Frequency shifting

If $x(t) \longleftrightarrow X(\omega)$

$$x(t)\,e^{j\omega_0 t} \longleftrightarrow X(\omega-\omega_0)$$

Proof:

$$F.T^{-1}\left[X(\omega-\omega_0)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\omega_0)\,e^{j\omega t}\,d\omega$$

Let $\omega^* = \omega - \omega_0 \Rightarrow \omega = \omega^* + \omega_0$

$$d\omega = d\omega^*$$

$$F.T\left[X(\omega-\omega_0)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega^*)\,e^{j(\omega^*+\omega_0)t}\,d\omega^*$$

$$= e^{j\omega_0 t}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega^*)\,e^{j\omega^* t}\,d\omega^*$$

$$= e^{j\omega_0 t}\,X(\omega^*)$$

Example 5.9

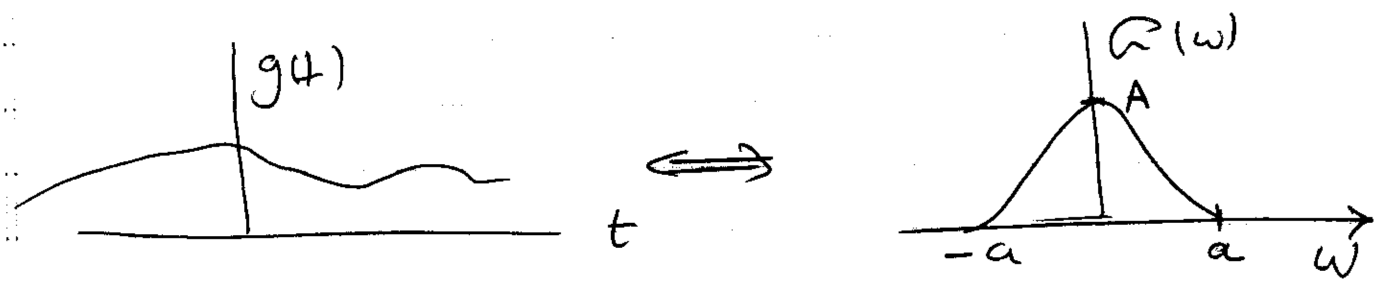
In Communication, a mean of transmitting information say speech, is by multiplying the infomation signal by a carrier ~~with freq~~ as

$$g(t) \cdot \cos(w_c t)$$

were $\cos(w_c t)$ is the carrier signal and w_c is the carrier frequency

This technique is called Amplitude Modulation or (AM) (EE 370 will cover it in detail).

let, $g(t) \leftrightarrow G(\omega)$ be as shown



Now let $s(t) = g(t) \cdot \cos(w_c t)$ (AM)

how $S(\omega)$ will ~~be~~ look like?

$$s(t) = g(t) \cdot \cos(w_c t)$$

$$= g(t) \tfrac{1}{2}\left[e^{jw_c t} + e^{-jw_c t}\right]$$

$$= \tfrac{1}{2} g(t) e^{jw_c t} + \tfrac{1}{2} g(t) e^{-jw_c t}$$

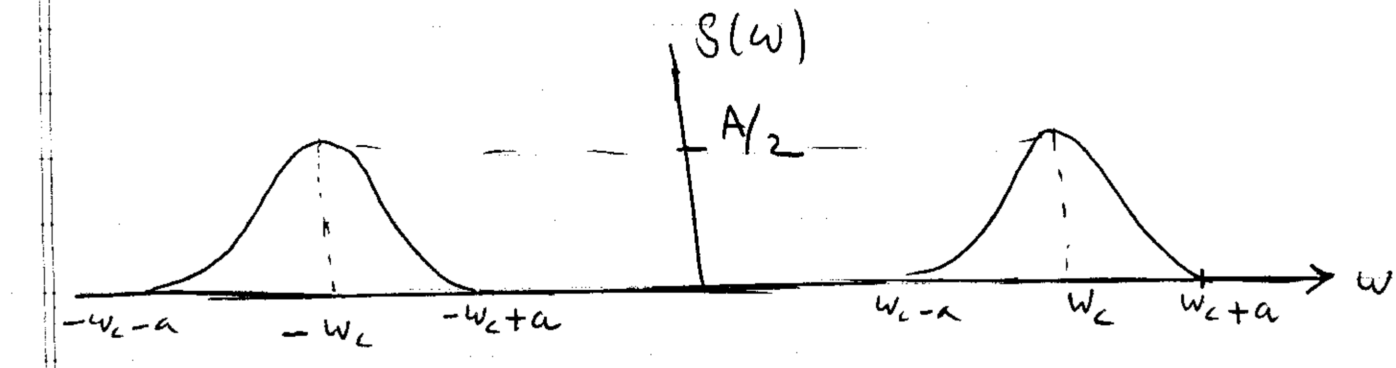
Now using the linearty propertey,

$$S(\omega) = F.T\{s(t)\} = \tfrac{1}{2} F.T\{g(t) e^{jw_c t}\}$$

$$+ \tfrac{1}{2} F.T\{g(t) e^{-jw_c t}\}$$

~~S(w)~~ Now using the frequency shift property,

$$S(\omega) = \tfrac{1}{2} G(\omega - w_c) + \tfrac{1}{2} G(\omega + w_c)$$



~~See Example~~ see Example 5.9

Time Differentiation

If $\quad f(t) \longleftrightarrow F(\omega)$

then $\quad \dfrac{d}{dt} f(t) \longleftrightarrow j\omega\, F(\omega)$

In general,

$$\frac{d^n f(t)}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$$

Proof

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d}{dt} f(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, \frac{d}{dt}\left(e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, (j\omega)\, e^{j\omega t}\, d\omega$$

$$= j\omega\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega\right]$$

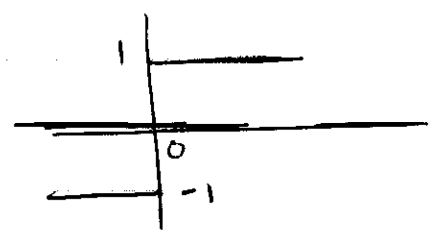
$$= j\omega\, F(\omega)$$

Time differentiation propirtey show that differention in time becom multiplication in Frequency

Differential Equation $\Longleftrightarrow$ Algebric Equation

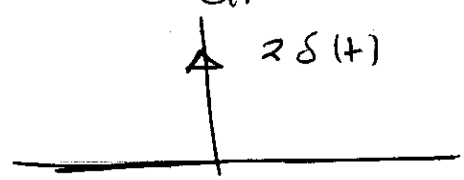
Example 5.10

$f(t) = \text{sgn}(t)$



Find $F(\omega)$?

$\frac{d}{dt}\text{sgn}(t) = 2\delta(t)$

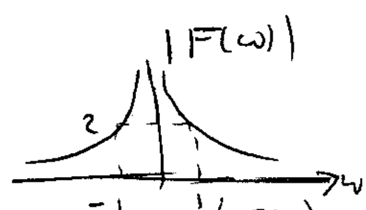


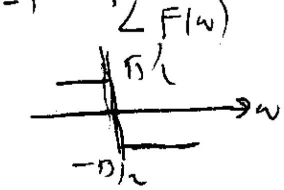
Solution

$$\frac{df(t)}{dt} = 2\delta(t)$$

$$j\omega F(\omega) = 2$$

$$\Rightarrow \quad F(\omega) = \frac{2}{j\omega}$$





$$\cancel{\text{sgn}(\omega)} \quad \text{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

(Entry #7 Table 5.2)

Fourier of a step function

$$u(t) \longleftrightarrow \cancel{F(\omega)}\ U(\omega)$$

We can write a step function $u(t)$ in terms of $\mathrm{sgn}(t)$ as

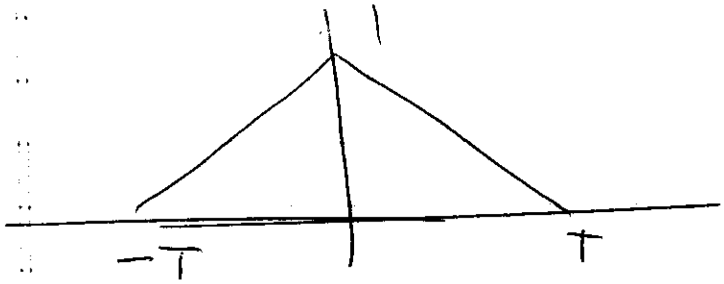
$$u(t) = \tfrac{1}{2}\left[1 + \mathrm{sgn}(t)\right]$$

$$= \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{sgn}(t)$$

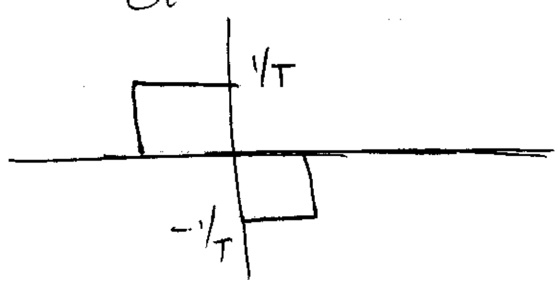
$$\Rightarrow \quad U(\omega) = \tfrac{1}{2}\left(2\pi\,\delta(\omega)\right) + \tfrac{1}{2}\left(\frac{2}{j\omega}\right)$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega} \qquad \text{(Entry \#5 Table 5.2)}$$

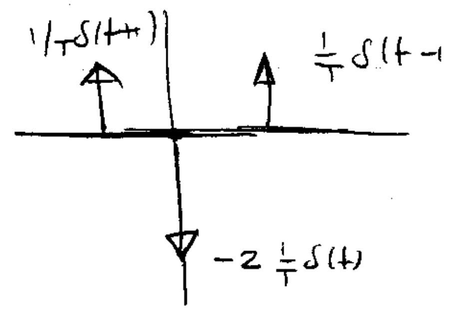
$f(t) = tr.(t/T)$



$\frac{df}{dt}$



$\frac{d^2 f(t)}{dt^2}$



$$\frac{d^2 f}{dt^2} = \frac{1}{T}\left[\delta(t+1) + \delta(t-1)\right] - \frac{2}{T}\delta(t)$$

$$(j\omega)^2 F(\omega) = \frac{1}{T}\left[e^{j\omega} + e^{-j\omega}\right] - 2$$

$$\Rightarrow \quad F(\omega) = \frac{1}{T}\,\frac{(e^{j\omega} + e^{-j\omega}) - 2}{(j\omega)^2}$$

$$= T\,\mathrm{sinc}^2(T\omega/2)$$ (entry 18 Table 5.1)

Time Integration

If $f(t) \longleftrightarrow F(\omega)$

then

$$g(t) = \int_{-\infty}^{t} f(\tau)\,d\tau \longleftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega) = G(\omega)$$

were $F(0) = F(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \Big|_{\omega=0}$

$$= \int_{-\infty}^{\infty} f(t)\,dt \quad \left(\begin{array}{c}\text{DC value} \\ \text{zero frequency}\end{array}\right)$$

Proof Since $f(t) * u(t) = \int_{-\infty}^{\infty} f(\tau)\, u(t-\tau)\, d\tau$

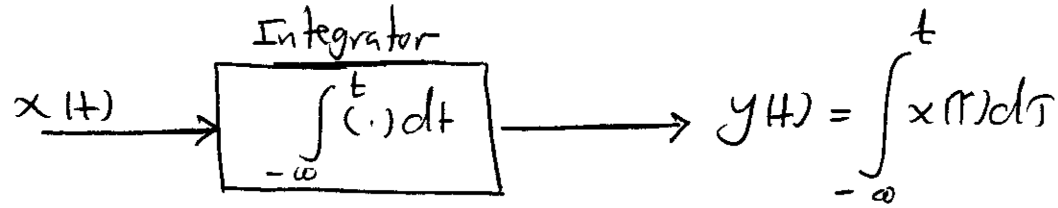
$$= \int_{-\infty}^{t} f(\tau)\,(1)\,d\tau = \int_{-\infty}^{t} f(\tau)\,d\tau$$

F.T of both side and using convolution property

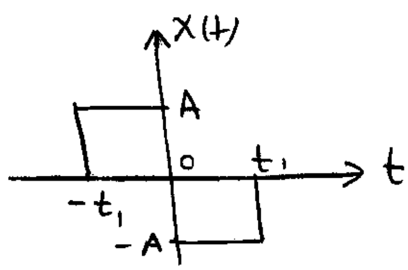
$$F(\omega) \cdot \left[\pi\delta(\omega) + \frac{1}{j\omega}\right] \leftrightarrow F.T\left[\int_{-\infty}^{t} f(\tau)\,d\tau\right]$$

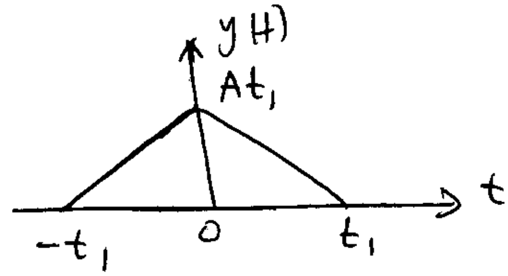
$$\Rightarrow \int_{-\infty}^{t} f(\tau)\,d\tau \longleftrightarrow \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$$

Example 5.12



If $x(t)$ is





Find $Y(\omega)$?
(using Integration property).

$$x(t) = A\,\mathrm{rect}\left[\frac{t + t_1/2}{t_1}\right] - A\,\mathrm{rect}\left[\frac{t - t_1/2}{t_1}\right]$$

$$\Rightarrow \quad X(\omega) = At_1 \,\mathrm{sinc}\left(\frac{t_1\omega}{2}\right)\left[e^{j\omega\frac{t_1}{2}} - e^{-j\omega\frac{t_1}{2}}\right]$$

$$= At_1 \,\mathrm{sinc}\left(\frac{t_1\omega}{2}\right)\left(2j \sin\left(\frac{\omega t_1}{2}\right)\right)$$

$$= j\omega A t_1^2 \,\mathrm{sinc}^2\left(\frac{t_1\omega}{2}\right)$$

Now $y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \Rightarrow Y(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$

$X(0) = X(\omega)\big|_{\omega=0} = 0$ or $\int_{-\infty}^{\infty} x(t)\,dt = 0 \Rightarrow Y(\omega) = \frac{1}{j\omega}X(\omega)$

$$\Rightarrow Y(\omega) = At_1^2 \,\mathrm{sinc}^2(t_1\omega/2)$$

Frequency Differentiation

If $\quad f(t) \Longleftrightarrow F(\omega)$

then,

$$(-jt)^n f(t) \longleftrightarrow \frac{d^n F(\omega)}{d\omega^n}$$

Proof

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, \frac{d}{d\omega}\left(e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{\infty} f(t)\, (-j\cancel{\omega} t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} (\cancel{f(t)}\,(-jt) f(t)\, e^{-j\omega t}\, dt$$

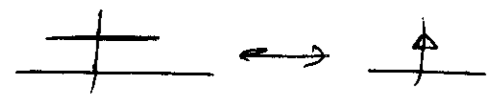
then $\quad (-jt)\, f(t) \Longleftrightarrow \dfrac{dF(\omega)}{d\omega}$

5.3 Fourier Transform of time function

DC level (both time and frequency)

$\cancel{1}$

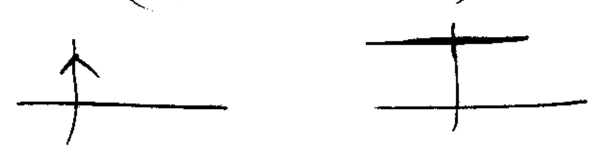
$$1 \longleftrightarrow 2\pi\,\delta(\omega)$$



$$1.\ e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$K \longleftrightarrow 2\pi k\,\delta(\omega)$$

$$\delta(t) \longleftrightarrow 1 \qquad \text{(by duality)}$$

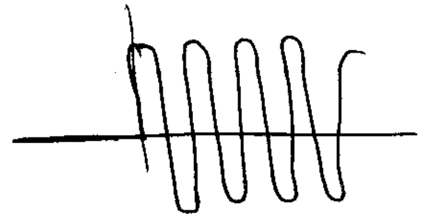


unite step

$$\cancel{u(t)}\ \mathrm{sgn}(t) \longleftrightarrow \frac{2}{j\omega}$$

$$u(t) = \tfrac{1}{2}[1+\mathrm{sgn}(t)] \longleftrightarrow \pi\,\delta(\omega) + \frac{1}{j\omega}$$

switched Cosine



$$f(t) = \cos(\omega_0 t)\, u(t)$$

$$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, u(t)$$

$$= \frac{1}{2} e^{j\omega_0 t} u(t) + \frac{1}{2} e^{-j\omega_0 t} u(t)$$

$$\frac{1}{2} e^{j\omega_0 t} u(t) \longleftrightarrow \frac{\pi}{2}\delta(\omega - \omega_0) + \frac{1}{2}\frac{1}{j(\omega-\omega_0)}$$

$$\frac{1}{2} e^{-j\omega_0 t} u(t) \longleftrightarrow \frac{\pi}{2}\delta(\omega + \omega_0) + \frac{1}{2}\frac{1}{j(\omega+\omega_0)}$$

Therefore

$$\cancel{\cos} \; \cos(\omega_0 t) u(t) \longleftrightarrow \frac{\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

$$+ \frac{1}{2}\frac{1}{j(\omega-\omega_0)} + \frac{1}{2}\frac{1}{j(\omega+\omega_0)}$$

$$\frac{j(\omega+\omega_0) + j(\omega - \omega_0)}{2j^2(\omega^2 - \omega_0^2)} = \frac{j\omega}{\omega_0^2 - \omega^2}$$

$$\Rightarrow \quad \cos(\omega_0 t) u(t) \longleftrightarrow \frac{\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

Pulse Cosing

$$f(t) = \text{rect}(t/T)\cos(\omega_0 t)$$

$$\text{rect}(t/T) \Leftrightarrow T\,\text{sinc}\left(\frac{\omega T}{2}\right) \qquad \cos(\omega_0 t) \rightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$\Rightarrow \quad F(\omega) = \frac{1}{2\pi}\left\{ T\,\text{sinc}\left(\frac{\omega T}{2}\right) * \left[\pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)\right]\right\}$$

$$= \frac{T}{2}\,\text{sinc}\left(\frac{\omega T}{2}\right) * \delta(\omega-\omega_0)$$

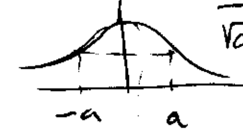
$$+ \frac{T}{2}\,\text{sinc}\left(\frac{\omega T}{2}\right) * \delta(\omega+\omega_0)$$

$$= \frac{T}{2}\,\text{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right) + \frac{T}{2}\,\text{sinc}\left(\frac{(\omega+\omega_0)T}{2}\right)$$

exponantial pulse

$$f(t) = e^{-at}u(t) \qquad \underline{\underline{a>0}}$$

$$F(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t}\,dt = \frac{1}{a+j\omega}$$



$$\frac{1}{\sqrt{a^2+\omega^2}}$$

$$\Downarrow \quad \frac{1}{a\sqrt{2}}$$

$$\angle F(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$