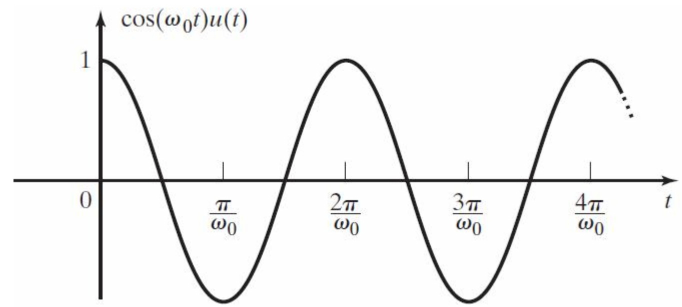
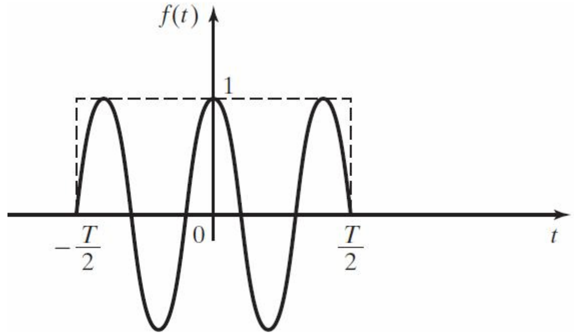
$$u(t) \overset{\mathscr{F}}{\longleftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}$$

Switched Cosine



$$f(t) = \cos(\omega_0 t)u(t)$$

$$f(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}u(t) = \frac{1}{2}e^{j\omega_0 t}u(t) + \frac{1}{2}e^{-j\omega_0 t}u(t)$$

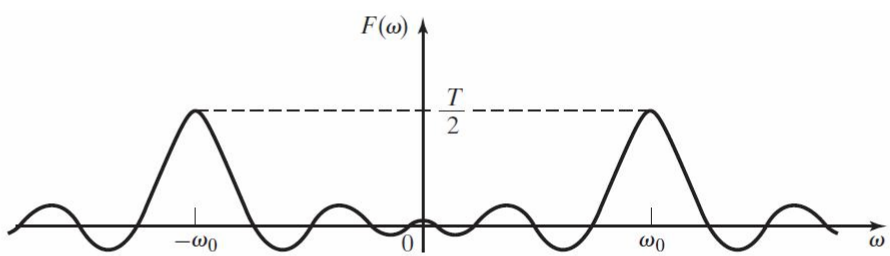


$$\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

Then, applying the convolution

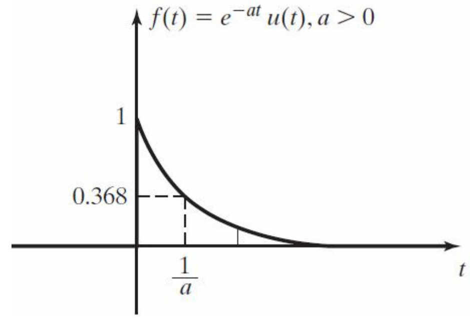
$$F(\omega) = \frac{T}{2}\int_{-\infty}^{\infty}[\delta(\omega - \lambda - \omega_0) + \delta(\omega - \lambda + \omega_0)]\operatorname{sinc}(\lambda T/2)\,d\lambda$$

$$\operatorname{rect}(t/T)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{T}{2}\left[\operatorname{sinc}\frac{(\omega - \omega_0)T}{2} + \operatorname{sinc}\frac{(\omega + \omega_0)T}{2}\right]$$



Exponential Pulse

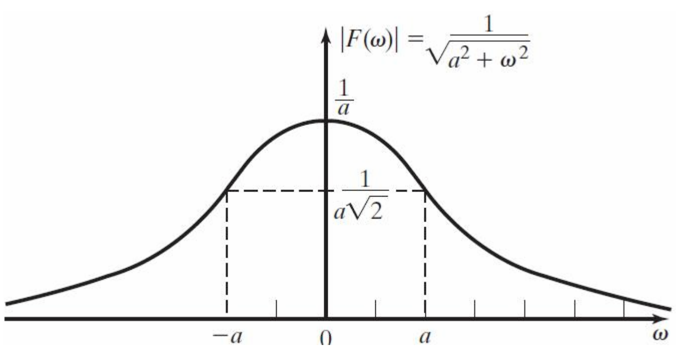
$$f(t) = e^{-at}u(t), a > 0$$

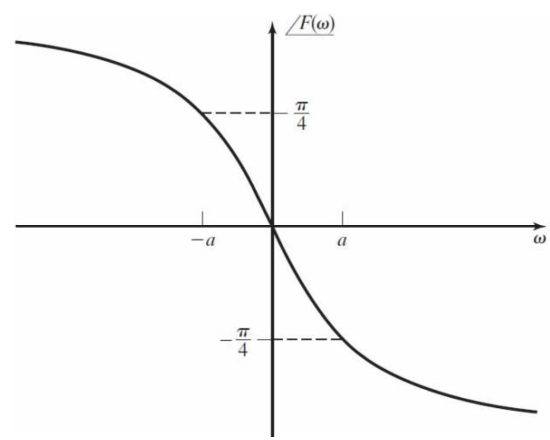


$$F(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \frac{1}{a + j\omega}$$

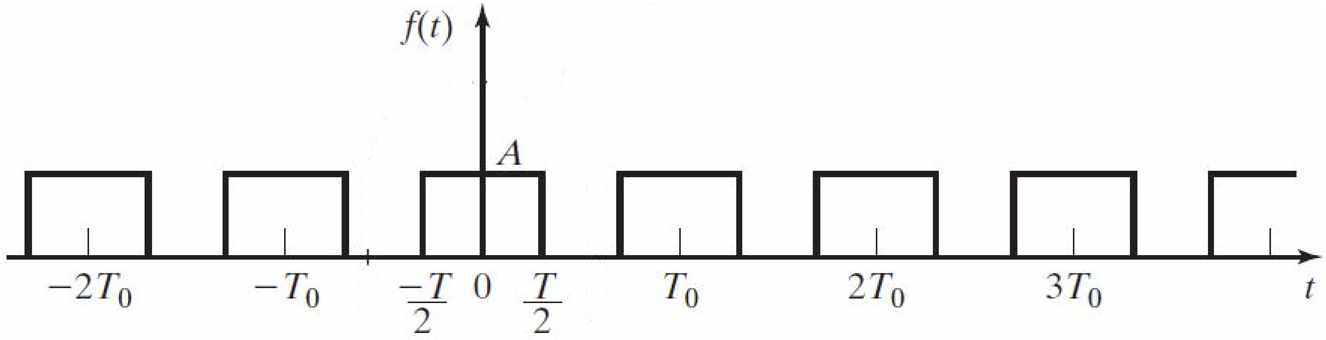
$$e^{-at}u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{a + j\omega}$$

$\mathrm{Re}\{a\} > 0$





The Fourier transform of a periodic signal



$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad C_k = \frac{1}{T_0}\int_{T_0} f(t)e^{-jk\omega_0 t}dt$$

the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\right]e^{-j\omega t}dt \qquad = \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{\infty}(e^{jk\omega_0 t})e^{-j\omega t}dt$$

$$2\pi\delta(\omega - k\omega_0)$$

$$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

Fourier Transform of periodical function is a series of weighted impulses by C_k

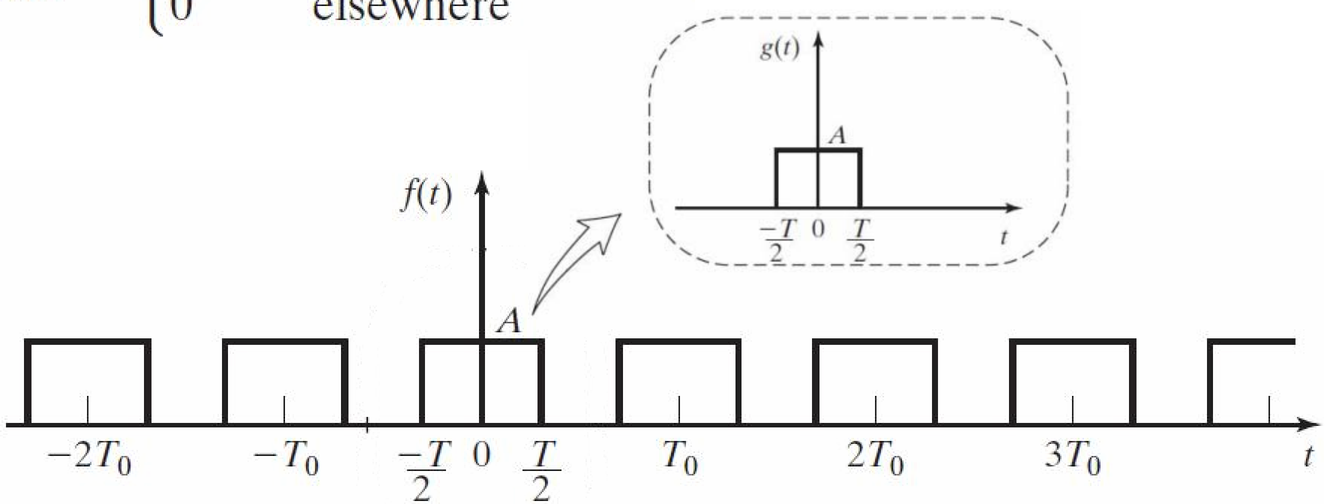
Located at multiple integer (harmonic) of the fundamental frequency ω_0

$$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

We now express the above relation in a different form which we will see can help Us find the Fourier Series coefficients C_k

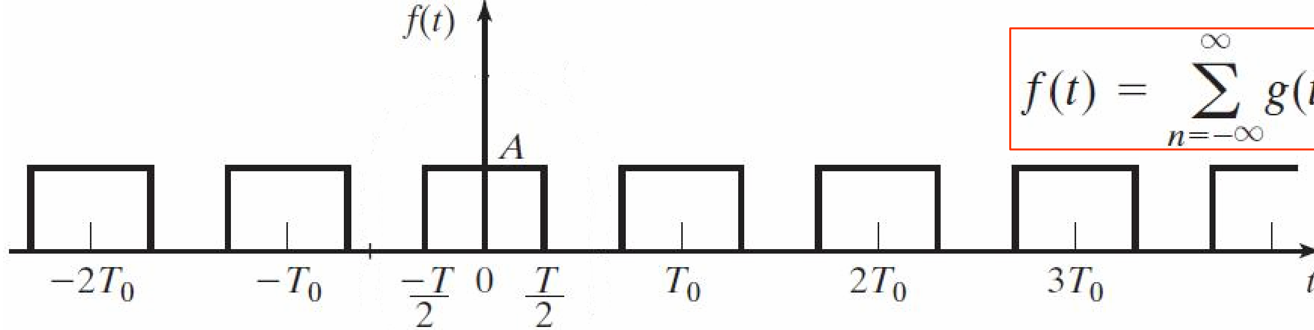
First, we define another function, which we will call the *generating function*

$$g(t) = \begin{cases} f(t), & -T_0/2 \leqq t \leqq T_0/2 \\ 0 & \text{elsewhere} \end{cases}$$



Now we can write the periodical function *f*(t) in terms of *g*(t) as

$$f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$$

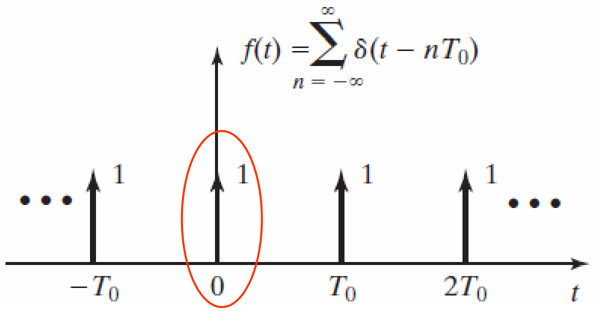


$$f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$$

Since $g(t)*\delta(t - t_0) = g(t - t_0)$ Then we can write $f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$

$$f(t) = \sum_{n=-\infty}^{\infty} g(t)*\delta(t - nT_0 = g(t)* \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

The train of impulse functions is expressed by its Fourier series

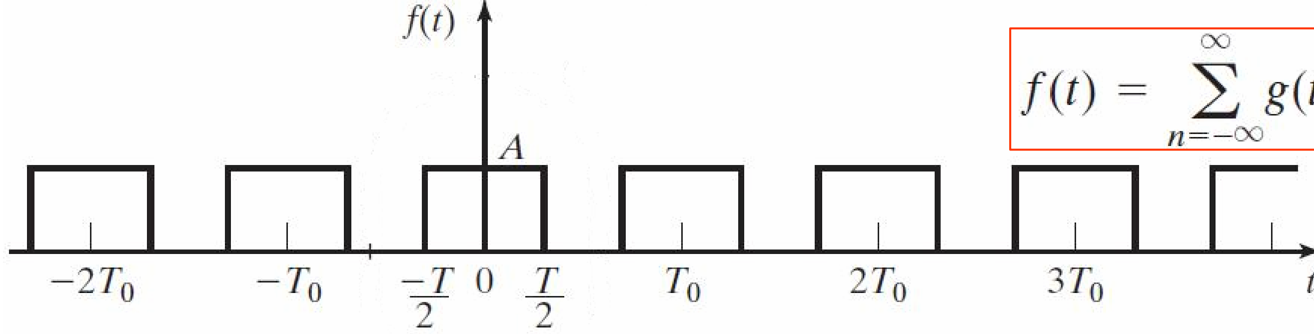


$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\left[\sum_{m=-\infty}^{\infty} \delta(t - mT_0)\right]e^{-jn\omega_0 t}dt$$

Since only $\delta(t)$ Can integrate between $-T_o/2$ and $T_o/2$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)dt = \frac{1}{T_0}$$



$$f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0)$$

$$f(t) = \sum_{n=-\infty}^{\infty} g(t) * \delta(t - nT_0 = g(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)dt = \frac{1}{T_0}$$

$$f(t) = g(t) * \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} g(t) * e^{jn\omega_0 t}$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} G(\omega) \bullet 2\pi\delta(\omega - n\omega_0)$$

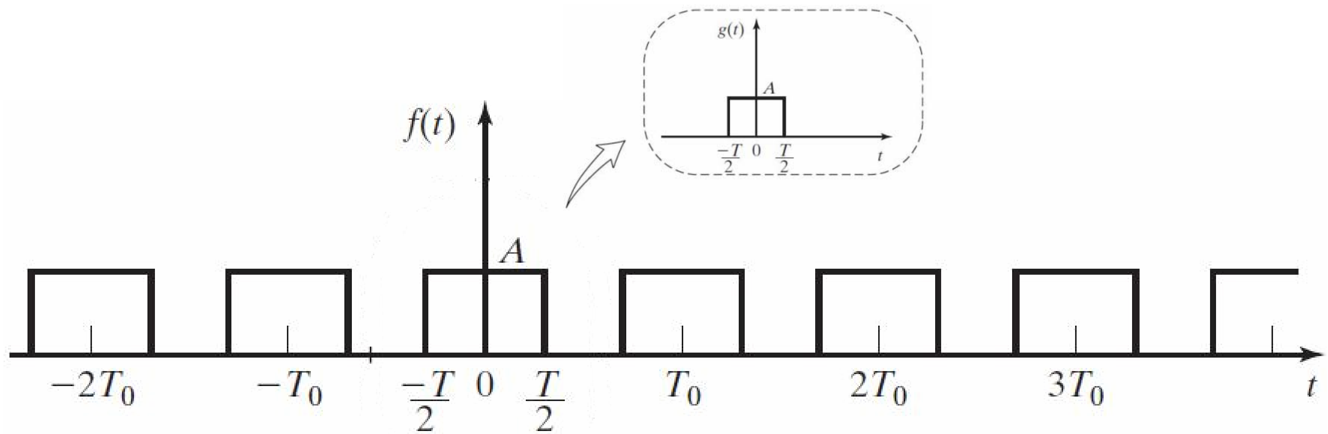
$$= \sum_{n=-\infty}^{\infty} \frac{2\pi}{T_0} G(n\omega_0) \bullet \delta(\omega - n\omega_0) = 2\pi \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} \delta(\omega - n\omega_0)$$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \overset{\mathscr{F}}{\longleftrightarrow} 2\pi \sum_{k=-\infty}^{\infty} C_k \delta(\omega - k\omega_0)$$

$$f(t) = \sum_{n=-\infty}^{\infty} g(t - nT_0) \longleftrightarrow 2\pi \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{T_0} \delta(\omega - n\omega_0)$$

$$C_n = \frac{G(n\omega_0)}{T_0}$$

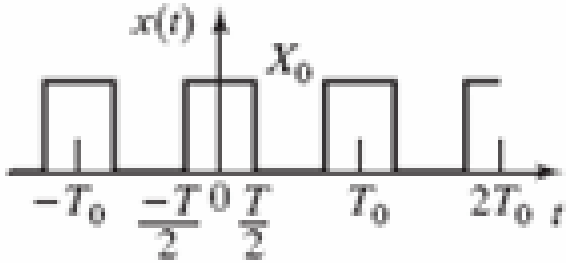
We now find the Fourier transform of the periodic train of rectangular pulses



$$g(t) = A\,\text{rect}(t/T) \quad \Leftrightarrow \quad G(\omega) = AT\,\text{sinc}(T\omega/2)$$

$$G(n\omega_0) = AT\ \text{sinc}\left(\frac{\text{T}n\omega_0}{2}\right) \quad \Rightarrow \quad C_n = \frac{G(n\omega_0)}{T_0}$$

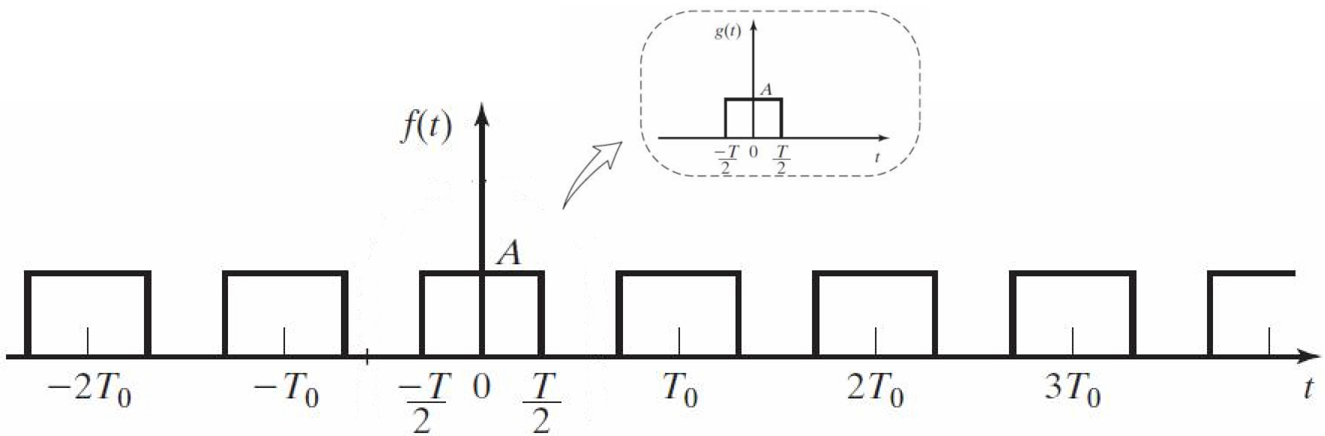
$$\Rightarrow \quad C_n = \frac{AT}{T_0}\ \text{sinc}\left(\frac{\text{T}n\omega_0}{2}\right)$$

6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\operatorname{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2}=\frac{\pi Tk}{T_0}$

$$C_n = \frac{AT}{T_0}\ \operatorname{sinc}\left(\frac{\mathrm{T}n\omega_0}{2}\right)$$

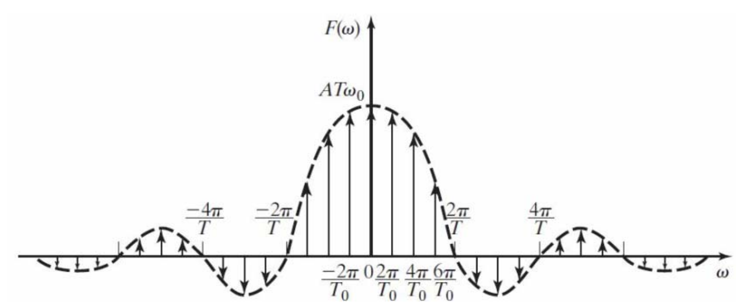
The Fourier transform of a periodic signal

We now find the Fourier transform of the periodic train of rectangular pulses



$$g(t) = A\,\mathrm{rect}(t/T) \quad \Longleftrightarrow \quad G(\omega) = AT\,\mathrm{sinc}(T\omega/2)$$

$$F(\omega) = \sum_{k=-\infty}^{\infty} AT\omega_0\,\mathrm{sinc}(k\omega_0 T/2)\delta(\omega - k\omega_0)$$

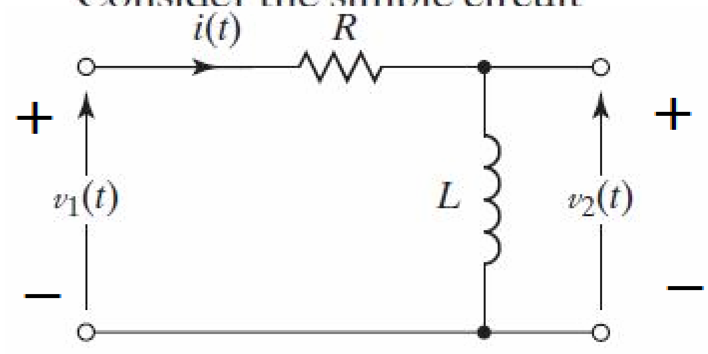


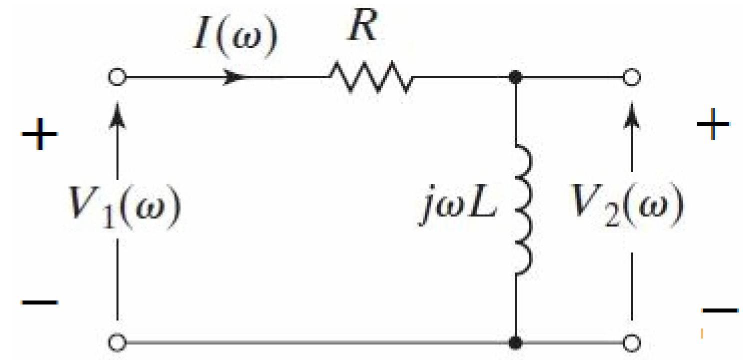
A periodic signal and its frequency spectrum

5.5 APPLICATION OF THE FOURIER TRANSFORM

Frequency Response of Linear Systems

Consider the simple circuit





$$v_1(t) = Ri(t) + L\frac{di(t)}{dt} \quad \text{and} \quad v_2(t) = L\frac{di(t)}{dt}$$

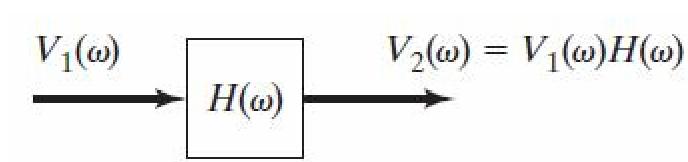
If we take the Fourier transform of each equation.

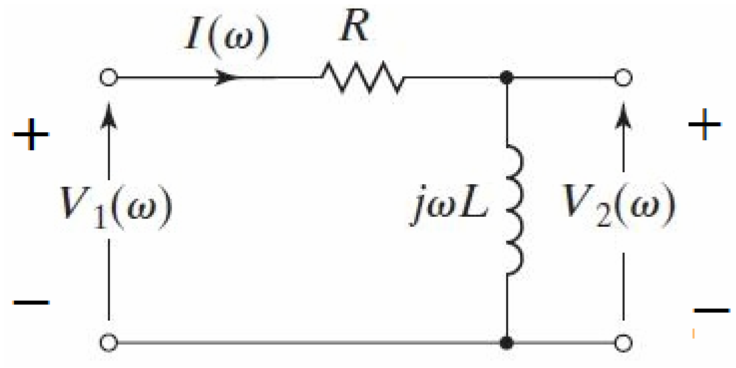
$$V_1(\omega) = RI(\omega) + j\omega L I(\omega) \quad \text{and} \quad V_2(\omega) = j\omega L I(\omega)$$

$$I(\omega) = \frac{1}{R + j\omega L} V_1(\omega).$$

$$V_2(\omega) = \frac{j\omega L}{R + j\omega L} V_1(\omega) \quad \Rightarrow \quad H(\omega) = \frac{V_2(\omega)}{V_1(\omega)} = \frac{j\omega L}{R + j\omega L}$$

Note: voltage division could have been used





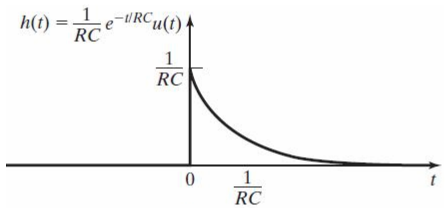
$$H(\omega) = \frac{V_2(\omega)}{V_1(\omega)} = \frac{j\omega L}{R + j\omega L}$$

$$H(\omega) = |H(\omega)| \angle \phi(\omega) = \frac{|V_2(\omega)| \angle V_2}{|V_1(\omega)| \angle V_1} = \frac{|V_2(\omega)|}{|V_1(\omega)|}(\angle V_2 - \angle V_1)$$

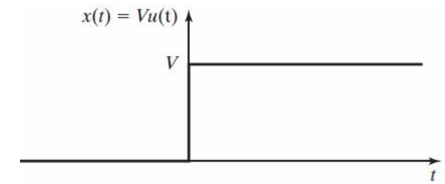
$$|H(\omega)| = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} \qquad \angle \phi(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

EXAMPLE 5.17 **Using the Fourier transform to find the response of a system to an input signal**

$$h(t) = (1/RC)e^{-t/RC}u(t)$$



$$x(t) = Vu(t)$$



$$y(t) = x(t)^*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$Y(\omega) = \mathscr{F}\{x(t)^*h(t)\} = X(\omega)H(\omega)$$

$$H(\omega) = \mathscr{F}\{(1/RC)e^{-t/RC}u(t)\} = \frac{1}{1+j\omega RC}$$

$$X(\omega) = \mathscr{F}\{Vu(t)\} = V\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

Therefore,

$$Y(\omega) = V\left[\frac{1}{1+j\omega RC}\right]\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = V\left[\frac{1}{j\omega(1+j\omega RC)} + \frac{\pi\delta(\omega)}{1+j\omega RC}\right]$$

$$Y(\omega) = V\left[\frac{-RC}{1+j\omega RC} + \frac{1}{j\omega} + \frac{\pi\delta(\omega)}{1+j\omega RC}\right] = V\left[\frac{-RC}{1+j\omega RC} + \frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$Y(\omega) = V\left[\frac{-RC}{1 + j\omega RC} + \frac{1}{j\omega} + \pi\delta(\omega)\right]$$

The time-domain representation of the output can now be found:

$$y(t) = \mathscr{F}^{-1}\{Y(\omega)\} = V\left[\mathscr{F}^{-1}\left\{\frac{1}{j\omega} + \pi\delta(\omega)\right\} - \mathscr{F}^{-1}\left\{\frac{1}{(1/RC) + j\omega}\right\}\right]$$

Using the transform pairs listed in Table 5.2, we find that

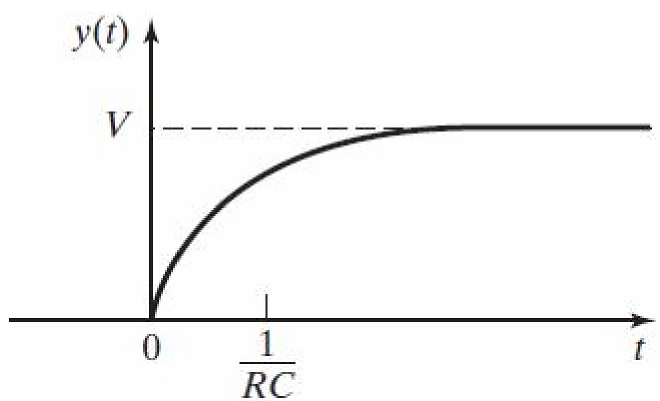
$$u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega) \qquad e^{-t/RC}u(t) \overset{\mathscr{F}}{\longleftrightarrow} \frac{1}{(1/RC) + j\omega}$$

Therefore, the time-domain expression for the output of the network is

$$y(t) = V(1 - e^{-t/RC})u(t)$$

Therefore, the time-domain expression for the output of the network is

$$y(t) = V(1 - e^{-t/RC})u(t)$$



5.6 ENERGY AND POWER DENSITY SPECTRA

An energy signal is defined $E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty$

If the signal is written in terms of its Fourier transform $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$

its energy equation can be rewritten as $E = \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right] dt$

$$F(-\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F(-\omega) d\omega$$

$$F(-\omega) = F^*(\omega)$$

where $F^*(\omega)$ is the complex conjugate of the function $F(\omega)$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F^*(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

The relationship is known as *Parseval's theorem*

Because the function $|F(\omega)|^2$ is a real and even function of frequency

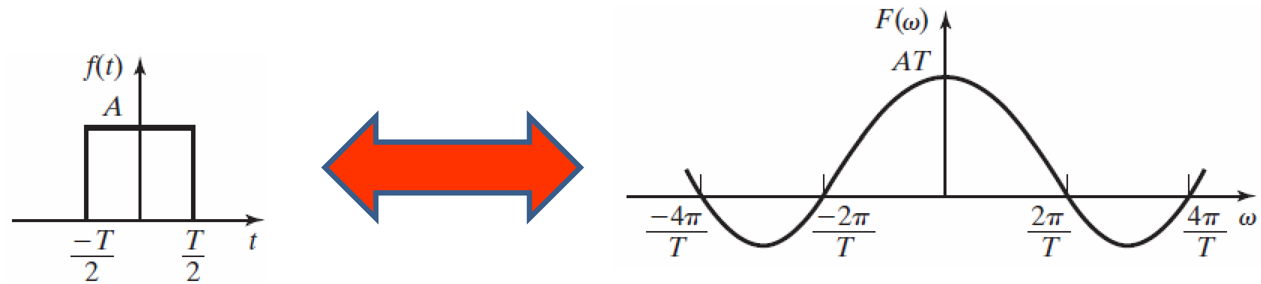
$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \frac{1}{\pi} \int_{0}^{\infty} |F(\omega)|^2 d\omega$$

The *energy spectral density* function of the signal $f(t)$ is defined as

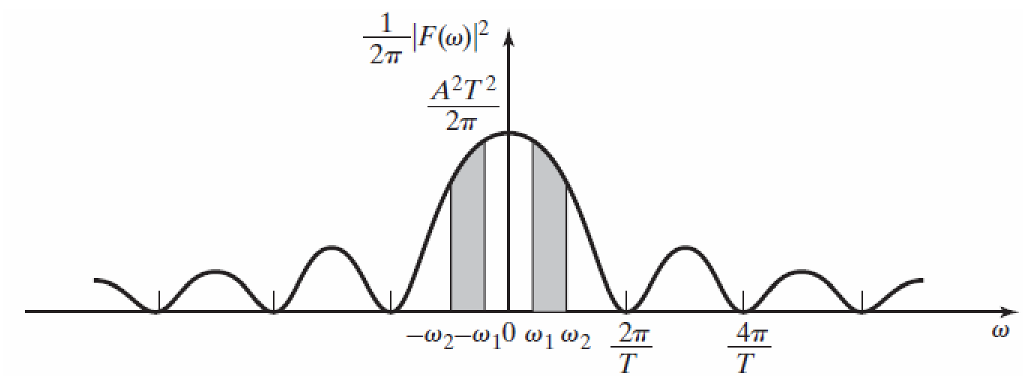
$$\mathcal{E}_f(\omega) \equiv \frac{1}{\pi} |F(\omega)|^2 = \frac{1}{\pi} F(\omega) F(\omega)^*$$

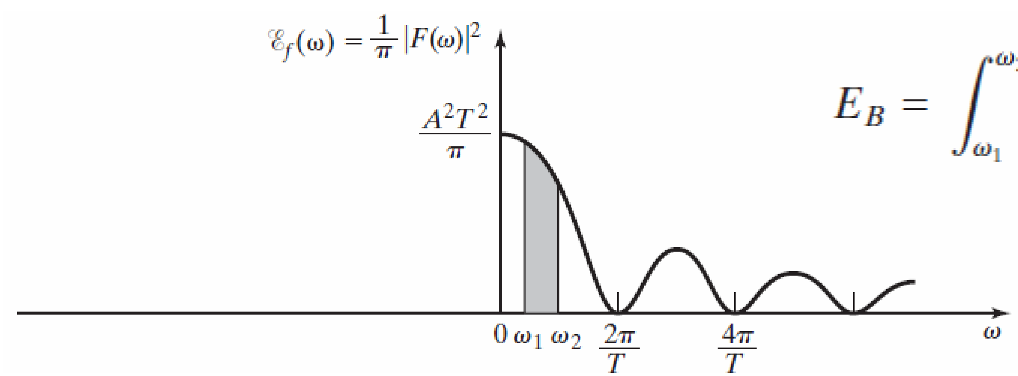
$$\mathcal{E}_f(\omega) \equiv \frac{1}{\pi} |F(\omega)|^2 = \frac{1}{\pi} F(\omega) F(\omega)^*$$

$$E = \int_{0}^{\infty} \mathcal{E}_f(\omega) d\omega$$



The amount of energy contained in the band of frequencies between ω_1 and ω_2





$$E_B = \int_{\omega_1}^{\omega_2} \mathscr{E}_f(\omega) d\omega$$

Power Density Spectrum

consider signals that have infinite energy but contain a finite amount of power

For these signals, the normalized average signal power is finite

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt < \infty$$

Such signals are called *power signals*

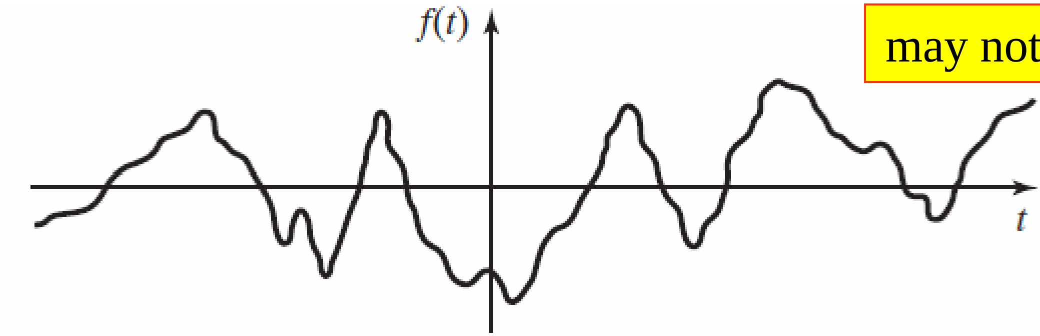
Examples The step function, the signum function, and all periodic functions are of power signals.

A problem with working in the frequency domain in the case of power signals

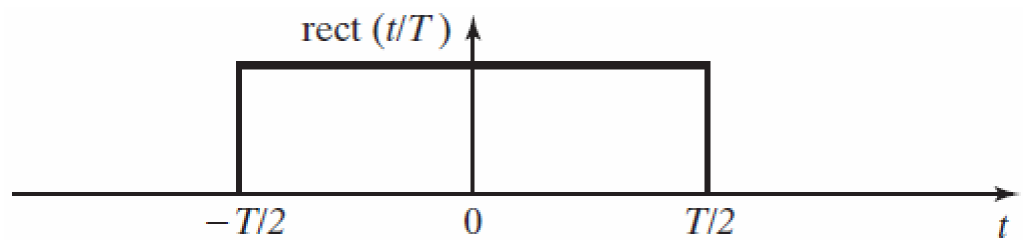
power signals have infinite energy and, therefore, may not be Fourier transformable

To overcome this problem, a version of the signal that is truncated in time is employed.

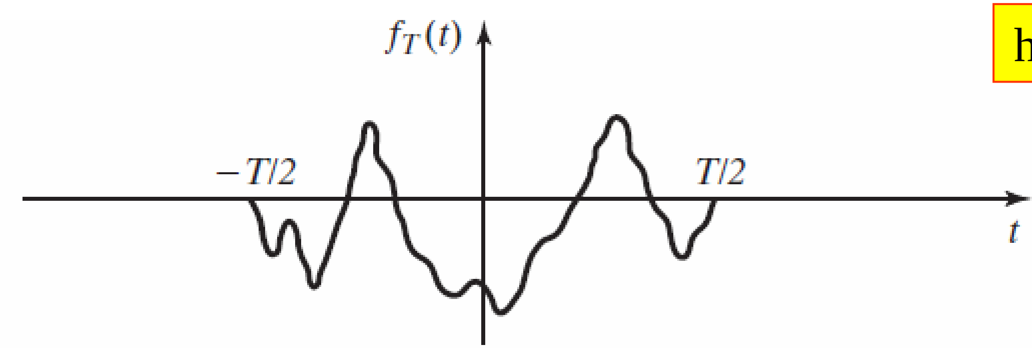
The signal (finite power and infinite energy)



may not have be Fourier transform



The truncated signal $f_T(t) = f(t)\,\mathrm{rect}(t/T)$ has finite energy



has a Fourier transform

$$f_T(t) \overset{\mathscr{F}}{\longleftrightarrow} F_T(\omega)$$

Power Spectral Density

In working with power signals, it is often desirable to know how the total power of the signal is distributed in the frequency spectrum.

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt$$

Sine $f_T(t)$ has zero magnitude for $|t| > T/2$ Then we can write P as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |f_T(t)|^2 dt$$

Because $f_T(t)$ has finite energy, the integral term can be recognized as the total energy contained in the truncated signal:

$$E = \int_{-\infty}^{\infty} |f_T(t)|^2 \, dt$$

$$E = \int_{-\infty}^{\infty} |f_T(t)|^2 \, dt$$

By Parseval's theorem $E = \int_{-\infty}^{\infty} |f_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F_T(\omega)|^2 \, d\omega$

Therefore $P = \lim_{T \to \infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} |F_T(\omega)|^2 \, d\omega$

Interchange the order of the limiting action on *T* and the integration

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{T} |F_T(\omega)|^2 d\omega$$

The integrand is called the power spectral density

$$\mathcal{P}_f(\omega) \equiv \lim_{T \to \infty} \frac{1}{T} |F_T(\omega)|^2$$

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{P}_f(\omega) d\omega$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \mathcal{P}_f(\omega) d\omega$$ because $\mathcal{P}_f(\omega)$ is an even function

Parsevals Theorem for periodical signal

The average power for **periodical signal** is defined as

$$P_{av} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{T_0}\int_{T_0} x(t)x(t)^* dt$$

Now we would like to express P_{av} in terms of the Fourier Coefficients of *x(t)*

$$P_{av} = \frac{1}{T_0}\int_{T_0} x(t)x(t)^* dt = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)^* dt$$

$$= \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} C_n^* e^{-jn\omega_0 t}\right) dt$$

$$P_{av} = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right)^{*} dt \quad = \frac{1}{T_0}\int_{T_0} x(t)\left(\sum_{n=-\infty}^{\infty} C_n^{*} e^{-jn\omega_0 t}\right) dt$$

The order of integration and summation can be inter changed

$$= \sum_{n=-\infty}^{\infty} C_n^{*} \underbrace{\left[\frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt\right]}_{C_n} = \sum_{n=-\infty}^{\infty} C_n^{*} C_n \quad = \sum_{n=-\infty}^{\infty} \left|C_n\right|^2$$

Parsevals Them (for periodical signal)

$$\underbrace{P_{av} = \frac{1}{T_0}\int_{T_0} \left|x(t)\right|^2 dt}_{\text{Time domain}} \quad \underbrace{= \sum_{n=-\infty}^{\infty} \left|C_n\right|^2}_{\text{Frequency domain}} \quad = \underbrace{C_0^2}_{\text{DC power}} + \underbrace{2\sum_{n=1}^{\infty} \left|C_n\right|^2}_{\text{Harmonic Power}}$$

$$Note \quad \left|C_n\right| = \left|C_{-n}\right|$$

Average power is the sum of DC power and harmonics power

Parsevals Them

Aperiodical (none periodical)

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Periodical

$$P_{av} = \frac{1}{T_0} \int_{T_0} \left| x(t) \right|^2 dt = \sum_{n=-\infty}^{\infty} \left| C_n \right|^2$$

$$= C_0^2 + 2 \sum_{n=1}^{\infty} \left| C_n \right|^2$$