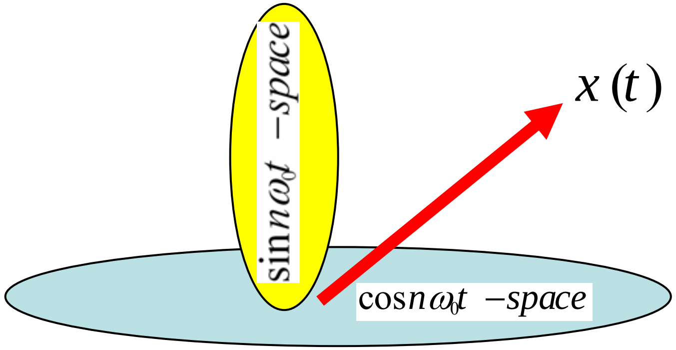
Fourier Series Expansion

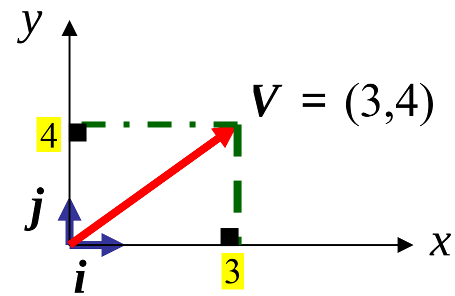
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$



$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

The average of x(t)

2D Vector expansion



$$V = 3i + 4j$$

$3 = V \bullet i$ **Proj of V on the direction of i**

$4 = V \bullet j$ **Proj of V on the direction of j**

$$a_n = x(t) \bullet \cos n\omega_0 t = \frac{2}{T_0}\int_{T_0} x(t) \cos n\omega_0 t\,dt \qquad n \neq 0$$

Projection of x(t) on the direction of $\cos \omega_0 t$

$$b_n = x(t) \bullet \sin n\omega_0 t = \frac{2}{T_0}\int_{T_0} x(t) \sin n\omega_0 t\,dt$$

Projection of x(t) on the direction of $\sin n\omega_0 t$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

Rewritting x(t) as

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2}+\sum_{n=1}^{\infty}b_n\frac{e^{jn\omega_0 t}+e^{-jn\omega_0 t}}{2j}$$

Rearranging terms as

$$x(t)=a_0+\sum_{n=1}^{\infty}\frac{1}{2}\underbrace{(a_n-jb_n)e^{jn\omega_0 t}}_{\text{term 1}}+\sum_{n=1}^{\infty}\frac{1}{2}\underbrace{(a_n+jb_n)e^{-jn\omega_0 t}}_{\text{term 2}}$$

term 1 and term 2 are complex conjugate of each other

Let $C_0=a_0$ $\quad C_n=\frac{1}{2}(a_n-jb_n)$ $\quad C_{-n}=\frac{1}{2}(a_n+jb_n)$ $\Rightarrow$ $C_n=C_{-n}^{*}$

$$\Rightarrow\quad x(t)=C_0+\sum_{n=1}^{\infty}C_n e^{jn\omega_0 t}+\sum_{n=1}^{\infty}C_{-n}e^{-jn\omega_0 t}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t$$

$$x(t) = C_0+\sum_{n=1}^{\infty}C_n e^{jn\omega_0 t}+\sum_{n=1}^{\infty}C_{-n}e^{-jn\omega_0 t}$$

$$C_0=a_0 \quad C_n=\frac{1}{2}(a_n-jb_n) \quad C_{-n}=\frac{1}{2}(a_n+jb_n) \quad \boxed{C_n=C_{-n}^{*}}$$

We can write x(t) as follows:

$$x(t)=\underbrace{\{\cdots+C_{-2}\,e^{-j2\omega_0 t}+C_{-1}\,e^{-j\omega_0 t}\}}_{\sum_{n=1}^{\infty}C_{-n}e^{-jn\omega_0 t}=\sum_{n=-1}^{-\infty}C_n e^{jn\omega_0 t}}+C_0+\underbrace{\{C_1\,e^{j\omega_0 t}+C_2\,e^{j2\omega_0 t}+\cdots\}}_{\sum_{n=1}^{\infty}C_n e^{jn\omega_0 t}}$$

$$x(t)=\sum_{n=-1}^{-\infty}C_n e^{jn\omega_0 t}+C_0+\sum_{n=1}^{\infty}C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty}C_n e^{jn\omega_0 t}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t = \sum_{n=-\infty}^{\infty}C_n e^{jn\omega_0 t}$$

$$x(t)=a_0+\sum_{n=1}^{\infty}a_n\cos n\omega_0 t+\sum_{n=1}^{\infty}b_n\sin n\omega_0 t=\sum_{n=-\infty}^{\infty}C_n e^{jn\omega_0 t}$$

How to find C_n ?

Method 1 (indirect)

Since $C_n=\frac{1}{2}(a_n-jb_n)$

$$C_n=\frac{1}{2}\left[\frac{2}{T_0}\int_{T_0}x(t)\cos n\omega_0 t\,dt-j\frac{2}{T_0}\int_{T_0}x(t)\sin n\omega_0 t\,dt\right]$$

$$=\frac{1}{T_0}\int_{T_0}x(t)\left[\cos n\omega_0 t-j\sin n\omega_0 t\right]dt=\frac{1}{T_0}\int_{T_0}x(t)e^{-jn\omega_0 t}dt$$

C_n is the projection of $x(t)$ on the direction of $e^{jn\omega_0 t}$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{2}(a_n - jb_n) \boxed{= \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt}$$

Since it is true for all m then it is true for all n

Method 2 (direct)

Multiplying both side of x(t) by $e^{-jm\omega_0 t}$ and integrating over T_0

$$\int_{T_0} x(t) e^{-jm\omega_0 t} dt = \int_{T_0} \left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right) e^{-jm\omega_0 t} dt = \sum_{n=-\infty}^{\infty} C_n \int_{T_0} e^{j(n-m)\omega_0 t} dt$$

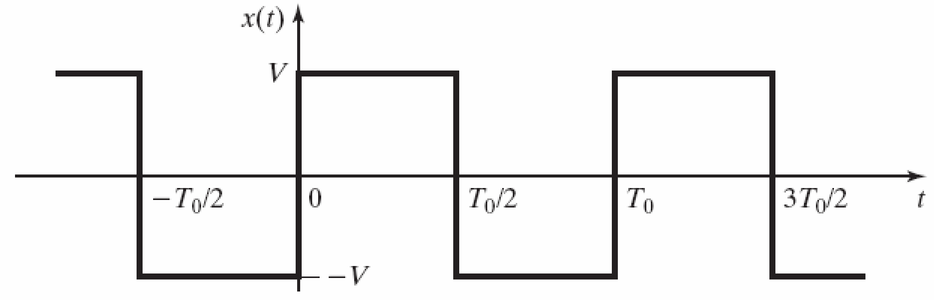
Since $\int_{T_0} e^{j(n-m)\omega_0 t} dt = \begin{cases} 0 & \text{if } m \neq n \\ T_0 & \text{if } m = n \end{cases}$

$$\int_{T_0} x(t) e^{-jm\omega_0 t} dt = \sum_{n=-\infty}^{\infty} C_n \int_{T_0} e^{j(n-m)\omega_0 t} dt = C_m (T_0)$$

$$\boxed{C_m = \frac{1}{T_0}\int_{T_0} x(t) e^{-jm\omega_0 t} dt}$$

EXAMPLE 4.2 **Fourier series of a square wave**

$$x(t) = \begin{cases} V, & 0 < t < T_0/2 \\ -V, & T_0/2 < t < T_0 \end{cases},$$



Solution

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt \quad = \frac{V}{T_0}\int_0^{T_0/2} e^{-jk\omega_0 t}\,dt - \frac{V}{T_0}\int_{T_0/2}^{T_0} e^{-jk\omega_0 t}\,dt = \frac{V}{T_0(-jk\omega_0)}\left[e^{-jk\omega_0 t}\Big|_0^{T_0/2} - e^{-jk\omega_0 t}\Big|_{T_0/2}^{T_0} \right]$$

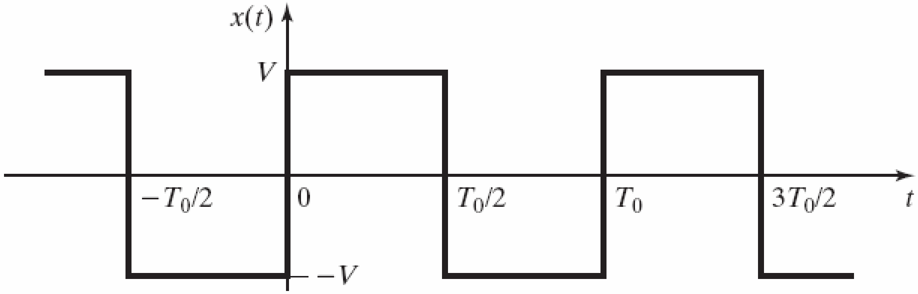
The values at the limits are evaluated as $\omega_0 t\Big|_{t=T_0/2} = \frac{2\pi}{T_0}\frac{T_0}{2} = \pi; \omega_0 T_0 = 2\pi$

Therefore,

$$C_k = \frac{jV}{2\pi k}(e^{-jk\pi} - e^{-j0} - e^{-jk2\pi} + e^{-jk\pi}) = \begin{cases} -\frac{2jV}{k\pi} = \frac{2V}{k\pi}\angle -90°, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

The value of C_0 is seen by inspection, $C_0 = 0$.

The exponential form of the Fourier series of the square wave is then $x(t) = \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{2V}{k\pi} e^{-j\pi/2} e^{jk\omega_0 t}$

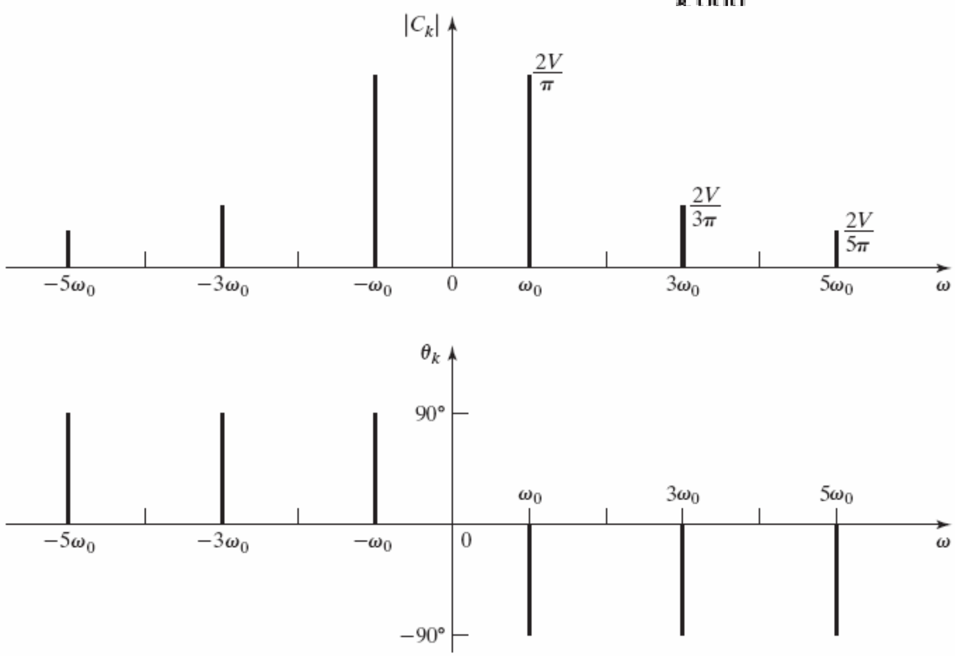


$$C_k = \begin{cases} -\dfrac{2jV}{k\pi} = \dfrac{2V}{k\pi}\angle -90^\circ, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

$$x(t) = \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{2V}{k\pi} e^{-j\pi/2} e^{jk\omega_0 t}$$

The combined trigonometric form is given by

$$x(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4V}{k\pi} \cos(k\omega_0 t - 90^\circ) = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{4V}{k\pi} \sin k\omega_0 t$$



$$x(t)=\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

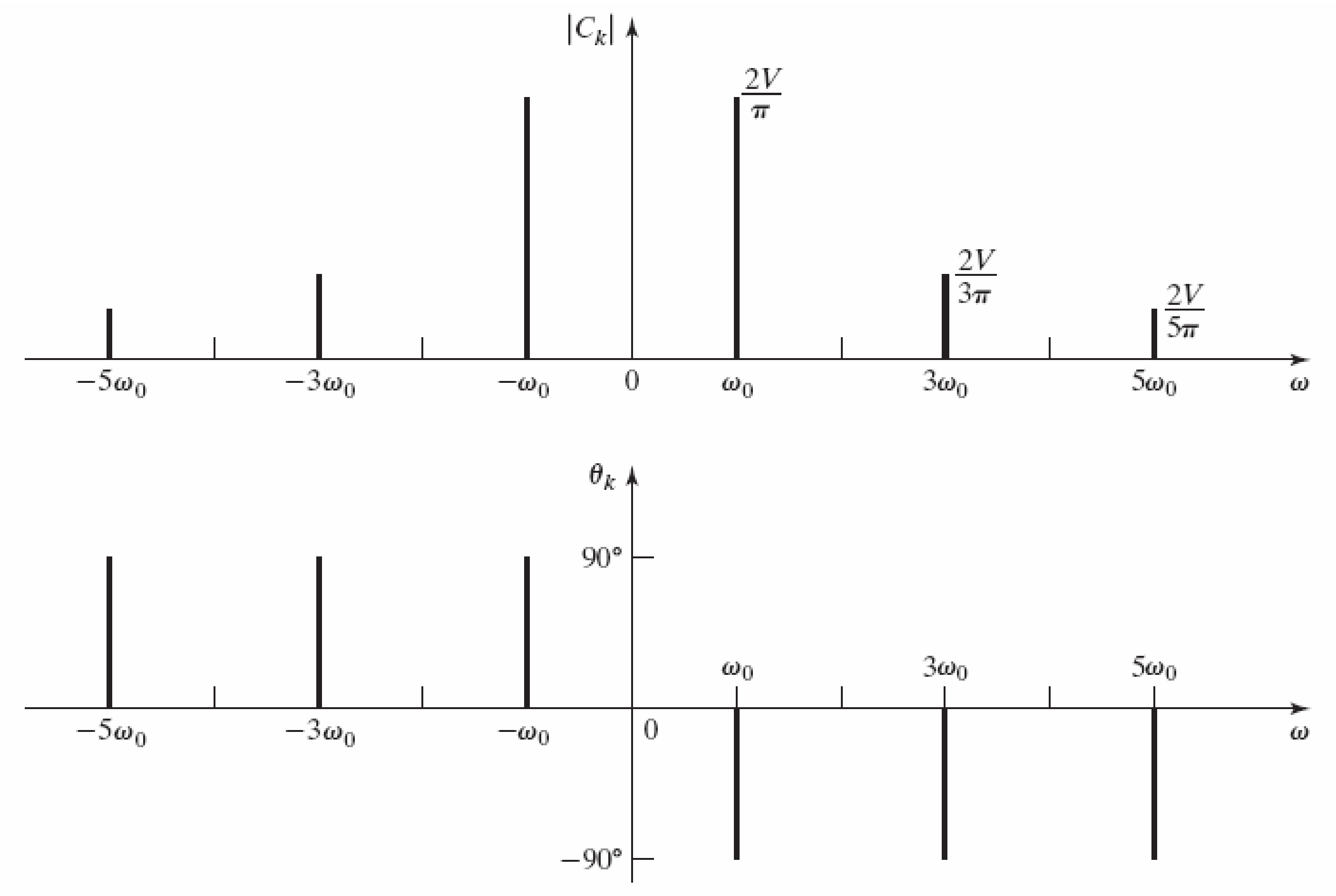
$$=\{\cdots+X_{-2}\,e^{-j2\omega_0 t}+X_{-1}\,e^{-j\omega_0 t}\}+X_0+\{X_1\,e^{j\omega_0 t}+X_2\,e^{j2\omega_0 t}+\cdots\}$$

For each nf_0 $(-\infty < n < \infty)$, $X_n = |X_n| \angle \theta_n$

$$|X_n| = |X_{-n}|$$ **even function**

$$\theta_n = -\theta_{-n}$$ **odd function**

$$C_k = \begin{cases} -\dfrac{2jV}{k\pi} = \dfrac{2V}{k\pi}\angle -90^\circ, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$



Frequency spectrum for a square wave

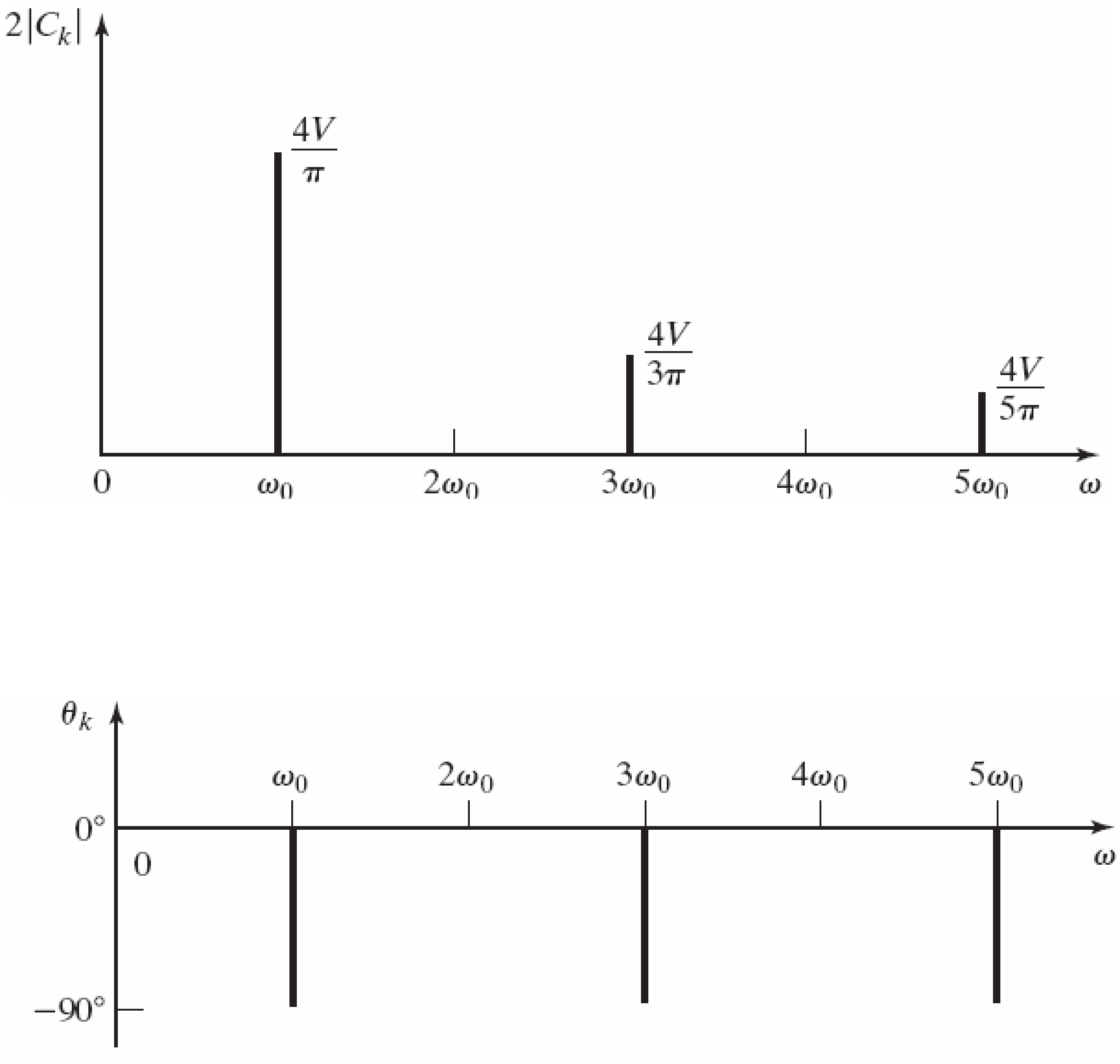
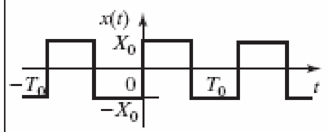
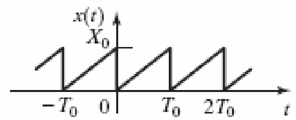
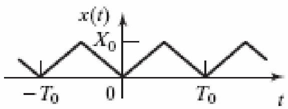
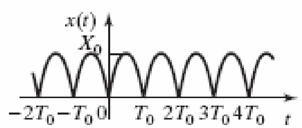
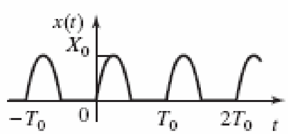
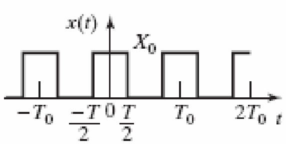
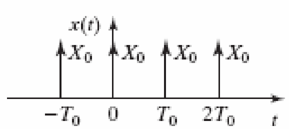


TABLE 4.3 Fourier Series for Common Signals

Name	Waveform	C_0	$C_k,\ k \neq 0$	Comments
1. Square wave		0	$-j\dfrac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\dfrac{X_0}{2}$	$j\dfrac{X_0}{2\pi k}$	
3. Triangular wave		$\dfrac{X_0}{2}$	$\dfrac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\dfrac{2X_0}{\pi}$	$\dfrac{-2X_0}{\pi(4k^2-1)}$	
5. Half-wave rectified		$\dfrac{X_0}{\pi}$	$\dfrac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\dfrac{X_0}{4}$ and $C_{-1} = j\dfrac{X_0}{4}$
6. Rectangular wave		$\dfrac{TX_0}{T_0}$	$\dfrac{TX_0}{T_0}\,\mathrm{sinc}\,\dfrac{Tk\omega_0}{2}$	$\dfrac{Tk\omega_0}{2} = \dfrac{\pi Tk}{T_0}$
7. Impulse train		$\dfrac{X_0}{T_0}$	$\dfrac{X_0}{T_0}$	