fed to a filter; new parameters are estimated using the filtered data. The process is repeated till convergence is acheived. A rectangular notch filter followed by a square law detector, and a decision device is swept through the desired bandwidth (BW); the parameters of the method are chosen to maximize the probability of detection (7). Pisarenko's method for harmonic retrieval (8,9) is formulated in an adaptive filtering framework. Here, the coefficients of the filter are determined by minimizing a certain cost function.

Another approach utilizes the autocorrelation function. Sinusoids are determined by searching the peaks of the windowed autocorrelation (10). The resolution is limited by the window length, and the finite amount of lags. Iterative autocorrelation is attractive in the sense that, convergence is inherent, provided, the SNR is high enough (11). The method is speeded up by quantizing the signal into two levels, and replacing the required multiplications by simple logic operations (12).

Techniques for separating signal space from noise were suggested. An autoregressove technique for sinusoidal estimation (13), nullifies, the noise subspace, and preserves the signal space, using an appropriate function of the autocorrelation matrix, apriori knowledge of the spread of the eigen values, and a threshold between the sinusoids and noise. Proney's method uses least square approximation to solve the linear prediction equations for the amplitudes and frequencies (14). A more stable solution is obtained using Total Least Squares (15). Nutall (16) extended the covariance method to work in the presence of noise. Super resolution techniques utilizing signal space, such as, Music (17), and Minimum norm (18), were also used.

In the previous techniques, apriori assumptions were made to separate sinusoids from noise. Assumptions regarding the number of sinusoids or the difference in power level are impractical and could adversely affect performance. The complexity and computational

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effort required by most techniques is prohibitively large. Some techniques are incapable of estimating all parameters. And the performance of most techniques deteriorates as SNR becomes low.

In this paper, a simple method is proposed for estimating the number and parameters of sinusoids in AWGN. The heart of the technique is a statistical rule that functions to separate the sinusoids from noise. The method bypasses some of the problems encountered by others. The only assumption required is an upper bound on the expected number of sinusoids. The technique requires more than one discrete measurement of the noisy amplitude spectrum inorder to achieve separation.

The estimation technique

The statistical rule used for discrimination exploits the detersive nature of the sinusoids and the random nature of noise. It is obvious that independent observations of the amplitude spectrum are identical for the deterministic sinusoids; but, totally different for noise. In case of noisy sinusoids,the amplitude spectrum is expected to fluctuate less at the location of the sinusoids. Consequently, by isolating the least fluctuating components of the spectrum, there is a good chance of filtering the sinusoids. The following steps are recommended for implementing the above scheme:

1- Using an appropriate mean, partition the bandwidth into N slots, and measure the amplitude and center frequency of each.

2- Repeat the above process Nf times creating the vector set:

$$A_j(i) \ , \ F_j(i) \qquad i = 1,\dots,N \quad j = 1,\dots,Nf$$

amplitudes are stored in the A's, and frequencies in the F's.

3- From each measurement, collect the largest Nm sinusoids, and store the corresponding frequencies in the vectors:

$$E_j(l) \qquad l = 1,\dots,Nm \quad j = 1,\dots,Nf$$

Nm has to be greater than or equal to the number of sinusoids.

4- Specify Ns (the minimum number of times a frequency component must appear to be considered as a sinusoid). Ns Nf.

5- Scan the Ei vectors and isolate any component that appears Ns times or more.

6- After estimating the number and frequencies, the phases ϕ_i and aplitudes a_i could be estimated as follow:

$$X_i = \langle S(t)\cdot COS(w_i t)\rangle \quad , \quad Y_i = \langle S(t)\cdot SIN(w_i t)\rangle$$
$$a_i = (X_i^{\,2} + Y_i^{\,2})^{1/2} \quad , \quad \phi_i = Atan(Y_i/X_i)$$
$$i = 1,\dots,M \tag{1}$$

where w_i's are the estimated frequencies, M the estimated number of sinusoids, S(t) the noisy sinusoids, and the brackets denotes the inner product. Note that these amplitudes are already estimated in step (1).

The above scheme was found capable of filtering the sinusoids with a controllable probability of error (Pe), and probability of false alarm (Pf). The performance judged by Pe, Pf, and the resolution is controlled by the parameters Nf,Nm,Ns, and N.

Analysis

In this section, the probability of error (Pe), and probability of false alarm (Pf) are derived.

a. Probability of error :

The probability of losing J out of M sinusoids Pe_J is derived in terms of the parameters of the rule. The noise is assumed to be AWGN with zero mean and variance σ^2. In (19) approximate distribution for the amplitude spectrum are proposed. A satisfactory approximation is to consider the distribution to be a two-degree of freedom Chi-square (20:P. 57) :

$$f_Y(y) = \begin{cases} y/\sigma^{-2}\exp(-y^2/2\sigma^2) & y \;\; 0 \\ 0 & \text{else where} \end{cases} \tag{2}$$

Assume that M sinusoids are present, each having an apmplitude a_i. Since the noise is uncorrelated and independent of the sinusoids, the probability that n noise components exceeds the amplitude of the i'th sinusoid is equal to :

$$(1 - \int_0^{ai} f_Y(y)\, dy)^n = (\exp(-SNR_i))^n$$

$$SNR_i = ai^2/2\sigma^2 \quad (3)$$

For convienience, the amplitude are arranged in a descending order, that is :

$$a1 \geq a2 \geq \ \ldots \ \geq aM$$

Let Ki be the number of noise components with amplitudes greater than a_i and less than a_{i-1}

$$\underline{K1}\ a1 \text{—} \ \cdots \text{—}\ a_i \ \underline{\ Ki\ }\ a_{i-1} \ \underline{\ KM\ }\ aM$$

from the above it is obvious that J sinusoids will be lost if the amplitude of sinusoid M-J+1 is exceeded by Nm-(M-J) noise component. Therefore, the probability of losing J sinusoids from one measurement is equal to:

$$(\exp(-SNR_{M-J+1})^{Nm-(M-J)}) \quad (4)$$

since this must repeat at least Nf-(Ns-1) times for the sinusoid to be completely lost, Pe_J will be :

$$Pe_J \leq (\exp(-SNR_{M-J+1})^{Nm-(M-J)})^{Nf-Ns+1} \quad (5)$$

in fig 1, and 2 the probability of error Pe for the special case of one sinusoid M=1 is plotted as a function of Ns, Nf.

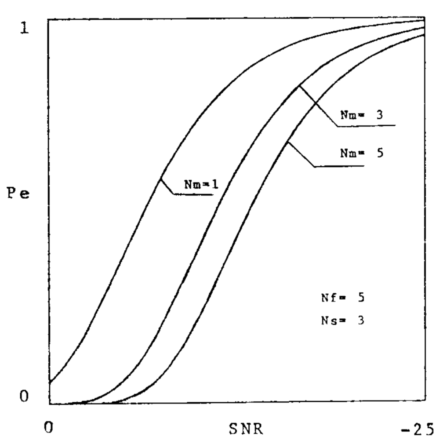

Fig 1

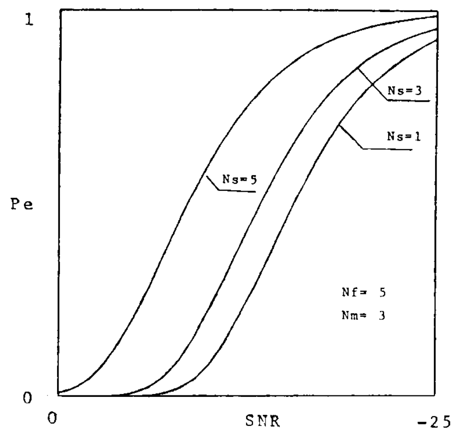

Fig 2

b. Probability of false alarm:

Erronous estimates occurs when a noise component appears Ns times or more in the Ei vectors. Since the presence of the sinusoids will only reduce this chance, an upper bound on (Pf) is established by neglecting their presence. As a result of the identical distribution of the noise, a component will lie within the Nm maxima with probability equal to Nm divided by the number of slots N. Inorder to be considered as a sinusoid, a component must appear at least Ns times; Making the probability of false alarm equal to:

$$Pf \leq (Nm/N)^{Ns} \quad (6)$$

The upper bound on Pf is independent of the noise level. Such a property is considered as an advantage in the sense that, the performance is still reliable at low SNR; instead of giving erronous estimates, the method will give an indication of its inability to handle the present noise level. Such an indication is by estimating zero number of sinusoids.

Results

Simulation results are presented in table 1,2. Three sinusoids are mixed with noise, the number,frequencies, and amplitudes are estimated at different noise levels. The frequency measuring device has a resolution of .05 Hz. The scanned frequency range lies between zero and 120 Hz. N=2400, Nf=7, Nm=10, Ns=3, and Pf= $7x10^{-8}$.

SNR dB	#	S1 Hz	S2 Hz	S3 Hz
0	3	5.10	5.15	30.10
-10	3	5.10	5.15	30.10
-20	3	5.10	5.15	30.10
-25	3	5.10	5.15	30.10
-26	2	5.10	30.1	xxxxx
-28	1	5.10	xxxx	xxxxx
-30	0	xxxx	xxxx	xxxxx

Table 1. Frequency estimates
F1=5.12, F2=5.15, F3=30.1 Hz

SNR dB	#	S1	S2	S2
0	3	0.998	0.998	0.998
-10	3	1.023	0.991	0.993
-20	3	1.127	1.026	0.921
-25	3	1.264	1.165	1.289
-26	2	1.177	1.242	xxxxx
-28	1	1.502	xxxxx	xxxxx
-30	0	xxxxx	xxxxx	xxxxx

nTable 2. Amplitude estimates
a1 = a2 = a3 = 1.00

as can be noticed, amplitude estimates deteriorates as the noise level increases; while frequency estimates remains unaffected. The number of estimated sinusoids starts to decrease with SNR, till a limit is reached wher no sinusoids could be estimated. Such a behaviour is understood by noticing that the method does not directly estimate frequency rather, it figures out which slot contains the sinusoids, and substitute the corresponding center frequency. Such a scheme makes the resolution totally dependent on the way the spectrum is partitioned. The probability of choosing a wrong slot is independent of the noise level; while it is almost sure that the sinusoids will be lost at high noise level; which explains the reliable performance at low SNR. The overall performance is better than that anticipated. Such a discripancy is due to neglecting the enhancing effect introduced by the frequency measuring device.

Conclusions

A new method is proposed for estimating the number and parameters of sinusoids in AWGN. The method requires the ability to make more than one independent measurement of the amplitude spectrum. The fact that the upper bound on Pf is independent of noise allows reliable performance at low SNR. The flexibility of the technique permits the use of efficient search stratigies that could significantly reduce the search time. The method could be easily modified to detect the absence of the sinusoids instead of their presence by collecting minima instead of maxima. Extension to the multidimensional case is easily achieved by scanning the whole signal space with no modification introduced to the separation technique. The simplicity of the technique facilitates implementation using a simple microprocessor based system.

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