KING FAHD UNIVERSITY OF PETROLEUM & MINERALS
Electrical Engineering Department

EE 380 - Control Engineering   **Experiment # 5**

_____ _____

Simulation of a Prototype Second Order System

OBJECTIVE

To develop an improved understanding of the very important class of 2nd order control systems and to become familiar with its typical characteristic response.

APPARATUS   IBM Compatible PC - MATLAB/ SIMULINK Software - Printer.

BACKGROUND

Although true second-order control systems are rare in practice, their analysis generally helps to form a basis for the understanding of analysis and design of higher-order systems.

The second-order system shown in Fig. 1 is defined as the **prototype second-order system.** The closed-loop transfer function of the system is:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For a unit-step input, $R(s) = \frac{1}{s}$, the output response of the system is

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$
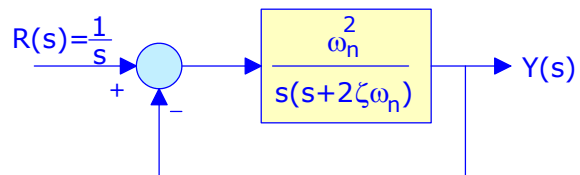

Fig. 1 Prototype second-order system

The dynamic behavior of second-order systems can be described in terms of the two parameters ζ and ω_n.

where :
ω_n undamped natural frequency
ζ damping ratio

(1)   **Underdamped Case ($0 < \zeta < 1$)**

   In this case, the system output for a unit-step input, can be written as :

$$Y(s) = \frac{\omega_n^2}{s(s + \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2})}$$

1

$$y(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta) \; ; \theta = \cos^{-1}\zeta$$

Note that the system has a pair of complex conjugate poles at :
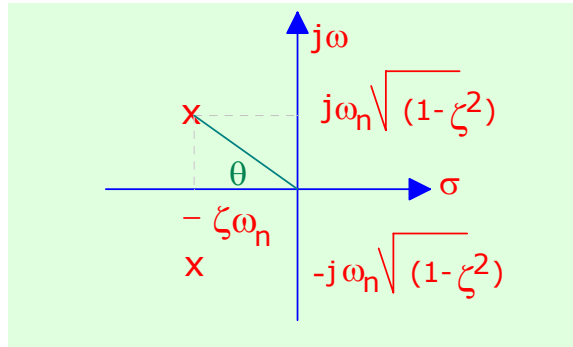$s = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2} \; = -a \pm j\omega$ as shown in Fig. 2.

Fig. 2   Poles location: underdamped system

The **maximum overshoot** and the **time** it occurs (peak time) are given by:-

$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$ and $\frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$ sec respectively.

The underdamped system response is a damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid ω, is equal to the imaginary part of the pole.

(2)   **Critically damped Case ($\zeta = 1$)**

In this case, the system output for a unit-step input, can be written as :
$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

Note that the system has 2 repeated real poles at
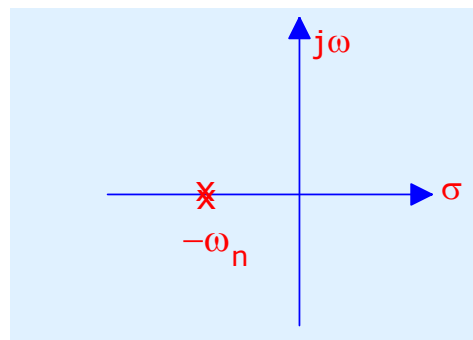$s = -\omega_n = $ as shown in Fig. 3.

Fig. 3 Poles location: critically-damped system

2

(3) Overdamped Case ($\zeta > 1$)

In this case, the system output for a unit-step input, can be written as :

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s\left(s + \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}\right)}$$

$$y(t) = 1 + \frac{\omega_n}{2\sqrt{1 - \zeta^2}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) ;$$
$$s_1 = \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} , s_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

or

$$y(t) = 1 + \frac{e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t}}{2\sqrt{\zeta^2 - 1}\left[\zeta + \sqrt{\zeta^2 - 1}\right]} - \frac{e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}}{2\sqrt{\zeta^2 - 1}\left[\zeta - \sqrt{\zeta^2 - 1}\right]}$$

Note that the system has 2 distinct real poles at :

$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$ as shown in Fig. 4.
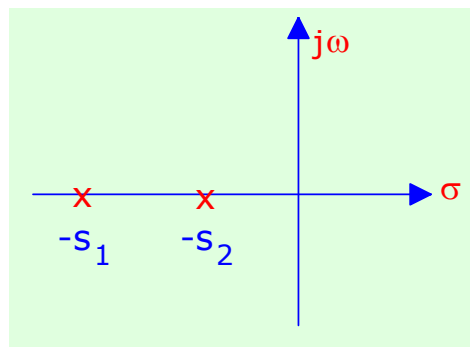

Fig. 4 Poles location: overdamped system

(4) Undamped Case ($\zeta = 0$)

In this case, the system output for a unit-step input, can be written as :

$$Y(s) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)} = \frac{\omega_n^2}{s(s \pm j\omega_n)}$$

$$y(t) = 1 - \cos(\omega_n t)$$

Note that the system has a pair of imaginary poles at $s = \pm j\omega_n$ as shown in Fig. 5.
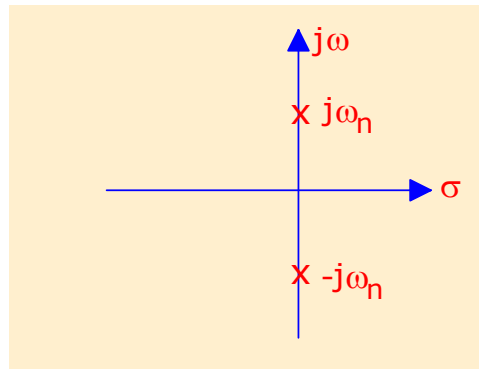
3

Fig. 5 Poles location: undamped system

The system response is an undamped sinusoid with radian frequency equal to the imaginary part of the poles.

Typical step responses of the prototype second-order system, for the four cases considered, are shown in Fig. 6. Notice that the critically damped case is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.
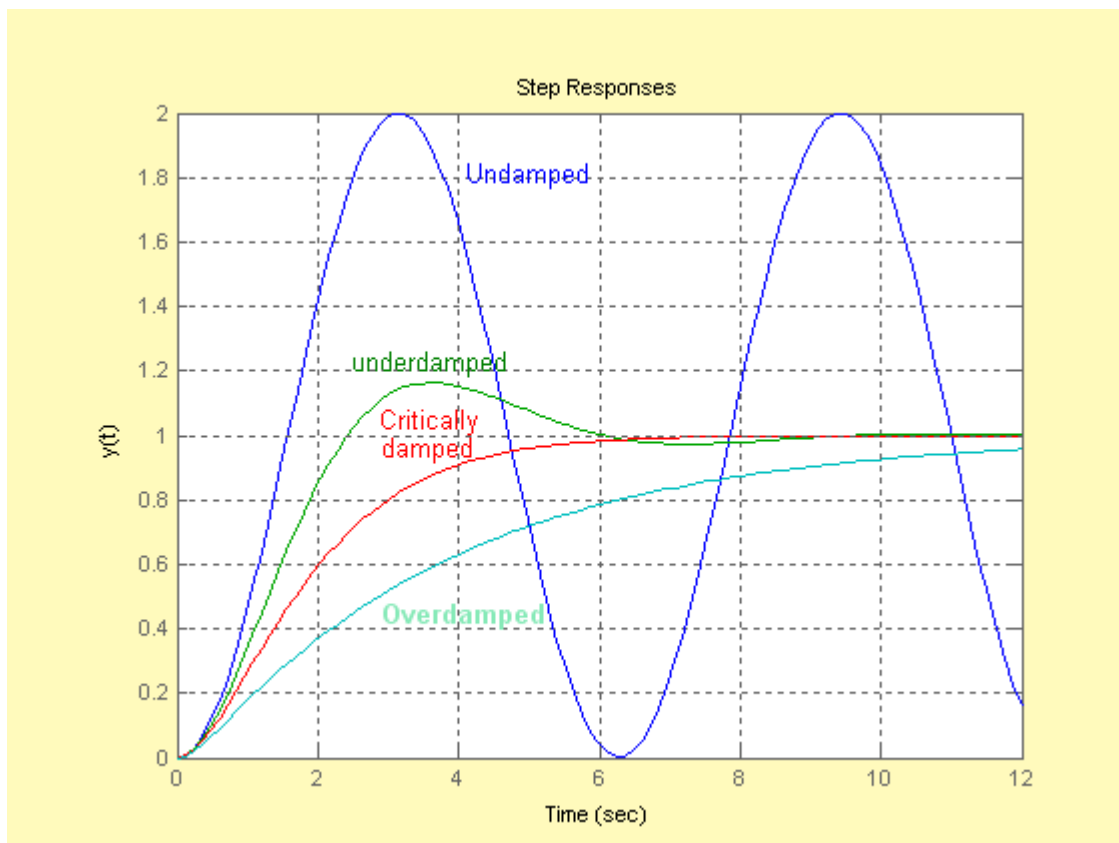

Fig. 6 Step responses of a prototype second-order system.

4

PROCEDURE:

1. Write a MATLAB/SIMULINK program to simulate the step-response of the prototype second -order system . Note that the equation describing the system is given by:

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = R u(t)$$

2. The following numerical parameters can be assumed:

R (step-input magnitude)	10
$y(0)$	0
$\dot{y}(0)$	0
ω_n	1 rad/sec

3. Obtain plots of $y(t)$ and $\dot{y}(t)$ for values of
$\zeta = 0, 0.1, 0.3, 0.4, 0.5, 0.6, 0.707, 1.0, 3.0,$ and 5.0

DISCUSSION:

1. Solve the differential equation describing the system (see (1) in PROCEDURE) for $\zeta = 0.3$ and other parameters as in the simulation. Show your work and express your solution in the form:

$$y(t) = k_1 + k_2 e^{-at} \sin(\omega t + \theta)$$

2. Evaluate $y(2.0)$ from the analytical expression obtained in (1) and compare it with the corresponding value obtained from the SIMULINK simulation.

3. For $\zeta = 0.1, 0.3, 0.5,$ and $0.7,$ compare the value of the **damped frequency** ω and the **maximum overshoot** obtained from the differential equation solution and the SIMULINK solution.

4. For $\zeta = 0$ compare the values of the **undamped natural frequency** ω_n obtained from the differential equation and the SIMULINK solution.

5. For the critically damped case, find the roots of the characteristics equation. Examine the SIMULINK solution for $\zeta = 1.$ Assuming the natural part of the solution is a simple decaying exponential, estimate the time constant τ of the natural response (use the 63.2% of change rule) .
Does $\frac{1}{\tau}$ = root of characteristic equation? Explain any difference obtained. Should there be any difference? (hint: Examine the analytical solution for $\zeta = 1;$ is this expression the same as the solution of a first-order differential equation?)

6. If the system has $\omega_n \neq 1$ rad/sec, can you still use your graphs for $y(t)$? Justify your answer. How do you obtain the proper time axis?

7. From the simulation results, and for the underdamped cases considered (i.e. $\zeta = 0.1, 0.3, 0.4, 0.5, 0.6,$ and 0.7), obtain:
   - a plot of the maximum overshoot versus the damping ratio.
   - a plot of the peak time versus the damping ratio.

8. Repeat part (7) for $\omega_n = 2$ rad/sec. What is the effect of increasing ω_n on the maximum overshoot, and on the peak time? Is this expected?. Comment on your results.

9. For parts (7) and (8), obtain a plot of the normalized peak time ($\omega_n \times peaktime$) versus the damping ratio. What do you observe?. Explain the results.

NOTE: Your plots should be well displayed, titled and labeled. Make use of MATLAB labeling commands and ask your instructor for help if you do not know how!