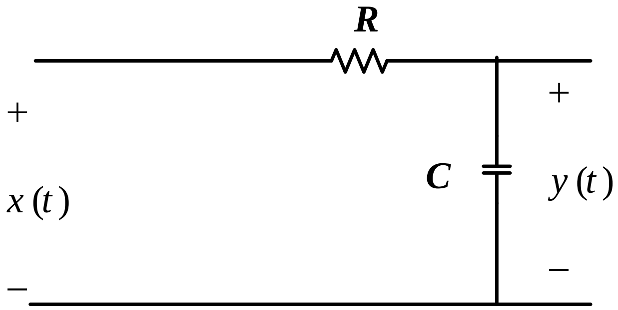
Chapter 7 The Laplace Transform

Consider the following RC circuit (**System**)



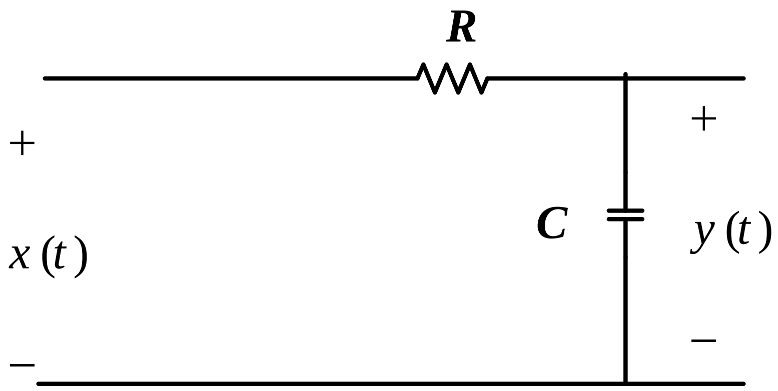
System analysis in the time domain involves (finding y(t)):

Solving the differential equation $RC\frac{dy(t)}{dt} + y(t) = x(t)$

OR

Using the convolution integral $y(t) = x(t)*h(t)$

Both Techniques can results in tedious (ممل)mathematical operation



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Fourier Transform provided an alternative approach

Differential Equation → Algebraic Equation

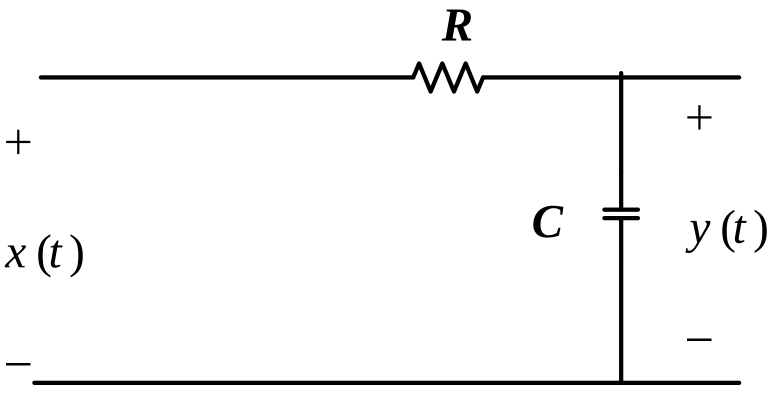
$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$RC(j2\pi f)Y(\mathrm{f}) + Y(\mathrm{f}) = X(\mathrm{f})$$

solve for $Y(\mathrm{f})$

$$Y(\mathrm{f}) = \frac{X(\mathrm{f})}{[(j2\pi fRC) + 1]}$$

$y(t)$ ← **Inverse Back**



$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$RC(j2\pi f)Y(\mathrm{f}) + Y(\mathrm{f}) = X(\mathrm{f})$$

$$Y(\mathrm{f}) = \frac{X(\mathrm{f})}{[(j2\pi fRC) + 1]}$$

Inverse Back

$$y(t)$$

Unfortunately , there are many signals of interest that arise in system analysis for which the Fourier Transform does not exists

A more general transform is needed

Fourier Transform pairs was defined

$$x(t)=\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \qquad X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Now multiply $x(t)$ by $e^{-\sigma t}$ and takes the Fourier Transform

$$FT\left[e^{-\sigma t}x(t)\right]=\int_{0}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}d=\int_{0}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt=X(\sigma+j\omega)$$

Let $s \square \sigma+j\omega$ Complex Frequency

$$\Rightarrow \quad FT\left[e^{-\sigma t}x(t)\right]=\int_{0}^{\infty} x(t)e^{-st}d=X(s) \square \mathrm{L}\left[x(t)\right]$$

were $\mathrm{L}\left[\quad\right]$ Denotes the operation of obtaining the Laplace Transform

7.1 DEFINITIONS OF LAPLACE TRANSFORMS

The Laplace transform of a time function is given by the integral

$$\mathcal{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

where $\mathcal{L}_b[\cdot]$ indicates the Laplace transform

This definition is called the **bilateral**, or **two-sided**, Laplace transform—hence, the subscript b

Notice that the bilateral Laplace transform integral becomes the Fourier transform integral if s is replaced by $j\omega$

The Laplace transform variable is complex $s = \sigma + j\omega$

$$F(s) = \int_{\infty}^{\infty} f(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} (f(t)e^{-\sigma t})e^{-j\omega}dt$$

This show that the bilateral Laplace transform of a signal can be interpreted as the Fourier transform of that signal multiplied by an exponential function $e^{-\sigma t}$

The inverse Laplace transform

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$ **where** $j = \sqrt{-1}$

where $\mathscr{L}^{-1}[\cdot]$ indicates the inverse Laplace transform

$$\mathscr{L}_b[f(t)] = F_b(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_{-\infty}^{0} f(t)e^{-st}dt + \int_{0}^{\infty} f(t)e^{-st}\,dt$$

We define f(t) to be zero for t < 0

$$\mathscr{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

where $\mathscr{L}[\cdot]$ denotes the unilateral Laplace transform

This transform is usually called, simply, the **Laplace transform** , the subscript *b* dropped

$$f(t) \stackrel{\mathscr{L}}{\longleftrightarrow} F(s)$$ **Laplace transform pairs**

Unilateral (single sided) Laplace transform pairs

$$f(t) \overset{\mathscr{L}}{\longleftrightarrow} F(s)$$

$$\mathscr{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$$

Forward transform

$$f(t) = \mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

Inverse transform

$s = \sigma + j\omega$

Complex domain

Because of the difficulty in evaluating the complex inversion integral

Simpler procedure to find the inverse of **Laplace Transform** (i.e f(t)) will be presented later

Using Properties and **table** of known transform (similar to Fourier)

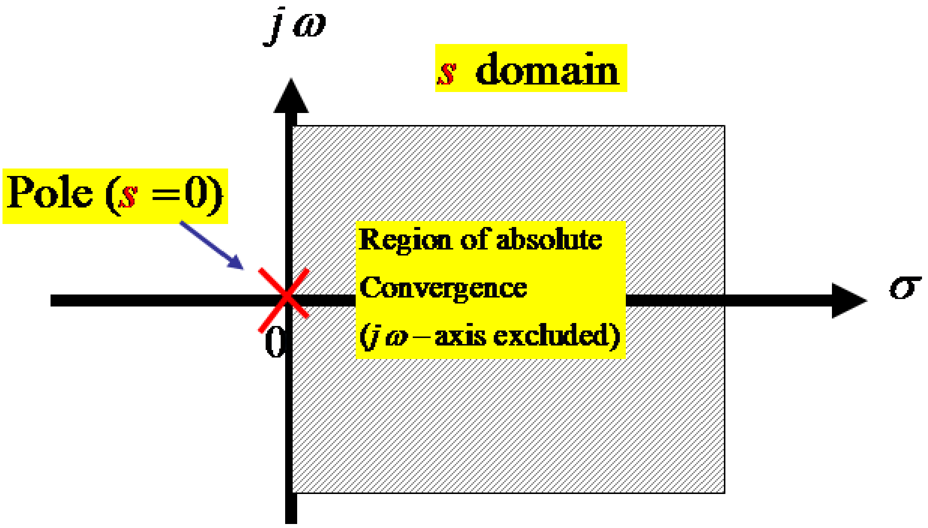
EXAMPLE 7.1 **Laplace transform of a unit step function**

$$\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}dt = \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{-1}{s}\left[\lim_{t\to\infty} e^{-st} - 1\right]$$

$$= \frac{-1}{s}\left[\lim_{t\to\infty} e^{-(\sigma+j\omega)t} - 1\right] = \frac{-1}{s}\left[\lim_{t\to\infty} e^{-\sigma t} e^{-j\omega t} - 1\right]$$

For this to converge then $\lim_{t\to\infty} e^{-\sigma t} \to 0 \Rightarrow \sigma > 0 \Rightarrow \underbrace{\mathrm{Re}(s) > 0}_{\text{region of convergence (ROC)}}$

$$\mathcal{L}[u(t)] = \frac{1}{s}, \quad \mathrm{Re}(s) > 0$$



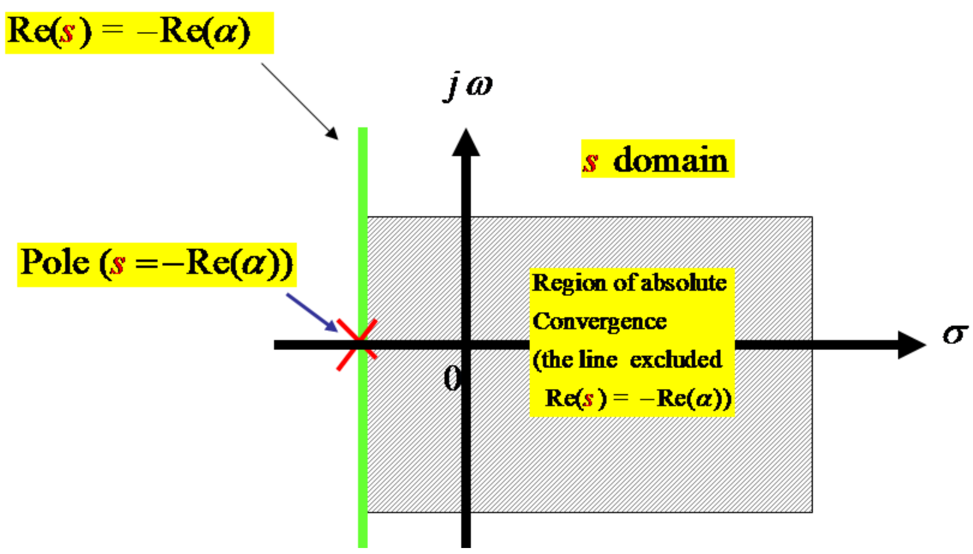
EXAMPLE 7.2 Laplace transform of an exponential function

We derive the Laplace transform of the exponential function $f(t) = e^{-at}$

$$F(s) = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt = \frac{e^{-(s+a)t}}{-(s+a)}\bigg|_0^\infty = \frac{-1}{s+a}\left[\lim_{t\to\infty} e^{-(s+a)t} - 1\right]$$

This transform exists only if $\text{Re}(s + a)$ is positive

$$\mathscr{L}[e^{-at}] = \frac{1}{s+a}, \quad \text{Re}(s + a) > 0$$



$$e^{-at} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s+a}$$

A short table of Laplace transforms is constructed from Examples 7.1 and 7.2 and is given as Table 7.1

TABLE 7.1 Two Laplace Transforms

$f(t), t > 0$	$F(x)$
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$

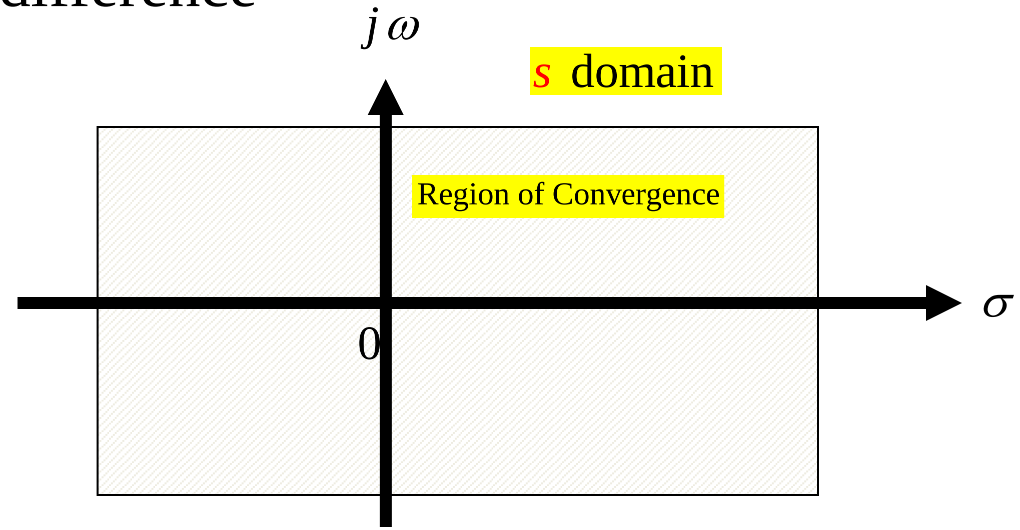
$$\mathcal{L}[\ \delta(t)\ ] = \int_{0}^{\infty} \delta(t) e^{-st} dt$$

Assuming the lower limit is 0^-

$$\mathcal{L}[\ \delta(t)\ ] = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = e^{-st}\Big|_{t=0} = 1$$

The value of s make no difference

Region of Convergence



$$\mathscr{L}[\delta(t - t_0)] = \int_0^{\infty} \delta(t - t_0)e^{-st}dt = e^{-st}\Big|_{t=t_0} = e^{-t_0 s}$$

$$\delta(t - t_0) \overset{\mathscr{L}}{\longleftrightarrow} e^{-t_0 s}$$

We also can drive it from the shifting in time property that will be discussed later

We know derive the Laplace Transform (LT) for a cosine function

$$\cos bt = \frac{e^{jbt} + e^{-jbt}}{2}$$

$$\mathscr{L}[\cos bt] = \tfrac{1}{2}[\mathscr{L}[e^{jbt}] + \mathscr{L}[e^{-jbt}]]$$

Since $$e^{-at} \overset{\mathscr{L}}{\longleftrightarrow} \frac{1}{s + a}$$

$$\mathscr{L}[\cos bt] = \frac{1}{2}\left[\frac{1}{s - jb} + \frac{1}{s + jb}\right] = \frac{s + jb + s - jb}{2(s - jb)(s + jb)} = \frac{s}{s^2 + b^2}$$

By the same procedure, because $\sin bt = (e^{jbt} - e^{-jbt})/2j$

$$\mathscr{L}[\sin bt] = \frac{1}{2j}[\mathscr{L}[e^{jbt}] - \mathscr{L}[e^{-jbt}]] = \frac{1}{2j}\left[\frac{1}{s - jb} - \frac{1}{s + jb}\right]$$

$$= \frac{s + jb - s + jb}{2j(s - jb)(s + jb)} = \frac{b}{s^2 + b^2}$$

7.4 LAPLACE TRANSFORM PROPERTIES

linearity $\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s)$

$$\mathcal{L}[f(t)] = \mathcal{L}[f_1(t) + f_2(t)] = \int_0^\infty [f_1(t) + f_2(t)]e^{-st}dt$$

$$= \int_0^\infty f_1(t)e^{-st}dt + \int_0^\infty f_2(t)e^{-st}\,dt$$

$$= \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] = F_1(s) + F_2(s)$$

$$\mathcal{L}[af_5(t)] = \int_0^\infty af_5(t)e^{-st}dt = a\int_0^\infty f_5(t)e^{-st}dt = a\mathcal{L}[f_5(t)] = aF_5(s)$$

$$\mathcal{L}[a_1f_1(t) + a_2f_2(t)] = a_1F_1(s) + a_2F_2(s).$$

complex shifting $\mathscr{L}[e^{-at}f(t)] = F(s)\Big|_{s\leftarrow s+a} = F(s+a)$

Proof

$$L[e^{-at}f(t)] = \int_0^\infty [e^{-at}f(t)]e^{-st}dt = \int_0^\infty f(t)e^{-(s+a)t}dt$$

Let s*= s+a $\Rightarrow$ $L[e^{-at}f(t)] = \int_0^\infty f(t)e^{-s^*t}d \quad = F(s^*) = F(s+a)$

$$L[\cos\omega_0 t] = \frac{s}{s^2+\omega_0^2} \Rightarrow \mathscr{L}[e^{-\alpha t}\cos\omega_0 t] = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

$$L[\sin\omega_0 t] = \frac{\omega_0}{s^2+\omega_0^2} \Rightarrow \mathscr{L}[e^{-\alpha t}\sin\omega_0 t] = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

Example

$$\mathcal{L}[\cos\omega_0 t] = \mathcal{L}\left[\frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}\right] = \frac{1}{2}\mathcal{L}[e^{j\omega_0 t}] + \frac{1}{2}\mathcal{L}[e^{-j\omega_0 t}]$$

Since $\mathcal{L}[e^{-\alpha t}u(t)] = \dfrac{1}{(\alpha + s)}$

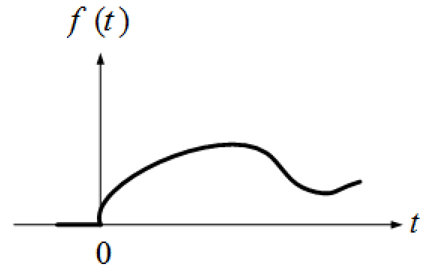
Then $\mathcal{L}[e^{j\omega_0 t}] = \dfrac{1}{(j\omega_0 + s)}$ $\qquad$ $\mathcal{L}[e^{-j\omega_0 t}] = \dfrac{1}{(-j\omega_0 + s)}$

$$\mathcal{L}[\cos\omega_0 t] = \frac{1}{2}\frac{1}{(s + j\omega_0)} + \frac{1}{2}\frac{1}{(s - j\omega_0)} = \frac{s}{s^2 + \omega_0^2}$$

Similarly $\mathcal{L}[\sin\omega_0 t] = \dfrac{\omega_0}{s^2 + \omega_0^2}$

Real Shifting

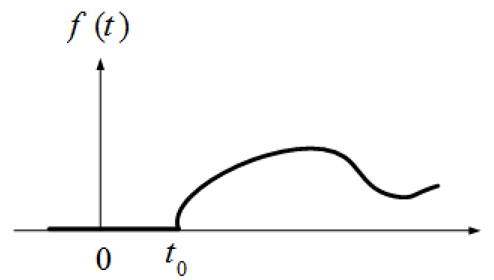
$$f(t)u(t) = \begin{cases} f(t), & t > 0 \\ 0, & t < 0 \end{cases}$$



$\Leftrightarrow$ $F(s)$

Let $f(t - t_0)u(t - t_0) = \begin{cases} f(t - t_0), & t > t_0 \\ 0, & t < t_0 \end{cases}$

where $t_0 \geqq 0$



The Laplace transform of the function

$$\mathscr{L}[f(t - t_0)u(t - t_0)] = \int_0^{\infty} f(t - t_0)u(t - t_0)e^{-st}\, dt = \int_{t_0}^{\infty} f(t - t_0)e^{-st} dt$$

We make the change of variable $(t - t_0) = \tau$ $\Rightarrow$ $t = (\tau + t_0)$ $\quad dt = d\tau$

$$\mathscr{L}[f(t - t_0)u(t - t_0)] = \int_0^{\infty} f(\tau)e^{-s(\tau + t_0)}\, d\tau = e^{-t_0 s} \underbrace{\int_0^{\infty} f(\tau)e^{-s\tau} d\tau}_{F(s)}$$

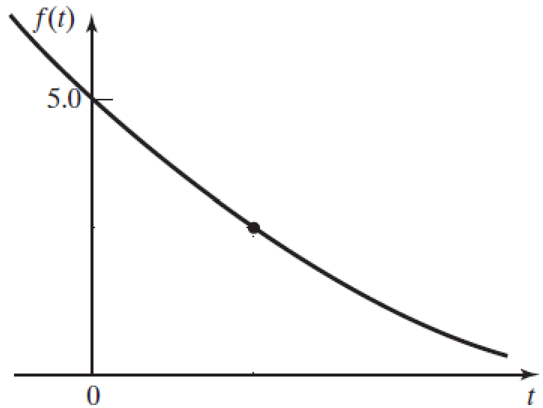
$$\mathscr{L}[f(t - t_0)u(t - t_0)] = e^{-t_0 s} F(s)$$

EXAMPLE 7.4 **Laplace transform of a delayed exponential function**

Consider the exponential function

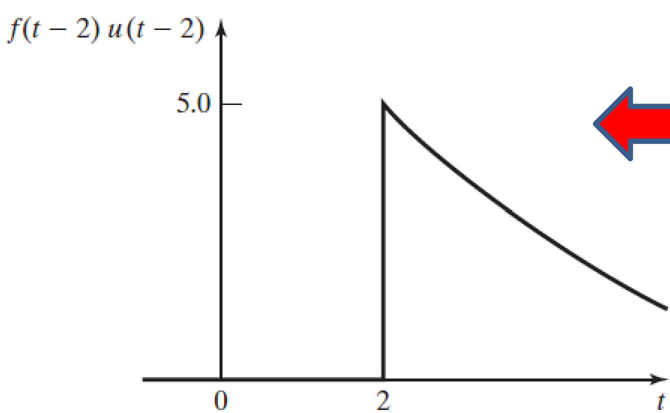
$$e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s + a}$$

$$f(t) = 5e^{-0.3t}$$



$$F(s) = \frac{5}{s + 0.3}$$

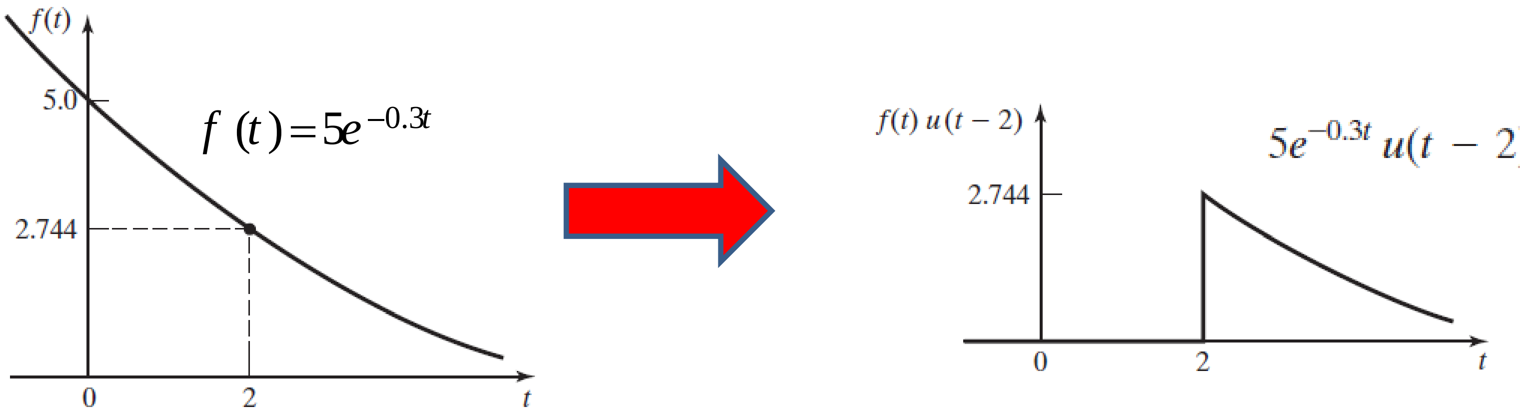
$$f_1(t) = 5e^{-0.3(t-2)}u(t - 2)$$



$$\mathcal{L}[f_1(t)] = F_1(s) = e^{-2s}F(s) = \frac{5e^{-2s}}{s + 0.3}$$

EXAMPLE 7.5 Laplace transform of a more complex delayed function

Consider now the function $f_2(t) = 5e^{-0.3t}\, u(t - 2)$



$$f_2(t) = 5e^{-0.3t}\, u(t - 2)$$

$$= 5e^{-0.3t}\, u(t - 2)[e^{0.3(2)}e^{-0.3(2)}]$$

$$= (5e^{-0.6})e^{-0.3(t-2)}u(t - 2)$$

$$= 2.744e^{-0.3(t-2)}u(t - 2)$$

$$F_2(s) = \mathcal{L}[f_2(t)] = \frac{2.744e^{-2s}}{s + 0.3}$$

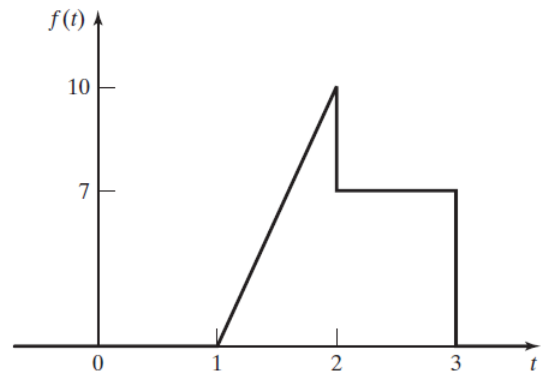
EXAMPLE 7.3 **Laplace transform of a unit ramp function**

We now find the Laplace transform of the unit ramp function $f(t) = t$:

$$\mathscr{L}[t] = \int_0^{\infty} t e^{-st} dt$$

EXAMPLE 7.6 **Laplace transform of a straight-line-segments function**

It is sometimes necessary to construct complex waveforms from simpler waveforms



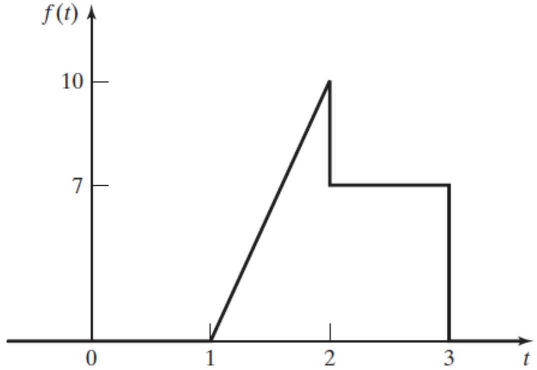
1. The slope of the function changes from 0 to 10 at $t = 1$: $f_1(t) = 10(t - 1)u(t - 1)$

2. The slope of the function changes from 10 to 0 at $t = 2$: $f_2(t) = f_1(t) - 10(t - 2)u(t - 2)$

3. The function steps by -3 at $t = 2$: $f_3(t) = f_2(t) - 3u(t - 2)$

4. The function steps by -7 at $t = 3$:

$$f(t) = f_3(t) - 7u(t - 3)$$
$$= 10(t - 1)u(t - 1) - 10(t - 2)u(t - 2) - 3u(t - 2) - 7u(t - 3)$$



$$f(t) = 10(t-1)u(t-1) - 10(t-2)u(t-2) - 3u(t-2) - 7u(t-3)$$

Laplace transform of the unit ramp function $f(t) = t$

$$\mathcal{L}[t] = \int_0^{\infty} te^{-st}dt = \frac{1}{s^2}$$ (See example 7.3)

Since $\mathcal{L}[f(t-t_0)u(t-t_0)] = e^{-t_0 s}F(s)$ (Shift in time property)

$$F(s) = \frac{10e^{-s}}{s^2} - \frac{10e^{-2s}}{s^2} - \frac{3e^{-2s}}{s} - \frac{7e^{-3s}}{s} = \frac{10e^{-s} - 10e^{-2s} - 3se^{-2s} - 7se^{-3s}}{s^2}$$

Transform of Derivatives

$$\mathscr{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}\,dt$$

We evaluate this integral by parts $$\int u\,dv = uv - \int v\,du$$

$$\mathscr{L}[f(t)] = \int_0^{\infty} \boxed{f(t)}\boxed{e^{-st}\,dt}$$

$$u = f(t), \quad dv = e^{-st}\,dt$$

$$\Rightarrow \quad du = \frac{df(t)}{dt}dt \qquad v = \frac{e^{-st}}{-s}$$

$$\mathscr{L}[f(t)] = F(s) = -\frac{1}{s}f(t)e^{-st}\Big|_0^{\infty} + \int_0^{\infty} \frac{df(t)}{dt}\frac{e^{-st}}{s}\,dt$$

$$F(s) = \frac{1}{s}[-0 + f(0)] + \frac{1}{s}\underbrace{\int_0^{\infty} \frac{df(t)}{dt}e^{-st}dt}_{\mathscr{L}\left[\frac{df(t)}{dt}\right]}$$

$$\Rightarrow \quad \mathscr{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^-)$$

Proof

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \left[\frac{df(t)}{dt}\right] e^{-st}\, dt$$

Integrating by parts, $u = e^{-st} \quad dv(t) = df(t)$

$$\Rightarrow \quad du = -se^{-st} \quad v(t) = f(t)$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = e^{-st} \quad dv(t) = df(t) \qquad du = -se^{-st} \quad v(t) = f(t)$$

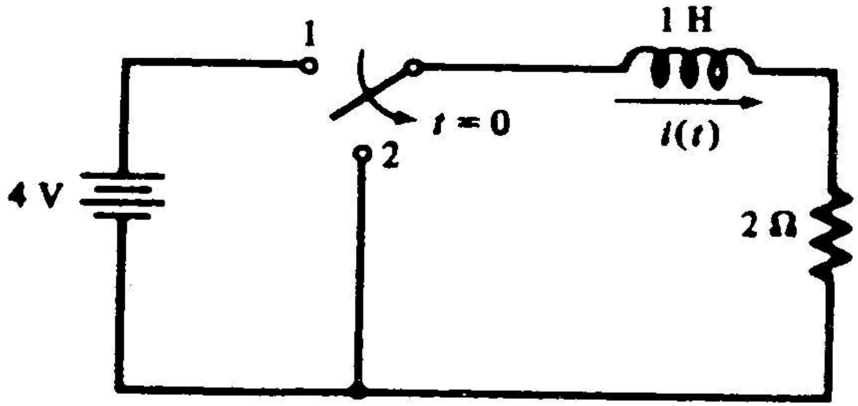
$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^{\infty}\left[\frac{df(t)}{dt}\right]e^{-st}\,dt = e^{-st}f(t)\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

$$= \left[e^{-s(\infty)}f(\infty) - e^{-s(0)}f(0^-)\right] + sF(s) = sF(s) - f(0^-)$$

($e^{-s(\infty)} \to 0$, $e^{-s(0)} \to 1$)

$$\Rightarrow \quad \frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0^-)$$

Consider the circuit shown



$$\frac{di(t)}{dt} + 2i(t) = \begin{cases} 4, & t \le 0 \\ 0, & t > 0 \end{cases}$$

Taking the Laplace transform of both sides starting at $t = 0^-$

$$sI(s) - i(0^-) + 2I(s) = 0$$

Assuming that the circuit was in steady state for $t < 0$ ⇒ $i(0^-) = \frac{4}{2} = 2$

⇒ $I(s)(s + 2) - 2 = 0$ ⇒ $I(s) = \frac{2}{s + 2}$ ⇒ $i(t) = 2e^{-2t}u(t)$

⇒ $$i(t) = \begin{cases} 2e^{-2t}, & t > 0 \\ 2, & t \le 0 \end{cases}$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s[sF(s) - f(0^+)] - f'(0^+) = s^2F(s) - sf(0^+) - f'(0^+)$$

where $f'(0^+)$ is the value of $df(t)/dt$ as $t \to 0^+$

$$\mathcal{L}\left[\frac{d^3f(t)}{dt^3}\right] = s^3F(s) - s^2f(0^+) - sf'(0^+) - f''(0^+)$$

where $f''(0^+)$ is the value of $d^2f(t)/dt^2$ as $t \to 0^+$

$$\mathcal{L}\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - \cdots - sf^{(n-2)}(0^+) - f^{(n-1)}(0^+)$$

where $f^{(i)}(0^+)$ is the value of $d^if(t)/dt^i$ as $t \to 0^+$

$$\frac{df(t)}{dt} \Leftrightarrow sF(s) - f(0^-)$$

$$\frac{d^2 f(t)}{dt^2} \Leftrightarrow s^2 F(s) - s f(t)\Big|_{t=0} - \frac{df(t)}{dt}\Big|_{t=0}$$

$$\frac{d^3 f(t)}{dt^3} \Leftrightarrow s^3 F(s) - s^2 f(t)\Big|_{t=0} - s\frac{dx(t)}{dt}\Big|_{t=0} - \frac{d^2 f(t)}{dt^2}\Big|_{t=0}$$

$$\frac{d^n f(t)}{dt^n} \Leftrightarrow s^n F(s) - s^{n-1} f(t)\Big|_{t=0} -$$

$$s^{n-2}\frac{df(t)}{dt}\Big|_{t=0} - s^{n-3}\frac{d^2 f(t)}{dt^2}\Big|_{t=0} - \cdots - \frac{d^{n-1} f(t)}{dt^{n-1}}\Big|_{t=0}$$

Integration

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

Proof

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \int_0^\infty\left[\int_0^t f(\tau)d\tau\right]e^{-st}\,dt$$

We integrate this expression by parts

$$u = \int_0^t f(\tau)d\tau, \quad dv = e^{-st}dt \quad \Rightarrow \quad du = f(t)dt, \quad v = \frac{e^{-st}}{-s}$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{e^{-st}}{-s}\int_0^t f(\tau)d\tau\Bigg|_{t=0}^{\infty} + \underbrace{\frac{1}{s}\int_0^\infty f(t)e^{-st}dt}_{F(s)}$$

$$= -[0 - 0] + \frac{1}{s}F(s)$$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

EXAMPLE 7.8 **Illustration of the integration property**

Consider the following relationship, for $t > 0$

$$\int_0^t u(\tau)d\tau = \tau\Big|_0^t = t$$

The Laplace transform of the unit step function is $1/s$ $\mathscr{L}[u(t)] = \frac{1}{s}, \quad \mathrm{Re}(s) > 0$

$$\mathscr{L}[t] = \mathscr{L}\left[\int_0^t u(\tau)\,d\tau\right] = \frac{1}{s}\mathscr{L}[u(t)] = \frac{1}{s}\frac{1}{s} = \frac{1}{s^2}$$

which is the Laplace transform of $f(t) = t$

find the Laplace transforms of the following integrals:

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

$$\mathcal{L}\left\{\int_0^t \cos at\,dt\right\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\left\{\int_0^t e^{at}\cos bt\,dt\right\} = \frac{s-a}{(s-a)^2+b^2} = \frac{s-a}{s((s-a)^2+b^2)}$$

$$\mathcal{L}\left\{\int_0^t te^{-3t}dt\right\} = \frac{1}{s} \times \frac{1}{(s+3)^2} = \frac{1}{s(s+3)^2}$$

Multiplication by t

$$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$$

Proof (See the book)

EXAMPLE 7.9 **Illustration of the multiplication-by-t property**

We now derive the transform of $f(t) = t\cos t$

Since $$\mathscr{L}[\cos bt] = \frac{s}{s^2 + b^2}$$

$$\mathscr{L}(t\cos bt] = -\frac{d}{ds}\left[\frac{s}{s^2 + b^2}\right] = -\frac{(s^2 + b^2)(1) - s(2s)}{(s^2 + b^2)^2} = \frac{s^2 - b^2}{(s^2 + b^2)^2}$$

Two more examples

$$\mathscr{L}[t^2] = -\frac{d}{ds}\mathscr{L}[t] = -\frac{d}{ds}\left[\frac{1}{s^2}\right] = \frac{2}{s^3} = \frac{2!}{s^3}$$

$$\mathscr{L}[t^3] = -\frac{d}{ds}\left[\frac{2}{s^3}\right] = \frac{2\cdot 3}{s^4} = \frac{3!}{s^4}$$

TABLE 7.2 Laplace Transforms

$f(t), t \geqq 0$	$F(s)$	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
3. t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
4. t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
5. e^{-at}	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
6. te^{-at}	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -a$
7. $t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}(s) > -a$
8. $\sin bt$	$\frac{b}{s^2+b^2}$	$\text{Re}(s) > 0$
9. $\cos bt$	$\frac{s}{s^2+b^2}$	$\text{Re}(s) > 0$
10. $e^{-at} \sin bt$	$\frac{b}{(s+a)^2+b^2}$	$\text{Re}(s) > -a$
11. $e^{-at} \cos bt$	$\frac{s+a}{(s+a)^2+b^2}$	$\text{Re}(s) > -a$
12. $t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$	$\text{Re}(s) > 0$
13. $t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$\text{Re}(s) > 0$

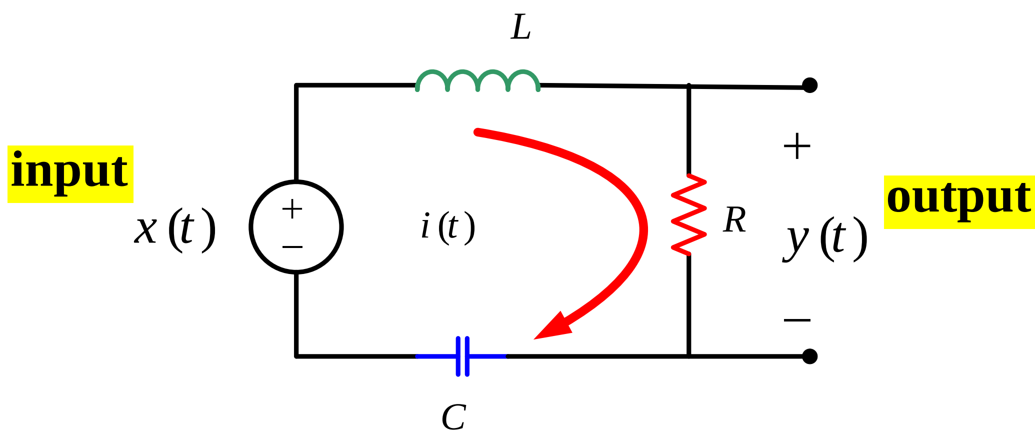
TABLE 7.3 Laplace Transform Properties

Name	Property
1. Linearity, (7.10)	$\mathscr{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$
2. Derivative, (7.15)	$\mathscr{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0^+)$
3. nth-order derivative, (7.29)	$\mathscr{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0^+)$ $- \cdots - s f^{(n-2)}(0^+) - f^{(n-1)}(0^+)$
4. Integral, (7.31)	$\mathscr{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$
5. Real shifting, (7.22)	$\mathscr{L}[f(t - t_0)u(t - t_0)] = e^{-t_0 s} F(s)$
6. Complex shifting, (7.20)	$\mathscr{L}[e^{-at} f(t)] = F(s + a)$
7. Initial value, (7.36)	$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$
8. Final value, (7.39)	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\, F(s)$
9. Multiplication by t, (7.34)	$\mathscr{L}[t f(t)] = -\frac{dF(s)}{ds}$
10. Time transformation, (7.42) ($a > 0; b \geqq 0$)	$\mathscr{L}[f(at - b)u(at - b)] = \frac{e^{-sb/a}}{a} F\left(\frac{s}{a}\right)$
11. Convolution	$\mathscr{L}^{-1}[F_1(s)F_2(s)] = \int_0^t f_1(t - \tau) f_2(\tau)\, d\tau$ $= \int_0^t f_1(\tau) f_2(t - \tau) d\tau$
12. Time periodicity	$\mathscr{L}[f(t)] = \frac{1}{1 - e^{-sT}} F_1(s)$, where
$[f(t) = f(t + T)], t \geqq 0$	$F_1(s) = \int_0^T f(t) e^{-st}\, dt$

7.6 **Response of LTI Systems**

Transfer Functions

Consider the following circuit

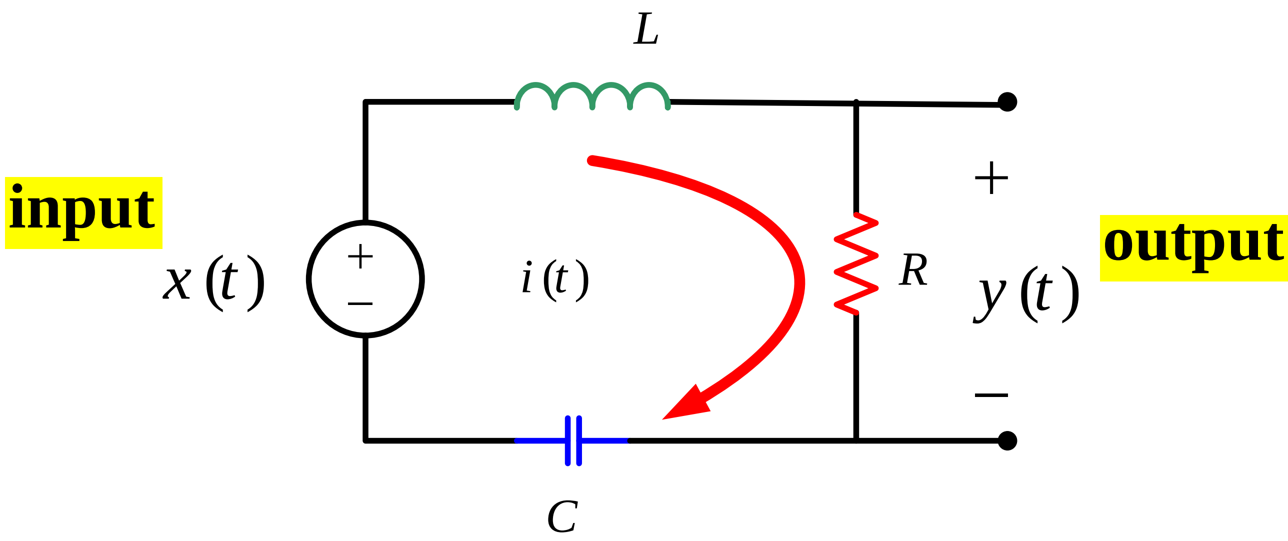


We want a relation (an equation) between the input *x*(t) and output *y*(t)

KVL

$$x(t) = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$$

$$\frac{dx(t)}{dt} = L\frac{di^2(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C}$$

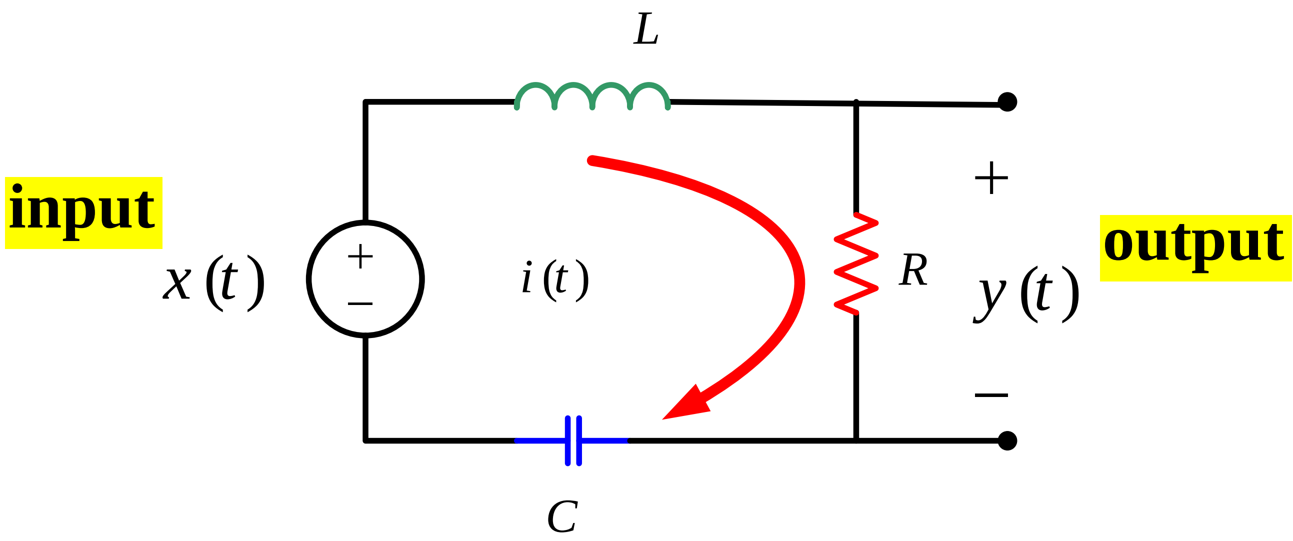


$$\frac{dx(t)}{dt} = L\frac{di^{2}(t)}{dt^{2}} + R\frac{di(t)}{dt} + \frac{i(t)}{C}$$

Since $i(t) = \dfrac{y(t)}{R}$ $\Rightarrow$ $\dfrac{dx(t)}{dt} = \dfrac{L}{R}\dfrac{dy^{2}(t)}{dt^{2}} + R\dfrac{dy(t)}{dt} + \dfrac{y(t)}{RC}$

Writing the differential equation as

$$RC\frac{dx(t)}{dt} = LC\frac{dy^{2}(t)}{dt^{2}} + RC\frac{dy(t)}{dt} + y(t)$$



$$\underbrace{\{RC\}}\frac{dx(t)}{dt} = \underbrace{\{LC\}}\frac{dy^2(t)}{dt^2} + \underbrace{\{RC\}}\frac{dy(t)}{dt} + \underbrace{\{1\}}y(t)$$

Real coefficients, **non negative** which results from system components R, L, C

In general,

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1}\frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{dx^m(t)}{dt^m} + b_{m-1}\frac{dx^{m-1}(t)}{dt^{m-1}} + \cdots + b_0 y(t)$$

were a_n's , b_m's **are real**, **non negative** which results from system components R, L, C

Now if we take the Laplace Transform of both side (Assuming Zero initial Conditions)

$$a_n s^n Y(s) + a_{n-1}s^{n-1}Y(s) + \cdots + a_0 Y(s) = b_m s^m X(s) + b_{m-1}s^{m-1}X(s) + \cdots + b_0 X(s)$$

We now define the transfer function $H(s)$,

$$H(s) \triangleq \left.\frac{Y(s)}{X(s)}\right|_{\text{all initial conditions are zero}} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$$a_n \frac{dy^n(t)}{dt^n} + a_{n-1} \frac{dy^{n-1}(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{dx^m(t)}{dt^m} + b_{m-1} \frac{dx^{m-1}(t)}{dt^{m-1}} + \cdots + b_0 x(t)$$

$$H(s) \triangleq \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \triangleq \frac{N(s)}{D(s)}$$

Since a_n's , b_m's **are real, non negative**

The roots of the polynomials $N(s)$, $D(s)$ are either real or occur in complex conjugate

The roots of $N(s)$ are referred to as the zero of $H(s)$ ($H(s) = 0$)

The roots of $D(s)$ are referred to as the pole of $H(s)$ ($H(s) = \pm\infty$)

$$H(s) \triangleq \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \triangleq \frac{N(s)}{D(s)}$$

The Degree of $N(s)$ (which is related to input) must be less than or Equal of $D(s)$ (which is related to output) for the system to be Bounded-input, bounded-output (**BIBO**)

Example : $$H(s) = \frac{4s^3 + 2s^2 + s + 1}{s^2 + 6s + 8}$$

Using polynomial division , we obtain $$H(s) = 4s + 2 + \frac{-19s + 17}{s^2 + 6s + 8}$$

Now assume the input $x(\text{t}) = u(\text{t})$ (**bounded input**) $\Rightarrow X(s) = \frac{1}{s}$

$$Y(s) = X(s)H(s) = 4 + \frac{2}{s} + \frac{1}{s}\left(\frac{-19s + 17}{s^2 + 6s + 8}\right)$$

$$\Rightarrow y(t) = \underbrace{4\delta(t)}_{\text{unbounded } (\to \infty)} + 2 + L^{-1}\left(\frac{-19s + 17}{s(s^2 + 6s + 8)}\right)$$

We see that for finite bounded Input (i.e x(t) =u(t))

We get an infinite (unbounded) output

$\Rightarrow$ $m \le n$ for **BIBO**

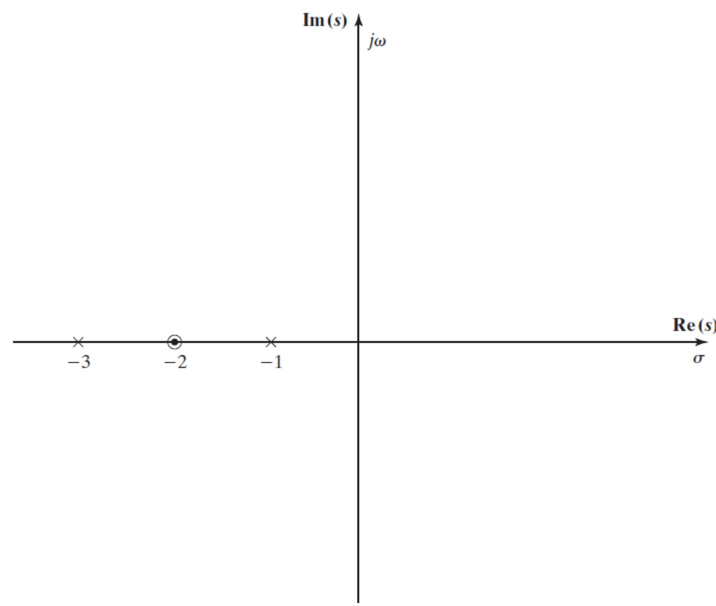
$$H(s) \triangleq \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \triangleq \frac{N(s)}{D(s)}$$

The poles of $H(s)$ must have real parts which are negative

→ The poles must lie in the left half of the s-plan

EXAMPLE 7.13 **Poles and zeros of a transfer function**

$$H(s) = \frac{4s + 8}{2s^2 + 8s + 6} = \frac{2(s + 2)}{(s + 1)(s + 3)}$$



Convolution

$$h(t)*x(t) \Rightarrow H(s)X(s)$$

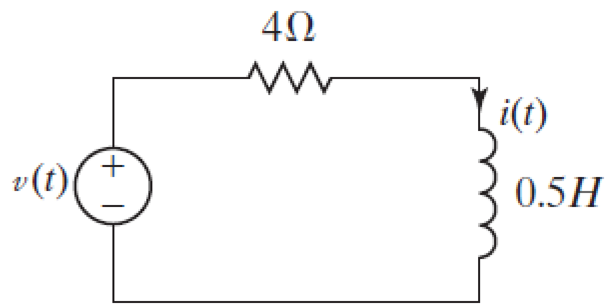
$$\mathscr{L}[x(t)*h(t)] = H(s)X(s)$$

Proof (See the book)

(It is very similar to the Fourier Transform Property)

EXAMPLE 7.12 LTI system response using Laplace transforms

Consider again the RL circuit



The loop equation (**KVL**) for this circuit is given by $$0.5\frac{di(t)}{dt} + 4i(t) = v(t)$$

The Laplace transform of the loop equation (ignoring initial conditions) is given by

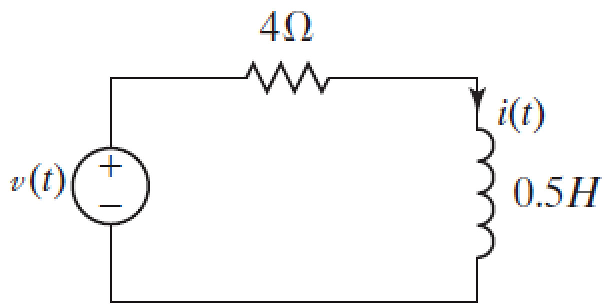
$$(0.5s + 4)I(s) = V(s)$$

We define the circuit input to be the voltage $v(t)$ and the output to be the current $i(t)$

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{0.5s + 4}$$

If $v(t) = 12u(t)$,find $i(t)$?

$$I(s) = H(s)V(s) = \frac{1}{0.5s + 4}\frac{12}{s} = \frac{24}{s(s + 8)}$$



$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{0.5s + 4}$$

$v(t) = 12u(t)$

$$I(s) = \frac{24}{s(s + 8)}$$

Now using the Laplace Table 7.2 to find *i*(t)

The partial-fraction expansion of $I(s)$ is then

$$I(s) = \frac{24}{s(s + 8)} = \frac{k_1}{s} + \frac{k_2}{s + 8}$$

$$k_1 = s\left[\frac{24}{s(s + 8)}\right]_{s=0} = \frac{24}{s + 8}\bigg|_{s=0} = 3$$

$$k_2 = (s + 8)\left[\frac{24}{s(s + 8)}\right]_{s=-8} = \frac{24}{s}\bigg|_{s=-8} = -3$$

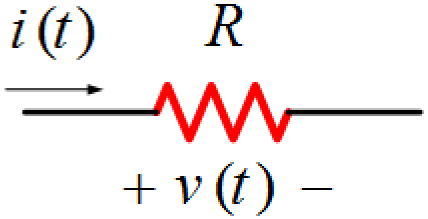
$$I(s) = \frac{24}{s(s + 8)} = \frac{3}{s} + \frac{-3}{s + 8}$$

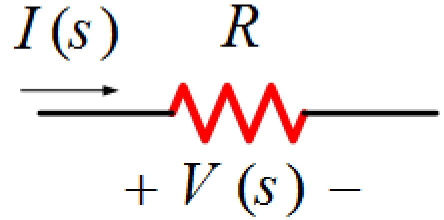
The inverse transform, from **Table 7.2**

3

$-3e^{-8t}$

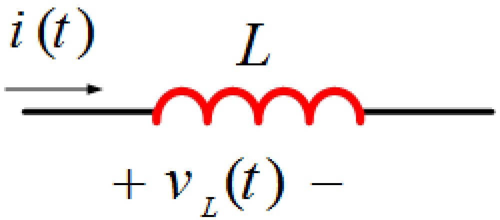
$$i(t) = 3[1 - e^{-8t}]$$

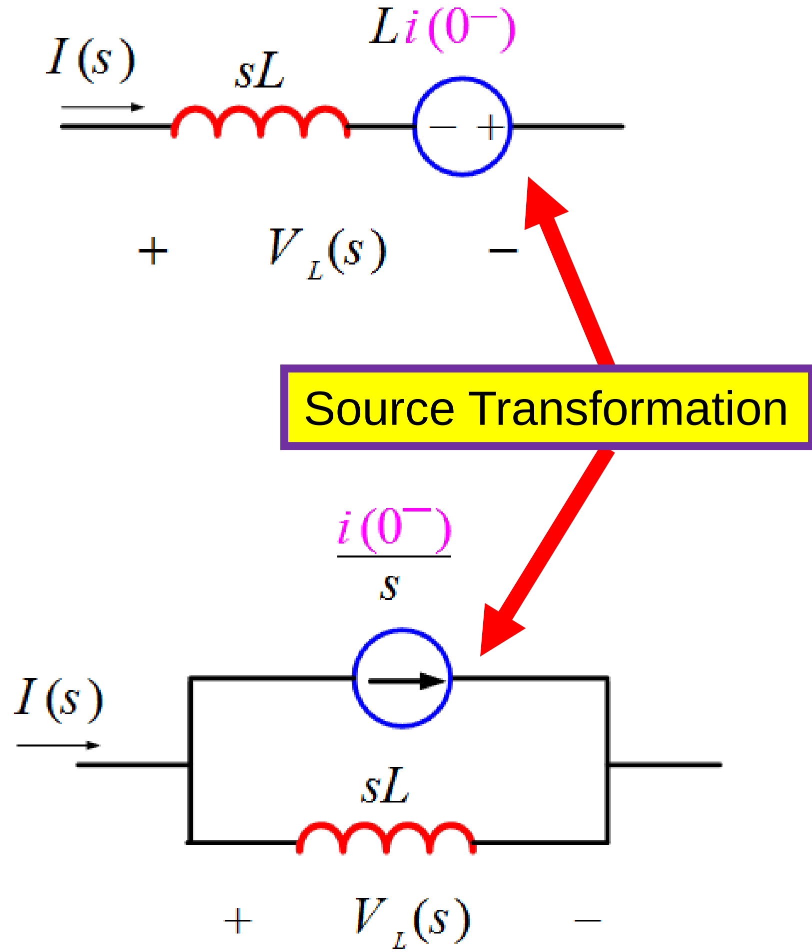




$$v(t)=Ri(t) \Leftrightarrow V(s)=RI(s)$$

$$\boxed{Z_R=R\ \Omega}$$



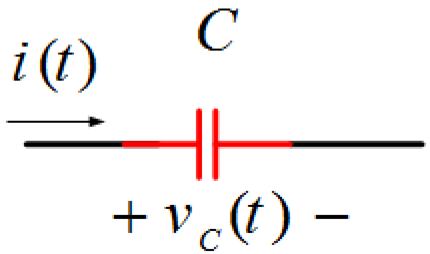


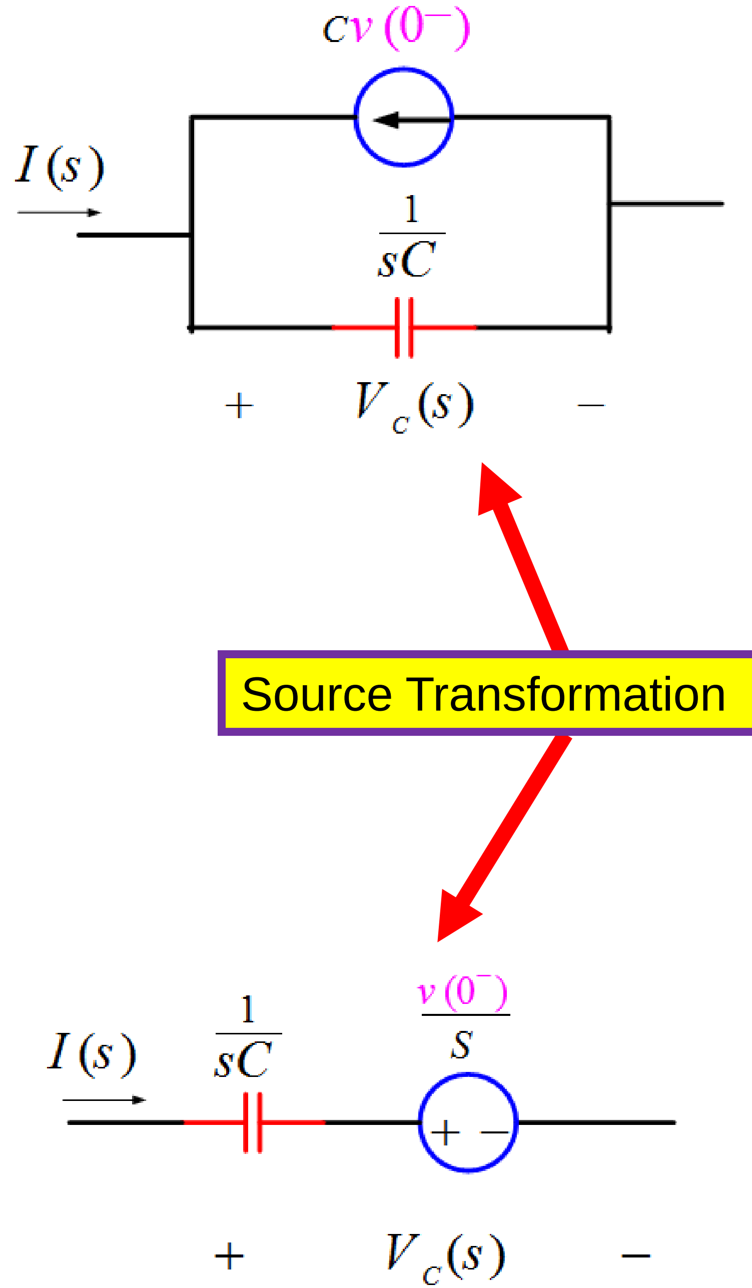
$$v_L(t)=L\frac{di(t)}{dt} \Leftrightarrow V_L(s)=L[sI(s)-i(0^-)]$$

$$= sLI(s)-Li(0^-)$$

$$\boxed{Z_L=sL\ \Omega}$$

$$I(s)=\frac{1}{sL}V_L(s)+\frac{i(0^-)}{s}$$





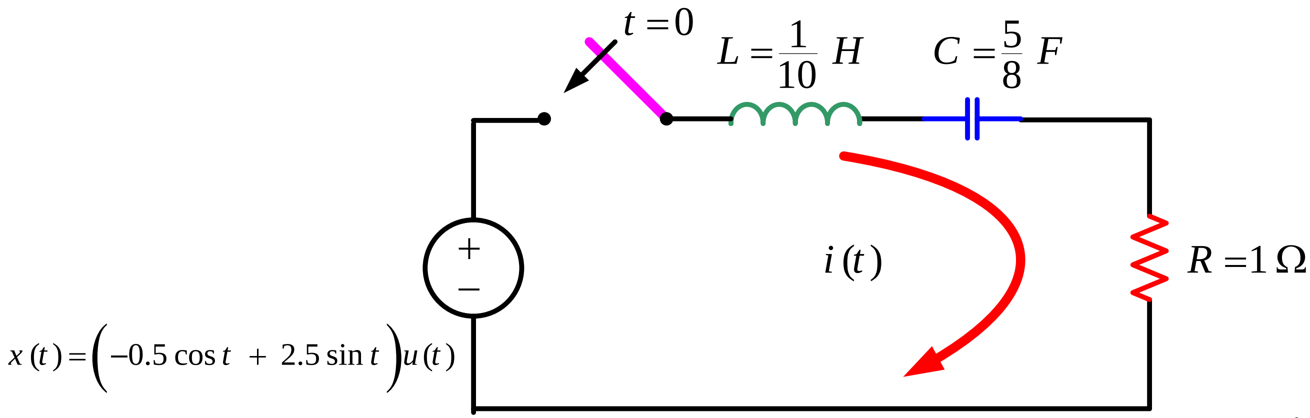
$$i(t) = C\frac{dv_C(t)}{dt} \Leftrightarrow I(s) = C[sV_C(s) - v(0^-)]$$

$$= sCV_C(s) - Cv(0^-)$$

$$Z_C = \frac{1}{sC}\ \Omega$$

$$V_C(s) = \frac{1}{sC}I(s) - \frac{v(0^-)}{s}$$

Example Using Laplace Transform Method find *i*(t)



$$i_L(0^-)=\ 0\ V$$

$$v_C(0^-)=\ -2\ V$$

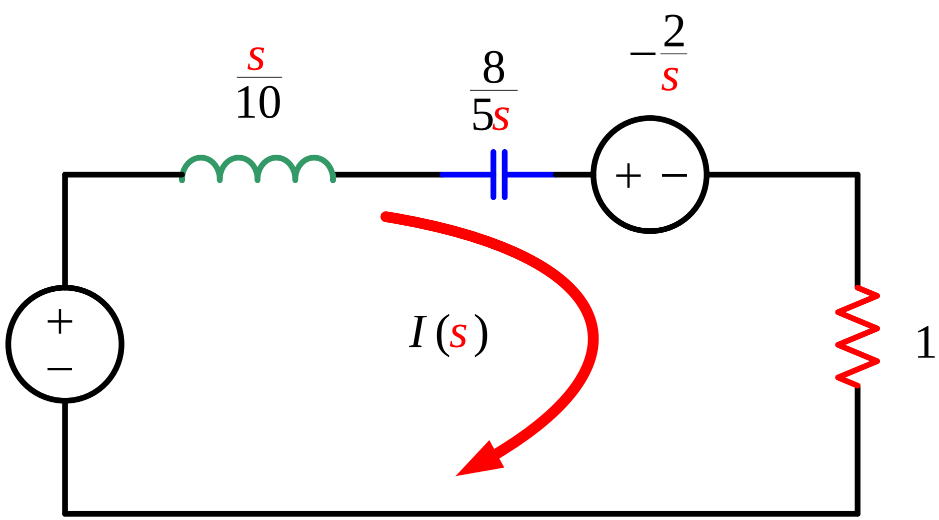
$$x(t)\ \Rightarrow\ X(s)=-\frac{0.5s}{s^2+1}+\frac{2.5s}{s^2+1}$$

$$L=\frac{1}{10}\ H\ \Rightarrow\ Z_L=\frac{s}{10}\ \Omega$$

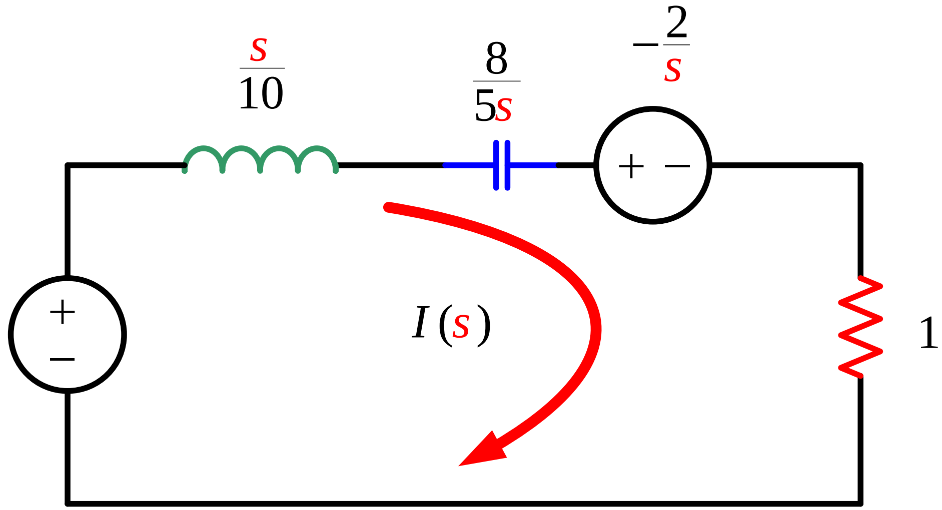
$$C=\frac{5}{8}\ F\ \Rightarrow\ Z_C=\frac{8}{5s}\ \Omega$$

$$R=1\,\Omega\ \Rightarrow\ Z_R=1\,\Omega$$

$$X(s)=-\frac{0.5s}{s^2+1}+\frac{2.5s}{s^2+1}$$



$$X(s)=-\frac{0.5s}{s^2+1}+\frac{2.5s}{s^2+1}$$



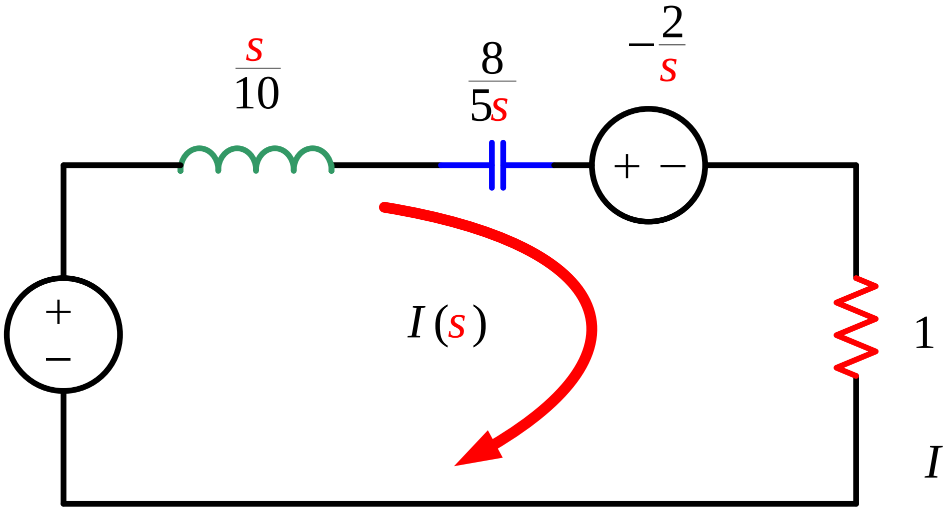
KVL

$$-X(s)+\left(\frac{s}{10}\right)I(s)+\left(\frac{8}{5s}\right)I(s)-\frac{2}{s}+(1)I(s)=0$$

$$-\frac{0.5s}{s^2+1}+\frac{2.5s}{s^2+1}=\left(\frac{s}{10}+\frac{8}{5s}+1\right)I(s)-\frac{2}{s}$$

$$I(s)=\frac{15s^2+25s+20}{(s^2+1)(s^2+10s+16)}$$

$$X(s) = -\frac{0.5s}{s^2+1} + \frac{2.5s}{s^2+1}$$



$$I(s) = \frac{15s^2+25s+20}{(s^2+1)(s^2+10s+16)}$$

$$I(s) = \frac{15s^2+25s+20}{(s^2+1)(s^2+10s+16)} = \frac{(15s^2+25s+20)}{(s+j)(s-j)(s+2)(s+8)}$$

$$= \frac{A_1}{(s+j)} + \frac{A_2}{(s-j)} + \frac{A_3}{(s+2)} + \frac{A_4}{(s+8)}$$

(Imaginary Roots)

$$i(t) = \left(\cos(t) + \sin(t) + e^{-2t} + e^{-8t}\right) u(t)$$

Inversion of Rational Function (Inverse Laplace Transform)

Let $Y(s)$ be Laplace Transform of some function $y(t)$.

We want to find $y(t)$ without using the inversion formula .

➡ We want to find $y(t)$ using the Laplace Transform known table and properties

Objective : Put $Y(s)$ in a form or a sum of forms that we know it is in the Laplace Transform Table

$Y(s)$ in general is a ratio of two polynomials ➡ Rational Function

Example $Y(s) = \dfrac{s^2 - 2s}{2s^3 - 5s^2 + 3s + 2}$

When the degree of the numerator of rational function is less the Degree of the dominator

⇨ *Proper Rational Function*

Highest Degree is **2**

$$\text{Example} \quad Y(s) = \frac{s^2 - 2s}{2s^3 - 5s^2 + 3s + 2}$$

Highest Degree is **3**

Examples of proper rational Functions

$$Y_1(s) = \frac{1}{s+1}$$

$$Y_2(s) = \frac{2s^2+6s+6}{(s+2)(s^2+2s^2+2)}$$

Examples of not proper rational Functions

$$Y_3(s) = \frac{s+2}{s+1}$$

However we can obtain a proper rational Function through long division

$$Y_3(s) = \frac{s+2}{s+1} = 1 + \frac{1}{s+1}$$

We will discuss different techniques of factoring $Y(s)$ into simple known forms

Simple Factors

$$\text{Let } Y(s) = \frac{10}{(s^2 + 10s + 16)}$$

If we check the Table , we see there is no form similar to $Y(s)$

However if we expand $Y(s)$ in partial fractions:

$$\frac{10}{(s^2 + 10s + 16)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

$\frac{A}{(s+2)}$ and $\frac{B}{(s+8)}$ Are available on the Table

Next we develop Techniques of finding A and B

Heavisdie's Expansion Theorem

$$Y(s) = \frac{10}{(s^2+10s+16)} = \frac{10}{(s+2)(s+8)} = \frac{A}{(s+2)} + \frac{B}{(s+8)}$$

Multiply (X) both side by $(s+2)$ and set $s = -2$

$$\left.\frac{10}{(s+2)(s+8)} \times (s+2)\right|_{s=-2} = \left.\frac{A}{(s+2)} \times (s+2)\right|_{s=-2} + \left.\frac{B}{(s+8)} \times (s+2)\right|_{s=-2}$$

$$\Rightarrow \frac{10}{(-2+8)} = A + \frac{B(-2+2)}{(-2+8)}$$

$$\Rightarrow \frac{10}{(6)} = A + 0 \Rightarrow A = \frac{5}{3}$$

Similarly Multiply both side by $(s+8)$ and set $s = -8$

$$\left.\frac{10}{(s+2)(s+8)} \times (s+8)\right|_{s=-8} = \left.\frac{A}{(s+2)} \times (s+8)\right|_{s=-8} + \left.\frac{B}{(s+8)} \times (s+8)\right|_{s=-8}$$

$$\Rightarrow B = -\frac{5}{3}$$

$$\Rightarrow y(t) = \frac{5}{3}e^{-2t}u(t) - \frac{5}{3}e^{-8t}u(t) = \frac{5}{3}\left(e^{-2t} - e^{-8t}\right)u(t)$$

Imaginary Roots

$$\text{Let } Y(s) = \frac{(15s^2+25s+20)}{(s^2+1)(s^2+10s+16)} = \frac{(15s^2+25s+20)}{(s+j)(s-j)(s+2)(s+8)}$$

$$= \frac{A_1}{(s+j)} + \frac{A_2}{(s-j)} + \frac{A_3}{(s+2)} + \frac{A_4}{(s+8)}$$

Using Heavisdie's Expansions, by multiplying the left hand side and Right hand side by the factors

$$(s+j)\ ,\ (s-j)\ ,\ (s+2)\ ,\ (s+8)$$

and substitute $s=-j\ ,\ s=j\ ,\ s=-2\ ,\ s=-8$ respectively

We obtain $A_1 = \frac{1}{2}(1+j)\ ,\ A_2 = \frac{1}{2}(1-j)\ ,\ A_3 = 1\ ,\ A_4 = -2$

From Table

$$e^{-2t}u(t) \qquad e^{-8t}u(t)$$

$$Y(s) = \frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)} + \frac{1}{(s+2)} - \frac{2}{(s+8)}$$

$$\frac{(1/2)(1+j)}{(s+j)} \text{ and } \frac{(1/2)(1-j)}{(s-j)}$$

Can be inverted in two methods:

(a)

$$\frac{(1/2)(1+j)}{(s+j)} \Rightarrow (1/2)(1+j)\,e^{-jt}u(t)$$

$$\frac{(1/2)(1-j)}{(s-j)} \Rightarrow (1/2)(1-j)\,e^{jt}u(t)$$

combine

$$\frac{(1/2)(1+j)}{(s+j)} \Rightarrow (1/2)(1+j)\, e^{-jt} u(t)$$

$$\frac{(1/2)(1-j)}{(s-j)} \Rightarrow (1/2)(1-j)\, e^{jt} u(t)$$

$$\frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)} \Rightarrow \frac{1}{2}(1+j)\, e^{-jt} u(t) + \frac{1}{2}(1-j)\, e^{jt} u(t)$$

$$= \frac{1}{2}(e^{jt} + e^{-jt})\, u(t) + \frac{j}{2}(e^{-jt} - e^{jt})\, u(t)$$

$$= \frac{1}{2}(e^{jt} + e^{-jt})\, u(t) + \frac{1}{2j}(e^{jt} - e^{-jt})\, u(t)$$

$$= \cos(t)\, u(t) + \sin(t)\, u(t)$$

(b) $\frac{(1/2)(1+j)}{(s+j)}$ and $\frac{(1/2)(1-j)}{(s-j)}$ **Can be combined as**

$$\frac{(1/2)(1+j)}{(s+j)} + \frac{(1/2)(1-j)}{(s-j)}$$

$$= \frac{(1/2)(1+j)(s-j) + (1/2)(1-j)(s+j)}{(s+j)(s-j)}$$

$$= \frac{s+1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2+1}$$

$\Rightarrow \cos(t)\,u(t) + \sin(t)\,u(t)$

$$\underbrace{\cos(t)\,u(t)+\sin(t)\,u(t)} \qquad e^{-2t}u(t) \qquad e^{-8t}u(t)$$

$$Y(s)=\frac{(1/2)(1+j)}{(s+j)}+\frac{(1/2)(1-j)}{(s-j)}+\frac{1}{(s+2)}-\frac{2}{(s+8)}$$

$$y(t)=\cos(t)\,u(t)+\sin(t)\,u(t)+e^{-2t}u(t)+e^{-8t}u(t)$$

$$y(t)=\left(\cos(t)+\sin(t)+e^{-2t}+e^{-8t}\right)u(t)$$

Repeated Linear Factor

If $$Y(s) = \frac{P(s)}{(s+\alpha)^n Q(s)}$$ $\left\{ \text{example } Y(s) = \frac{10s}{(s+2)^2(s+8)} \right\}$

Then its partial fraction

$$Y(s) = \frac{A_1}{(s+\alpha)} + \frac{A_2}{(s+\alpha)^2} + \frac{A_2}{(s+\alpha)^3} + \cdots + \frac{A_n}{(s+\alpha)^n} + \frac{R(s)}{Q(s)}$$

Were

$$A_m = \frac{1}{(n-m)!} \frac{d^{(n-m)}}{ds^{(n-m)}} \left[(s+\alpha)^n Y(s) \right]_{s=-\alpha}$$

Repeated Linear Factor

Let $$Y(s) = \frac{10s}{(s+2)^2(s+8)}$$

Also A_2 can be found using **Heavisdie's** by multiplying both sides by $(s+2)^2$

B can be found using **Heavisdie's** by multiplying both sides by $(s+8)$

$n=2$, $m=1$

$n=2$, $m=2$

Then $$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{B}{(s+8)}$$

$$A_m = \frac{1}{(n-m)!}\frac{d^{(n-m)}}{ds^{(n-m)}}\left[(s+\alpha)^n Y(s)\right]_{s=-\alpha}$$

$$A_1 = \frac{1}{(2-1)!}\frac{d^{(2-1)}}{ds^{(2-1)}}\left[(s+2)^2\frac{10s}{(s+2)^2(s+8)}\right]_{s=-2} = \frac{d}{ds}\left[\frac{10s}{(s+8)}\right]_{s=-2}$$

$$= \left[\frac{10(s+8)-10s}{(s+8)^2}\right]_{s=-2} = \left[\frac{10(-2+8)-10(-2)}{(-2+8)^2}\right] = \left[\frac{80}{36}\right] = \frac{20}{9}$$

$$Y(s)=\frac{A_1}{(s+2)}+\frac{A_2}{(s+2)^2}+\frac{B}{(s+8)}$$

$A_1 \rightarrow \frac{20}{9}$, $A_2 \rightarrow -\frac{10}{3}$

$$A_2=\frac{1}{(2-2)!}\frac{d^{(2-2)}}{ds^{(2-2)}}\left[(s+2)^2\frac{10s}{(s+2)^2(s+8)}\right]_{s=-2}=\left[\frac{10s}{(s+8)}\right]_{s=-2}$$

$$=\left[\frac{10(-2)}{(-2+8)}\right]=\left[\frac{-20}{6}\right]=-\frac{10}{3}$$

$$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{B}{(s+8)}$$

$A_1 \rightarrow \frac{20}{9}$, $A_2 \rightarrow -\frac{10}{3}$, $B \rightarrow -\frac{20}{9}$

To find B , we use Heaviside

$$(s+8)\frac{10s}{(s+2)^2(s+8)}\bigg|_{s=-8} = \frac{A_1}{(s+2)}\text{X}(s+8)\bigg|_{s=-8} + \frac{A_2}{(s+2)^2}\text{X}(s+8)\bigg|_{s=-8} + \frac{B}{(s+8)}\text{X}(s+8)\bigg|_{s=-8}$$

(the first two terms on the right → 0)

$$\Rightarrow \quad B = -\frac{20}{9} \qquad \Rightarrow Y(s) = \frac{(20/9)}{(s+2)} - \frac{(10/3)}{(s+2)^2} - \frac{(20/9)}{(s+8)}$$

$$Y(s) = \frac{20}{9}\left[-\frac{1}{(s+8)} + \frac{1}{(s+2)} - \frac{(3/2)}{(s+2)^2} \right]$$

$$y(t) = \frac{20}{9}\left(-e^{-8t} + e^{-2t} - \frac{3}{2}te^{-2t} \right)u(t)$$

Repeated Linear Factor

Let $$Y(s) = \frac{10s}{(s+2)^3(s+8)}$$

Can be found using Heaviside's expansion

Then $$Y(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{B}{(s+8)}$$

$$B = (s+8)\,Y(s)\Big|_{s=-8} = \frac{10}{27}$$

$$A_3 = (s+2)^3\,Y(s)\Big|_{s=-2} = -\frac{10}{3}$$

$$A_2 = \frac{1}{(3-2)!}\frac{d^{(3-2)}}{ds^{(3-2)}}\left[(s+2)^3 Y(s)\right]_{s=-2} = \frac{d}{ds}\left[\frac{10s}{(s+8)}\right]_{s=-2} = \frac{20}{9}$$

$$Y(s)=\frac{A_1}{(s+2)}+\frac{A_2}{(s+2)^2}+\frac{A_3}{(s+2)^3}+\frac{B}{(s+8)}$$

$A_1 \to -\frac{20}{27}$, $A_2 \to \frac{20}{9}$, $A_3 \to -\frac{10}{3}$, $B \to \frac{10}{27}$

A_1 Can be found using Heaviside differentiation techniques

$$A_1=\frac{1}{(3-1)!}\frac{d^{(3-1)}}{ds^{(3-1)}}\left[(s+2)^3 Y(s)\right]_{s=-2}=\frac{1}{2}\frac{d^2}{ds^2}\left[\frac{10s}{(s+8)}\right]_{s=-2}$$

$$=\frac{1}{2}\left[\frac{-160}{(s+8)^3}\right]_{s=-2}\qquad=-\frac{20}{27}$$

$$Y(s)=\frac{10}{27}\left[\frac{1}{(s+8)}-\frac{1}{(s+2)}\right]+\frac{20}{9}\frac{1}{(s+2)^2}-\frac{10}{3}\frac{1}{(s+2)^3}$$

$$\Rightarrow y(t)=\left[\frac{10}{27}\left(e^{-8t}-e^{-2t}\right)-\frac{5}{3}t\left(t-\frac{4}{3}\right)e^{-2t}\right]u(t)$$