

(1)

Frequency shifting

$$\text{If } x(t) \leftrightarrow X(\omega)$$

$$x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Proof:

$$\text{F.T.}^{-1} [X(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega - \omega_0) e^{j\omega t} d\omega$$

$$\text{Let } \omega^* = \omega - \omega_0 \Rightarrow \omega = \omega^* + \omega_0$$

$$d\omega = d\omega^*$$

$$\text{F.T.} [X(\omega - \omega_0)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega^*) e^{j(\omega^* + \omega_0)t} d\omega^*$$

$$= e^{j\omega_0 t} \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega^*) e^{j\omega^* t} d\omega^*$$

$$= e^{j\omega_0 t} X(\omega^*)$$

Example 5.9

(2)

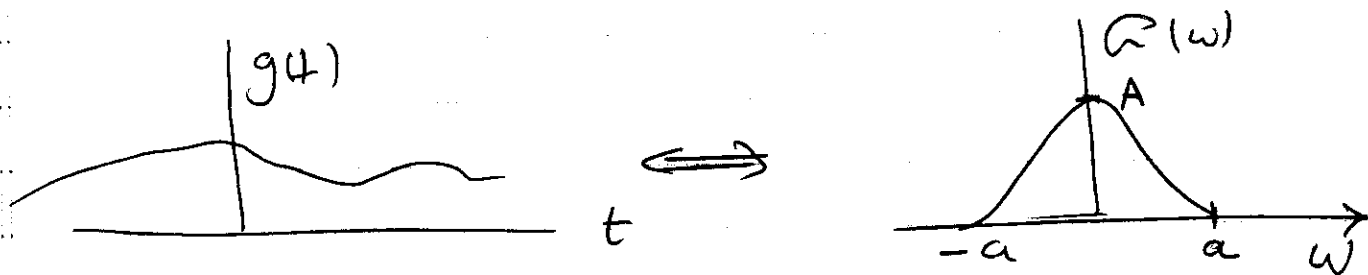
In communication, a mean of transmitting information say speech, is by multiplying the information signal by a carrier ~~with~~ as

$$g(t) \cdot \cos(\omega_c t)$$

where $\cos(\omega_c t)$ is the carrier signal and ω_c is the carrier frequency

This technique is called Amplitude Modulation or (AM) (EE 370 will cover it in detail).

Let, $g(t) \leftrightarrow G(\omega)$ be as shown



Now let $s(t) = g(t) \cdot \cos(\omega_c t)$ (AM)

how $S(\omega)$ will look like?

③

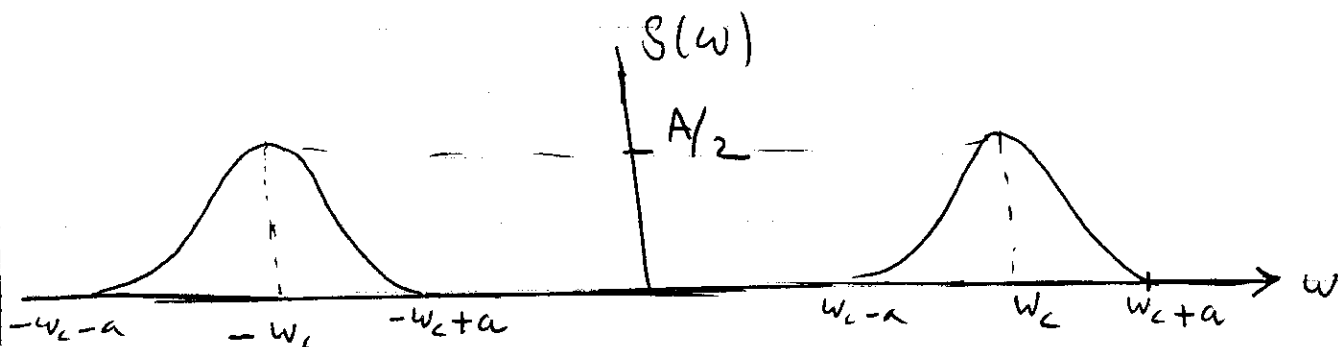
$$\begin{aligned} s(t) &= g(t) \cdot \cos(\omega_c t) \\ &= g(t) \frac{1}{2} [e^{j\omega_c t} + e^{-j\omega_c t}] \\ &= \frac{1}{2} g(t) e^{j\omega_c t} + \frac{1}{2} g(t) e^{-j\omega_c t} \end{aligned}$$

Now using the linearity property,

$$\begin{aligned} \mathcal{F}\{s(t)\} &= \mathcal{F}\left\{\frac{1}{2} g(t) e^{j\omega_c t} + \frac{1}{2} g(t) e^{-j\omega_c t}\right\} \\ &= \frac{1}{2} \mathcal{F}\{g(t) e^{j\omega_c t}\} + \frac{1}{2} \mathcal{F}\{g(t) e^{-j\omega_c t}\} \end{aligned}$$

~~$S(\omega)$~~ Now using the frequency shift property,

$$S(\omega) = \frac{1}{2} G(\omega - \omega_c) + \frac{1}{2} G(\omega + \omega_c)$$



See ~~Example~~ see Example 5.9

Time Differentiation

(4)

$$\text{If } f(t) \longleftrightarrow F(\omega)$$

$$\text{then } \frac{d}{dt} f(t) \longleftrightarrow j\omega F(\omega)$$

In general,

$$\frac{d^n f(t)}{dt^n} \longleftrightarrow (j\omega)^n F(\omega)$$

Proof

$$f(t) = \frac{1}{2\pi} \int_{-\omega}^{\omega} F(\omega) e^{j\omega t} d\omega$$

$$\frac{d}{dt} f(t) = \frac{d}{dt} \left[\frac{1}{2\pi} \int_{-\omega}^{\omega} F(\omega) e^{j\omega t} d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\omega}^{\omega} F(\omega) \frac{d}{dt} (e^{j\omega t}) d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega}^{\omega} F(\omega) (j\omega) e^{j\omega t} d\omega$$

$$= j\omega \left[\frac{1}{2\pi} \int_{-\omega}^{\omega} F(\omega) e^{j\omega t} d\omega \right]$$

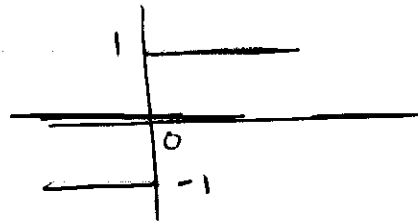
$$= j\omega F(\omega)$$

Time differentiation property show that (5)
 differentiation in time becomes multiplication in
 Frequency

~~Diff~~ Differential Equation \leftrightarrow Algebraic Equation

Example 5.10

$$f(t) = \text{sgn}(t)$$

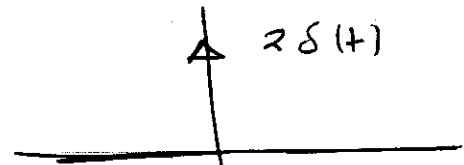


Find $F(\omega)$?

$$\frac{d}{dt} \text{sgn}(t) = 2\delta(t)$$

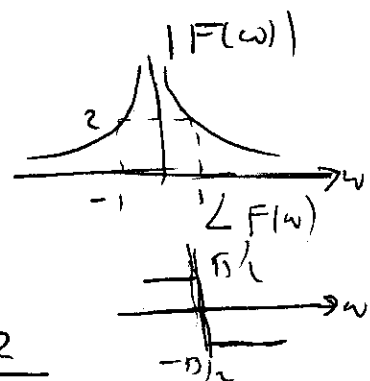
Solution

$$\frac{df(t)}{dt} = 2\delta(t)$$



$$j\omega F(\omega) = 2$$

$$\Rightarrow F(\omega) = \frac{2}{j\omega}$$



~~sgn(t)~~ $\text{sgn}(t) \leftrightarrow \frac{2}{j\omega}$

(Entry #7 Table 5.2)

6

Fourier of a step function

$$u(t) \longleftrightarrow \text{FT} \quad U(\omega)$$

We can write a step function $u(t)$ in terms of $\text{sgn}(t)$ as

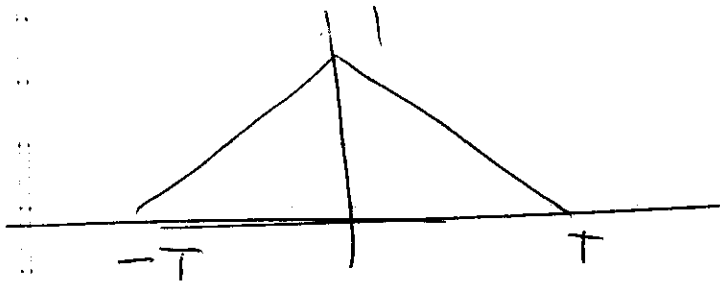
$$\begin{aligned} u(t) &= \frac{1}{2} [1 + \text{sgn}(t)] \\ &= \frac{1}{2} + \frac{1}{2} \text{sgn}(t) \end{aligned}$$

$$\begin{aligned} \Rightarrow U(\omega) &= \frac{1}{2} (2\pi \delta(\omega)) + \frac{1}{2} \left(\frac{2}{j\omega} \right) \\ &= \pi \delta(\omega) + \frac{1}{j\omega} \quad (\text{Entry \#5 Table 5.4}) \end{aligned}$$

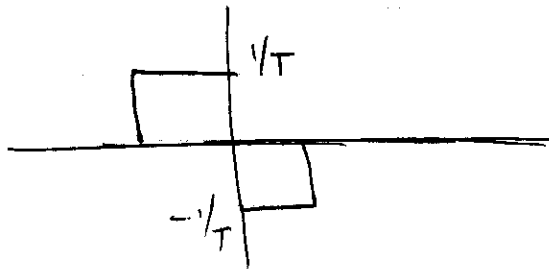
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$$f(t) = \text{tr.}(t/T)$$

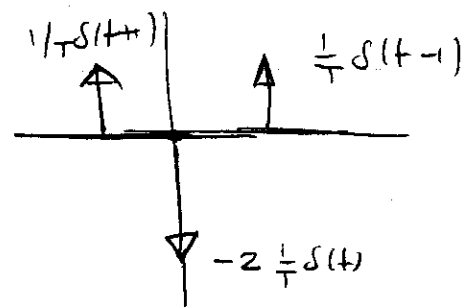
(7)



$$\frac{df}{dt}$$



$$\frac{d^2f(t)}{dt^2}$$



$$\frac{d^2f}{dt^2} = \frac{1}{T} [\delta(t+1) + \delta(t-1)] - \frac{2}{T} \delta(t)$$

$$(j\omega)^2 F(\omega) = \frac{1}{T} [e^{j\omega} + e^{-j\omega}] - 2$$

$$\Rightarrow F(\omega) = \frac{1}{T} \frac{(e^{j\omega} + e^{-j\omega}) - 2}{(j\omega)^2}$$

$$= T \text{sinc}^2(T\omega/2) \quad (\text{entry } \# \text{ Table 5.4})$$

Time Integration

(8)

$$\text{if } f(t) \leftrightarrow F(\omega)$$

then

$$g(t) = \int_{-\infty}^t f(\tau) d\tau \leftrightarrow \frac{F(\omega)}{j\omega} + \pi F(0) \delta(\omega) = G(\omega)$$

$$\begin{aligned} \text{where } F(0) &= F(\omega) \Big|_{\omega=0} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \Big|_{\omega=0} \\ &= \int_{-\infty}^{\infty} f(t) dt \quad \left(\begin{array}{l} \text{DC value} \\ \text{zero frequency} \end{array} \right) \end{aligned}$$

Proof

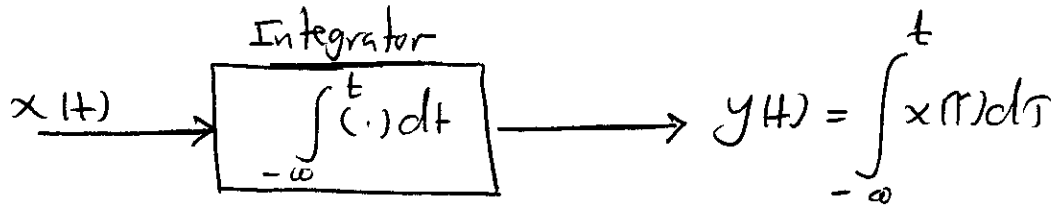
$$\begin{aligned} \text{since } f(t) * u(t) &= \int_{-\infty}^{\infty} f(\tau) u(t-\tau) d\tau \\ &= \int_{-\infty}^t f(\tau) u(t-\tau) d\tau = \int_{-\infty}^t f(\tau) d\tau \end{aligned}$$

F.T of both side and using convolution property

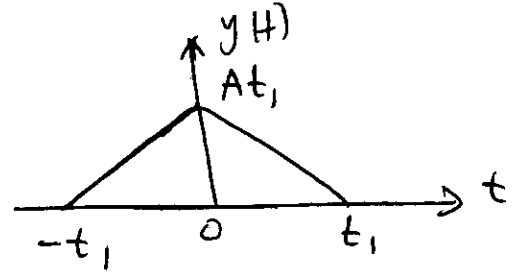
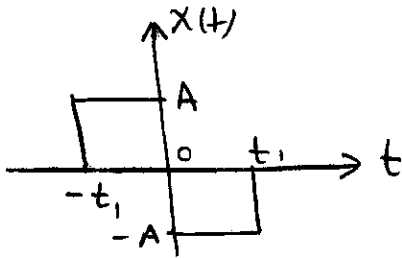
$$\begin{aligned} F(\omega) \cdot [\pi \delta(\omega) + \frac{1}{j\omega}] &\leftrightarrow \text{F.T} \left[\int_{-\infty}^t f(\tau) d\tau \right] \\ \Rightarrow \int_{-\infty}^t f(\tau) d\tau &\leftrightarrow \frac{1}{j\omega} F(\omega) + \pi F(0) \delta(\omega) \end{aligned}$$

Example 5.12

(9)



If $x(t)$ is



Find $Y(\omega)$?
(using Integration property).

$$x(t) = A \text{rect} \left[\frac{t + t_1/2}{t_1} \right] - A \text{rect} \left[\frac{t - t_1/2}{t_1} \right]$$

$$\Rightarrow X(\omega) = At_1 \text{sinc} \left(\frac{t_1 \omega}{2} \right) \left[e^{j\omega t_1/2} - e^{-j\omega t_1/2} \right]$$

$$= At_1 \text{sinc} \left(\frac{t_1 \omega}{2} \right) (2j \sin(\omega t_1/2))$$

$$= j\omega A t_1^2 \text{sinc}^2 \left(\frac{t_1 \omega}{2} \right)$$

Now $y(t) = \int_{-\infty}^t x(\tau) d\tau \Rightarrow Y(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0) \delta(\omega)$

$$X(0) = \left. X(\omega) \right|_{\omega=0} = 0 \text{ or } \int_{-\infty}^{\infty} x(t) dt = 0 \Rightarrow Y(\omega) = \frac{1}{j\omega} X(\omega)$$

$$\Rightarrow Y(\omega) = At_1^2 \text{sinc}^2 \left(\frac{t_1 \omega}{2} \right)$$

Frequency Differentiation

$$\text{If } f(t) \leftrightarrow F(\omega)$$

then,

$$(-jt)^n f(t) \leftrightarrow \frac{d^n F(\omega)}{d\omega^n}$$

Proof

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$\frac{dF(\omega)}{d\omega} = \frac{d}{d\omega} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f(t) \frac{d}{d\omega} (e^{-j\omega t}) dt$$

$$= \int_{-\infty}^{\infty} f(t) (-jt) e^{-j\omega t} dt$$

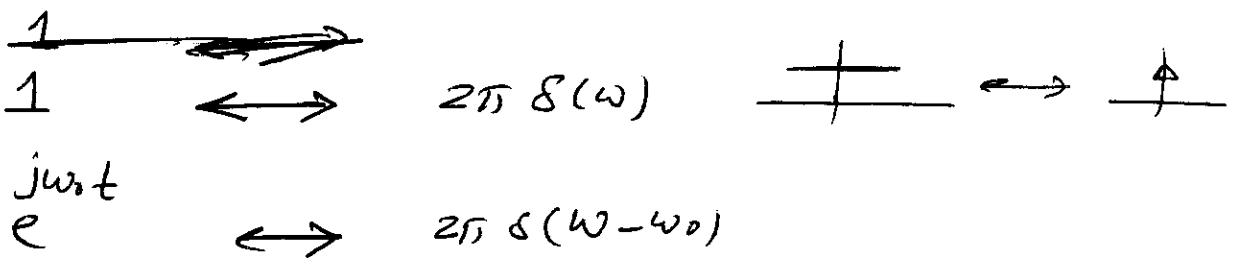
$$= \int_{-\infty}^{\infty} (-jt) f(t) e^{-j\omega t} dt$$

then $\rightarrow (-jt) f(t) \leftrightarrow \frac{dF(\omega)}{d\omega}$

5.3 Fourier Transform of time function

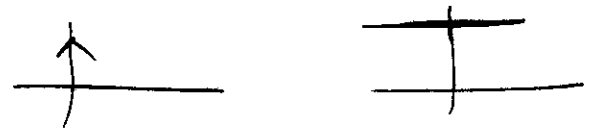
(11)

DC level (both time and frequency)

$$\begin{array}{l} 1 \xleftrightarrow{\quad} 2\pi \delta(\omega) \\ 1 \xleftrightarrow{\quad} 2\pi \delta(\omega) \\ 1 \cdot e^{j\omega t} \xleftrightarrow{\quad} 2\pi \delta(\omega - \omega_0) \end{array}$$


$$K \xleftrightarrow{\quad} 2\pi K \delta(\omega)$$

$$\delta(t) \xleftrightarrow{\quad} 1 \quad (\text{by duality})$$

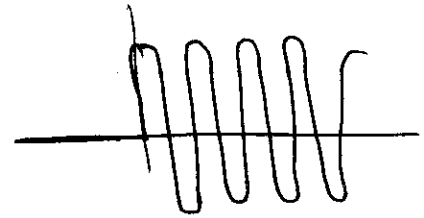


un.ite step

$$\text{sgn}(t) \xleftrightarrow{\quad} \frac{2}{j\omega}$$

$$u(t) = \frac{1}{2}[1 + \text{sgn}(t)] \xleftrightarrow{\quad} \pi \delta(\omega) + \frac{1}{j\omega}$$

switched Cosine



$$\begin{aligned}
 f(t) &= \cos(\omega_0 t) u(t) \\
 &= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} u(t) \\
 &= \frac{1}{2} e^{j\omega_0 t} u(t) + \frac{1}{2} e^{-j\omega_0 t} u(t)
 \end{aligned}$$

$$\frac{1}{2} e^{j\omega_0 t} u(t) \longleftrightarrow \frac{\pi}{2} \left[\delta(\omega - \omega_0) + \frac{1}{j(\omega - \omega_0)} \right]$$

$$\frac{1}{2} e^{-j\omega_0 t} u(t) \longleftrightarrow \frac{\pi}{2} \left[\delta(\omega + \omega_0) + \frac{1}{j(\omega + \omega_0)} \right]$$

Therefore

$$\cos(\omega_0 t) u(t) \longleftrightarrow \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$+ \frac{1}{2} \frac{1}{j(\omega - \omega_0)} + \frac{1}{2} \frac{1}{j(\omega + \omega_0)}$$



$$\frac{j(\omega + \omega_0) + j(\omega - \omega_0)}{2j^2(\omega^2 - \omega_0^2)} = \frac{j\omega}{\omega_0^2 - \omega^2}$$

$$\Rightarrow \cos(\omega_0 t) u(t) \longleftrightarrow \frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$$

Pulse Cosine

$$f(t) = \text{rect}(t/T) \cos(\omega_0 t)$$

$$\begin{matrix} \Downarrow & & \Downarrow \\ \frac{T}{2} \text{sinc}\left(\frac{\omega T}{2}\right) & & \sqrt{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] \end{matrix}$$

$$\Rightarrow F(\omega) = \frac{1}{\sqrt{2}} \left\{ \frac{T}{2} \text{sinc}\left(\frac{\omega T}{2}\right) * [\sqrt{2} (\delta(\omega - \omega_0) + \delta(\omega + \omega_0))] \right\}$$

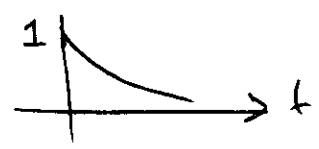
$$= \frac{T}{2} \text{sinc}\left(\frac{\omega T}{2}\right) * \delta(\omega - \omega_0)$$

$$+ \frac{T}{2} \text{sinc}\left(\frac{\omega T}{2}\right) * \delta(\omega + \omega_0)$$

$$= \frac{T}{2} \text{sinc}\left(\frac{(\omega - \omega_0) T}{2}\right) + \frac{T}{2} \text{sinc}\left(\frac{(\omega + \omega_0) T}{2}\right)$$

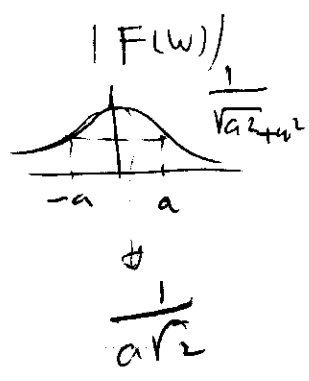
Exponential pulse

$$f(t) = e^{-at} u(t) \quad a > 0$$



$$F(\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a + j\omega}$$



$$\angle F(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$